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SOME PROOFS OF RELATIVE COMPLETENESS IN MODAL LOGIC

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In this paper we adopt for a number of modal systems a method of proving completeness used in [1] (to prove the completeness of S4 relative to T).[1] We prove the completeness of S7, S8 relative to S3, of S3, S6 relative to S2 and of S2 relative to E2.

The system E2 was proposed by E. J. Lemmon[2] and can be axiomatized as follows;

A1 *If* α *is a* **PC**-*tautology then* $\vdash \alpha$
A2 $Lp \supset p$
A3 $L(p \supset q) \supset (Lp \supset Lq)$

(with uniform substitution or replaced by the equivalent schemata)

A4 $\vdash \alpha \supset \beta \rightarrow \vdash L\alpha \supset L\beta$
A5 (**MP**) $\vdash \alpha$, $\vdash \alpha \supset \beta \rightarrow \vdash \beta$

E2 is in fact the system T (as axiomatized in [4], pp. 533-535) without the rule of necessitation. We obtain S2 by replacing A4 by,

A6 $\vdash L(\alpha \supset \beta) \rightarrow \vdash L(L\alpha \supset L\beta)$

and adding;

A7 *If* α *is a* **PC**-*tautology or an axiom then* $\vdash L\beta$[3]

For S3 we replace A6 and A3 by

A8 $L(p \supset q) \supset L(Lp \supset Lq)$

For S6 we add to S2, A9 MMp
For S7 we add to S3, A10 MMp
For S8 we add to S3, A11 $LMMp$
where A7 does not apply to A9 - A11.

An account of validity for these systems has recently been given by Saul Kripke [7] and it is essentially this account we shall use. We define an E2 model as an ordered quadruple $<VWRx_1>$ where W is a set of objects (worlds), $x_1 \varepsilon W$ and R a quasi-reflexive relation over W. By this is meant that for any $x_i \varepsilon W$ if any x_jRx_i then x_iRx_i. Members x_i of W such that x_iRx_i are called normal.[4] Quasi-reflexiveness ensures that no world is

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related to a non-normal world and that every world related to any other world is related to a normal world. V is a function from wffs and members of W to the set $\{1,0\}$ satisfying the following;

1.1 For propositional variable p and $x_i \; \varepsilon \; W$, $V(p\,x_i) = 1$ or 0
1.2 For wff α and $x_i \; \varepsilon \; W$, $V(\sim\alpha\, x_i) = 1$ iff $V(\alpha\, x_i) = 0$ otherwise 0.
1.3 For wffs α and β and $x_i \; \varepsilon \; W$ $V((\alpha \vee \beta)x_i) = 1$ iff either $V(\alpha\, x_i) = 1$ or $V(\beta\, x_i) = 1$, otherwise 0.
1.4 For wff α and $x_i \; \varepsilon \; W$, $V(L\alpha\, x_i) = 1$ iff x_i is normal (i.e. $x_i R x_i$) and for every $x_j R x_i$, $V(\alpha\, x_j) = 1$, otherwise 0. (Thus for non-normal x_i, $V(L\alpha\, x_i) = 0$)

A model $<VWRx_1>$ in which x_1 is normal is a normal model. An S2 model is a normal E2 model. An S3 model is an S2 model in which R is transitive. An S6 (S7) model is an S2 (S3) model in which there is some non-normal $x_i R x_1$. An S8 model is an S3 model in which for every normal $x_i R x_1$ there is some non-normal $x_j R x_i$. A formula α is true in a model $<VWRx_1>$ iff $V(\alpha\, x_i) = 1$. A formula is valid in a given system iff it is true in all models for that system.[5] To ensure that every theorem is valid (in the appropriate system) we need simply ensure that the axioms are valid and the rules validity-preserving. This can be carried out quite mechanically and will not be done here.[6]

We base our relative completeness proofs on the completeness of E2. As noted by M. Ohnishi ([8] p. 139) Anderson's decision procedure for T(M)[7] can be modified for E2 by simply dropping condition III from the conditions for an F-row.[8] It is clear that if I or II hold of an F-row then that F-row is inconsistent in E2 (in the sense of [1] p. 342). And the construction of the counter-model proceeds exactly as in [1] p. 342 except that we might have the case where some formula $L\beta$ has 0 in the table and $\vdash\beta$ where there are no L-constituents having 1. In such a case we simply let x_1 ([1] p. 343) be non-normal. Hence every E2-valid formula is an E2 theorem, i.e. E2 is complete.

