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SEMI-BOOLEAN LATTICES*

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An implicative semi-lattice is an algebraic system $\langle L, \leq, \wedge, *\rangle$ in which $\langle L, \leq, \wedge\rangle$ is a meet semi-lattice, and $*$ is a binary composition such that $x \leq y * z$ if and only if $x \wedge y \leq z$ for all elements x, y, z, of L. Every implicative semi-lattice has a greatest element, denoted by 1. If an implicative semi-lattice has a least element 0, then it is called bounded. In a bounded implicative semi-lattice L, elements of the form $x * 0$ are called "closed". The set of closed elements forms a Boolean algebra which is a sub-implicative semi-lattice of L but not necessarily a sub-lattice. By a sub-lattice of an implicative semi-lattice we shall mean a sub-implicative semi-lattice which is a lattice and such that the join of any two elements of the sub-lattice is also a join in the semi-lattice.

An implicative lattice is simply an implicative semi-lattice which happens to be a lattice. Birkhoff [1] identifies bounded implicative lattices with Brouwerian logics. In general, the join of an implicative lattice is not very closely related to the implication. An exception to this is the case of a Boolean algebra. Here the join of two elements a and b always equals the element $(a * b) * b$. With this as a starting point, we make the following definition.

Definition 1. By the *pseudo-join* ab, of two elements a and b of an implicative semi-lattice L, we shall mean the element $((a * b) * b) \wedge ((b * a) * a)$.

Theorem 1. Let L be an implicative semi-lattice, and let a and b be elements of L. Then

(1) $a \leq ab$, $b \leq ab$
(2) $a \leq b$ *if and only if* $ab = b$
(3) $aa = a$, $ab = ba$
(4) $a(a \wedge b) = a = a \wedge (ab)$.

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Proof. First note that $a * ((a * b) * b) = (a \wedge b) * b = 1$, so $a \leq (a * b) * b$. This suffices to prove 1. As for 2, it follows from 1 that if $ab = b$, then $a \leq b$. If $a \leq b$, then $(a * b) * b = b$ and as above $b \leq (b * a) * a$, so $ab = b$. 3 is obvious, and 4 follows immediately from 1 and 2.

There are implicative lattices in which the pseudo-join is not the join. If in an implicative semi-lattice the pseudo-join of any two elements is a join, then we say that the lattice is *semi-Boolean.*

Theorem 2. Let L be an implicative semi-lattice. Then the following six conditions are equivalent:

1. *L is a semi-Boolean lattice.*
2. *$a(bc) = (ab)c$ for a, b, and c any elements of L.*
3. *if $b \leq c$, then $ab \leq ac$ for a, b, and c any elements of L.*
4. *$a(b \wedge c) = (ab) \wedge (ac)$ for a, b, and c any elements of L.*
5. *$(ab) \wedge c = (a \wedge c)(b \wedge c)$ for a, b, and c any elements of L.*
6. *$(ab) * c = (a * c) \wedge (b * c)$ for a, b, and c any elements of L.*

Proof. Since all of the conditions 2 through 6 are true in an implicative lattice, clearly 1 implies 2 through 6. In view of 3 and 4 of theorem 1, 2 implies that the pseudo-join is idempotent, commutative, associative, and absorbtive with respect to meet, and so the pseudo-join is a join. Hence 2 implies 1. To show that any of the conditions 3 through 6 imply 1, let a, b, and c be elements of L such that $a \leq c$, and $b \leq c$. We want to show that any of these conditions implies that $ab \leq c$, or equivalently, that $ab \wedge c = ab$. Assuming 3 we have $ab \leq ac \leq cc = c$. Assuming 4 we have $ab = a(b \wedge c) = (ab) \wedge (ac) = (ab) \wedge c$. Assuming 5 we have $(ab) \wedge c = (a \wedge c)(b \wedge c) = ab$. Assuming 6 we have $(ab) * c = (a * c) \wedge (b * c) = 1$, so $ab \leq c$.

The Boolean algebra of closed elements of a bounded semi-Boolean lattice L is a sub-lattice of L, since the pseudo-join of closed elements is itself closed. Also L is a Stone lattice (every closed element is complemented). However these conditions are not equivalent to that of L being a semi-Boolean lattice, as the lattice of figure 1 shows.