The completeness of S2 and S6 relative to E2 follows easily. Clearly $\vdash_{E2} L(p \supset p) \supset \alpha \rightarrow \vdash_{S2}\alpha$[9] since S2 contains material detechment and $L(p \supset p)$. We shew that if α is S2-valid then $L(p \supset p) \supset \alpha$ is E2 valid. And hence, given the completeness of E2, $\vdash_{S2}\alpha$, i.e. the completeness of S2. We shew that if $L(p \supset p) \supset \alpha$ is not E2-valid then α is not S2-valid. If $L(p \supset p) \supset \alpha$ is not E2-valid then for some E2 model $<VWRx_1>$, $V((L(p \supset p) \supset \alpha)x_1) = 0$, i.e. $V(L(p \supset p)x_1) = 1$ and $V(\alpha\, x_1) = 0$. But since $V(L(p \supset p)x_1) = 1$ then x_1 is normal, hence α is false in a normal E2 model, i.e. α is not S2 valid. Thus if α is S2-valid then $\vdash_{S2}\alpha$, i.e. S2 is complete.[10]

For S6 we observe that $\vdash_{S2} MM(p.\sim p) \supset \alpha \rightarrow \vdash_{S6}\alpha$. Hence we shew that if $MM(p.\sim p) \supset \alpha$ is false in some S2 model then that model is an S6 model. It suffices to shew that for any S2 model $<VWRx_1>$ if $V(MM(p.\sim p)x_1) = 1$ then there is some non-normal $x_i R x_1$. Clearly for every normal x_i, $V(M(p.\sim p)x_i) = 0$ hence if every $x_i R x_1$ is normal then $V(MM(p.\sim p)x_1) = 0$ contrary to hypothesis. Hence there must be some non-normal $x_i R x_1$. I.e. $<VWRx_1>$ is an S6 model.

To get the completeness of S3 relative to S2 is more complicated. We shew that for every S2 model $<VWRx_1>$ which falsifies a given formula α there will be some substitution instances of A8 such that if $<VWRx_1>$ verifies all of them then we can construct an S3 model $<V'W'R'x_1'>$ which falsifies α. Given $<VWR\,x_1>$ we define $<V'W'R'x_1'>$ as follows; $x_1' = x_1$, $W' = W$, for normal worlds $R' = R_*$ (the ancestral of R), for non-normal x_i, $x_iR'x_j$ if for some x_k, x_iRx_k and $x_kR_*x_j$. For propositional variable p and $x_i \in W\,(W')$ let $V'(px_i) = V(px_i)$ and let V' be increased to an S3 assignment with respect to W', R', and x_1'. Clearly $<V'W'R'x_1>$ is an S3 model.

LEMMA *If* $<VWR\,x_1>$ verifies *every substitution instance* of A8[11] then for any wf part β of α and any $x_i \in W\;(W')$ $V(\beta x_i) = V'(\beta x_i)$.

Proof by induction on the construction of wf parts of α. For propositional variables the lemma holds from the definition of V'. Clearly the induction holds for truth functions.

We shew that for any wff β if $V(L\beta x_1) = 1$ then $V(\beta x_i) = 1$ for every $x_iR'x_1$. Given $V(L\beta x_1) = 1$ we have $V(L((p \supset p) \supset \beta)x_1) = 1$ hence (by A8) $V(L(L(p \supset p) \supset L\beta)x_1) = 1$ hence for every x_iRx_1 if $V(L(p \supset p)x_i) = 1$ then $V(L\beta x_i) = 1$ i.e. for every normal x_iRx_1, $V(L\beta x_i) = 1$. Now since $V(L\beta x_1) = 1$ then for every x_iRx_1, whether normal or not, $V(\beta x_i) = 1$. For non-normal x_i there are no x_iRx . Further since every substitution instance of A8 (in its strict form) is of the form $L\gamma$ we have A8 true in every x_iRx_1 and so may proceed as before to shew that in every normal x_jRx_i, $V(L\beta x_i) = 1$ and for non-normal x_jRx_i $V(\beta x_j) = 1$. Thus A8 is true in every $x_iR'x_1$ (though only in material form in non-normal worlds). And so given $V(L\beta x_i) = 1$ for any $x_iR'x_1$, (whether $V(L\beta x_i) = 1$ or not) we can, by the above method, ensure that $V(\beta x_j) = 1$ for every $x_jR'x_i$.

Hence if for wf part β of α, $V(L\beta x_i) = 1$ then $V(\beta x_j) = 1$ for every $x_jR'x_i$ hence (induction hypothesis) $V'(\beta x_j) = 1$, hence $V'(L\beta x_i) = 1$. If $V(L\beta x_i) = 0$ then if x_i is non-normal $V'(L\beta x_i) = 0$ and if x_i is normal then for some x_jRx_i and hence for some $x_jR'x_i$, $V(\beta x_j) = 0$. (Note that by definition of R' normal worlds are the same in both models), hence (induction hypothesis) $V'(\beta x_j) = 0$, hence $V'(L\beta x_i) = 0$. Hence the lemma holds (by induction on the construction of wf parts of β).

Now although we assumed that $<VWRx_1>$ verified every substitution instance of A8 the substitutenda for the variables actually used in the proof can all be specified as follows:

1. $(p \supset p)$ and $L(p \supset p)$ are A8-substitutenda.
2. If β is a wf part of α then β is an A8-substitutendum.
3. If β, γ are A8-substitutenda then $L(\beta \supset \gamma) \supset L(L\beta \supset L\gamma)$ and $L(L(\beta \supset \gamma) \supset L(L\beta \supset L\gamma))$ are A8-substitutenda.

An 'appropriate instance' of A8 is one in which p and q are replaced by A8-substitutenda. Now since any formula is of finite degree[12] (say n) we need only consider worlds n R-steps from x_1 and therefore only appropriate instances of A8 of degrée $\leqslant n + 2$. Hence only a finite number. Hence we can form the conjunction γ of these and shew that if α is S3-valid then $\gamma \supset \alpha$ is S2-valid hence $\vdash_{S2} \gamma \supset \alpha$ hence $\vdash_{S3} \alpha$.

The completeness of S7 relative to S3 is proved exactly as that of S6 to S2. For S8 we observe that if $V(LMM(p.\sim p)x_1) = 1$ in any S3 model then for every x_iRx_1, $V(\sim LL\sim(p.\sim p)x_i) = 1$, If x_i is normal this can only be so if for some x_jRx_i, $V(L(p \supset p)x_j) = 0$, i.e. if x_j is non-normal, i.e. if for every normal x_iRx_1 there is some non-normal x_jRx_i. Hence if α is S8 valid then $LMM(p.\sim p) \supset \alpha$ is S3-valid, hence $\vdash_{S3} LMM(p.\sim p) \supset \alpha$ hence $\vdash_{S8} \alpha$.

The completeness results given here have, of course, already been obtained in [7] and some of the decision procedures used occur in [3] and [8]. The interest of the present paper would seem to lie in combining these earlier results with the recent semantical analyses. The results clearly also constitute alternative decision procedures for the systems though (except possibly for E2) they are hardly practical ones.

NOTES

1. Familiarity with [1] will aid comprehension.

2. [2] p. 182. A system equivalent to Lemmon's E3 seems to have been proposed by Anderson in [3] and called by him Q. Lemmon intended the L of his E systems to represent 'It is Scientifically but not logically necessary that' ([2] p. 183) but the derivability of $Lq \supset L(p \supset p)$ would seem to rule out this interpretation.

3. [2] p. 177 (as (a')) In view of A2, A7 could replace A1. S3 was originally axiomatized in [4] pp. 294, 295 and S2 in [5] pp. 124-126 and p. 166. Bases for both systems are summarized in [5] p. 493. These original formulations are shown to be equivalent to the ones given in the present paper in [2].

4. [7] pp. 210-211.

5. Alternatively we could, as in [1], have simply had $<V\ W\ R>$ as the model and defined the validity of α as $V(\alpha x_i) = 1$ for every $x_i \varepsilon W$. For S2, S3 (and particularly for S6, S7) the present way (which more closely follows Kripke) proves more convenient.

6. Kripke proves ([7] pp. 214, 215) that every theorem is appropriately valid when the systems are axiomatized in his formulations and also proves (p. 218) that his axiomatizations are equivalent to Lemmon's.

7. As can his decision procedure for S4 be modified for E3. (v. [3] where the system in question is labelled Q).

8. This condition is set out in [9] p. 212 or (in our terminology) in [1] p. 342. The statement of Anderson's decision procedure when adapted for E2 is in [8] p. 139.

9. In [3] Anderson mentions analogous unpublished results of Halldén for S3, S7 and S8 (and a system S7.5 obtained by adding $MLMMp$ to S3) relative to E3 (Q). What is interesting about the present proofs is of course that they combine this method of shewing relations between systems with semantical analyses of those systems.

10. This shews that an adequate alternative formulation for S2 would be:

    1. *If* $\vdash_{E2} \alpha$ *then* $\vdash_{S2} \alpha$
    2. $L(p \supset p)$
    3. $\vdash \alpha,\ \vdash \alpha \supset \beta \rightarrow \vdash \beta$

    Note however that if $L(p \supset p)$ is added to Lemmon's formulation of E2 we obtain the system T (cf. [10] p. 80 where it is shown that changing A6 to A4 in the S2 axiomatization gives T).

11. The number of substitution instances will in fact be finite (v. *infra*). However the proof proceeds more easily by simply assuming for the present that $\langle V\ W\ R\ x_1 \rangle$ verifies every substitution instance of $L(L(p \supset q) \supset L(Lp \supset Lq))$.

12. For the notion of the 'modal degree' of a formula v. [1] footnote 4 (taken from [11] p. 144 footnote 11). For a precise definition of an 'R-step' v. [1] p. 342.

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