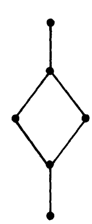

Figure 1

However it is possible to prove the following theorem.

Theorem 3. Let L be a bounded implicative semi-lattice with dense filter D. Then L is a semi-Boolean lattice if and only if the closed algebra of L is a sub-lattice of L and D is a semi-Boolean lattice.

(An element of L is dense if $x * 0 = 0$.)

Proof. Necessity is obvious. So suppose that the closed algebra of L is a sub-lattice of L and that D is a semi-Boolean lattice. Let a, b, and c be elements of L such that $a \leq c$, and $b \leq c$. It is shown in [4] that there are dense elements d_1, d_2, and d_3 such that

$$\begin{aligned} a &= a^{**} \wedge d_1, \ a^{**} * d_1 = d_1 \\ b &= b^{**} \wedge d_2, \ b^{**} * d_2 = d_2 \\ c &= c^{**} \wedge d_3, \ c^{**} * d_3 = d_3 \end{aligned}$$

where for any element x of L, $x^{**} = (x * 0) * 0$. Further, if we let $e_1 = b^{**} * d_1$, and $e_2 = a^{**} * d_2$, then

$$a * b = (a^{**} \wedge d_1) * (b^{**} \wedge d_2) = d_1 * (a^{**} * b^{**}) \wedge a^{**} * (d_1 * d_2) = a^{**} * b^{**} \wedge d_1 * e_2,$$

and so

$$(a * b) * b = (a * b) * (b^{**} \wedge d_2) = (a^{**} * b^{**}) * b^{**} \wedge (d_1 * e_2) * d_2,$$

and hence

$$ab = (a^{**} \vee b^{**}) \wedge (((d_1 * e_2) * d_2) \wedge ((d_2 * e_1) * d_1)).$$

Now if u_1 and u_2 are closed elements of L and v_1 and v_2 are any dense elements of L, then clearly

$$u_1 \wedge v_1 \leq u_2 \wedge v_2 \ \textit{if and only if} \ u_1 \leq u_2 \ \textit{and} \ u_1 * v_1 \leq u_1 * v_2.$$

Hence to prove the theorem, it suffices to show that if $u = a^{**} \wedge b^{**}$ and $v = ((d_1 * e_2) * d_2) \wedge ((d_2 * e_1) * d_1)$, then $u * v \leq u * d_3$. Since the closed algebra of L is a sub-lattice of L,

$$u = (a^{**} \wedge b^{*}) \vee (a^{*} \wedge b^{**}) \vee (a^{**} \wedge b^{**}),$$

where $x^* = x * 0$ for any element x of L. So

$$u * v = (a^{**} \wedge b^{**}) * v \wedge (a^{*} \wedge b^{**}) * v \wedge (a^{**} \wedge b^{*}) * v.$$

Now

$$(a^{**} \wedge b^{*}) * v = a^{**} * (b^{*} * d_1) = b^{*} * d_1,$$

and similarly

$$(a^{*} \wedge b^{**}) * v = a^{*} * d_2.$$

Also

$$(a^{**} \wedge b^{**}) * v = (e_1 * e_2) * e_2 \wedge (e_2 * e_1) * e_1 = e_1 \vee e_2.$$

Similarly,

$$u * d_3 = (a^{**} \wedge b^{*}) * d_3 \wedge (a^{*} \wedge b^{**}) * d_3 \wedge (a^{**} \wedge b^{**}) * d_3.$$

But since $a^{**} * d_1 \leq a^{**} * d_3$, we have that

$$b^{*} * d_1 \leq (a^{**} \wedge b^{*}) * d_3$$

and similarly

$$a^{*} * d_2 \leq (a^{*} \wedge b^{**}) * d_3.$$

Now

$$(a^{**} \wedge b^{**}) * d_3 = a^{**} * (b^{**} * d_3) \geq a^{**} * d_2 = e_2,$$

and similarly

$$(a^{**} \wedge b^{**}) * d_3 \geq e_1, \text{ so } e_1 \vee e_2 \leq (a^{**} \wedge b^{**}) * d_3.$$

Thus we have shown that $u * v \leq u * d_3$. This completes the proof.

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