Notre Dame Journal of Formal Logic
Volume XVI, Number 4, October 1975
NDJFAM

ON THE USE OF MORE THAN TWO-ELEMENT BOOLEAN VALUED MODELS

ALEXANDER ABIAN

The independence of the Continuum Hypothesis from the set-theoretical axioms of **ZF** is proved in [1] and [2] by making use of models in each of which a complete Boolean algebra other than $\{0, 1\}$ is used for the logical evaluation of formulas. In both proofs the underlying logical system is the first order Predicate Calculus. We observe that as long as the underlying logical system for a theory is the first order Predicate Calculus, then for that theory, models, in which formulas are evaluated over any complete Boolean algebra, are almost equally suitable. A main reason for this is the fact that over any complete Boolean algebra $\mathfrak{A}$, the usual rules for evaluation of formulas (with $\sim, \forall, \exists, \ldots$ interpreted respectively, as the *complement*, *infimum*, *supremum*, . . .) yield the unit $\mathsf{U}_{\mathfrak{A}}$ of $\mathfrak{A}$ for every logical tautology, and, yield the zero $0_{\mathfrak{A}}$ of $\mathfrak{A}$ for every logical contradiction. On the other hand, as shown in this paper, more than two-element Boolean valued models are often more convenient for handling seemingly unintuitive situations. Such is the situation, for instance, in connection with the proof of the independence of the Continuum Hypothesis where it is possible to construct models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ for **ZF** in such a way that the powerset of the "same" set $\aleph_0$ of **ZF** does not have the same "amount" of elements in $\mathfrak{M}_1$ as it does in $\mathfrak{M}_2$.

In this paper we reproduce the above situation in connection with an extremely modest Example of a set-theory (which is far from resembling **ZF**) and we hope that it will be useful to a reader interested in related matters.

We introduce our Example of a set-theory as a first order theory without equality whose nonlogical symbols consist of the elementhood binary predicate symbol "ϵ" and the six individual constants (sets) $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{n}$. Moreover, our Example has five axioms. However, prior to the description of our Example, we recall that a (complete) Boolean algebra $\langle A, +, \cdot\rangle$ is also a (complete) partially ordered set with respect to $\leqslant$ provided for every element x and y of A we write:

(1) $$x \leqslant y \textit{ if and only if } xy = x$$

Received January 17, 1973

For a first order set-theory S, a *Boolean valued model*, or simply a *model* (or more precisely a *matrix of a model*) $\mathfrak{M}$ with a domain whose individuals are $a_1, a_2, a_3, \ldots$ (not necessarily denumerably many) is meant *in this paper* to be essentially a matrix such as:

$$
(2) \qquad
\begin{array}{c|cccc}
\ni & a_1 & a_2 & a_3 & \cdots \\
\hline
a_1 & b_{11} & b_{12} & b_{13} & \cdots \\
a_2 & b_{21} & b_{22} & \cdots & \\
a_3 & b_{31} & \cdots & & \\
\vdots & \vdots & & &
\end{array}
$$

subject to the following conditions:

(3) *Every b_{ij}, is an element of a given complete Boolean algebra.*
(4) *The value $|H|$ of a formula H which is an axiom of S is equal to the unit $\mathsf{U}_{\mathfrak{A}}$ of $\mathfrak{A}$, where $|H|$ is evaluated with respect to* (2) *according to the following rules.*

Rules for evaluation of formulas Denoting the value of a formula P of S by $|P|$, we let

(5) *For every atomic formula $a_j \in a_i$ of S,*

$$|a_j \in a_i| = b_{ij}$$

(6) *For every formula P and Q of S,*

$$
\begin{aligned}
|\sim P| &= |P| + \mathsf{U}_{\mathfrak{A}} \\
|P \wedge Q| &= |P| \cdot |Q| = \inf\{|P|, |Q|\} \\
|P \vee Q| &= |P| + |Q| + |P| \cdot |Q| = \sup\{|P|, |Q|\} \\
|P \rightarrow Q| &= |P| \cdot |Q| + |P| + \mathsf{U}_{\mathfrak{A}} \\
|P \leftrightarrow Q| &= |P| + |Q| + \mathsf{U}_{\mathfrak{A}} \\
|(\forall x)P(x)| &= \inf\{|P(a_1)|, |P(a_2)|, |P(a_3)|, \ldots\} \\
|(\exists x)P(x)| &= \sup\{|P(a_1)|, |P(a_2)|, |P(a_3)|, \ldots\}
\end{aligned}
$$

Since every formula of S is built from atomic formulas of S in finitely many steps and according to well known rules, we see that, based on (5) and (6), matrix (2) assigns a unique value (i.e., logical value) $|P|$ to a formula P of S. Clearly, in our case $|P|$ is an element of the complete Boolean algebra $\mathfrak{A}$. Also, from (1) to (6) it follows that if formula V is a theorem of S then $|V| = \mathsf{U}_{\mathfrak{A}}$. Moreover, as usual, we say that a formula K is *true* in model $\mathfrak{M}$ if and only if $|K| = \mathsf{U}_{\mathfrak{A}}$.

In the top left corner of matrix (2) the elementhood predicate symbol "$\in$" is advertently written in reverse form "$\ni$" so that b_{ij} denotes the value of the atomic formula $a_j \in a_i$. In this way, the i-th row in matrix (2) describes the set a_i as a function from the domain of individuals of $\mathfrak{M}$ into the complete Boolean algebra $\mathfrak{A}$. This is a customary way of describing sets. It coincides with the familiar representation of sets by their elements in the case where $\mathfrak{A}$ is the two-element Boolean algebra $\{0, 1\}$. In this connection, let us mention that, without loss of generality, any Boolean algebra can be represented as a set of dyadic sequences (i.e., sequences made of 0's and 1's) of a certain ordinal type where addition and multiplication of sequences are performed coordinatewise modulo 2, and, where the unit and the zero of the Boolean algebra under consideration are given respectively by a sequence every coordinate of which is 1, and, by a sequence every coordinate of which is 0. Moreover, if for every sequence s_1 and s_2 we write

(7) $s_1 \leqslant s_2$ *if and only if at every coordinate that* s_1 *has* 1 *so does* s_2

then we see that (7) defines the same partial order which is given by (1).

In view of what we said about the first order Predicate Calculus, if $\mathfrak{M}$ is a model for a first order set-theory S then it can be easily verified that (1) to (6) imply that for every formula P of S,

(8) *If* $|P| = \mathsf{U}_{\mathfrak{A}}$ *then* P *is consistent with the axioms of* S.

Remark 1 From the sixth equality in (6) it follows that $|(\forall x)P(x)| = \mathsf{U}_{\mathfrak{A}}$ if and only if $\mathsf{U}_{\mathfrak{A}} = |P(a_1)| = |P(a_2)| = \ldots$ In other words, formula $(\forall x)P(x)$ is true in model (2) if and only if every instance of $P(x)$ is true. However, from the seventh equality in (6) it follows that the situation *is not so intuitive* in connection with $(\exists x)P(x)$. This is because in a Boolean algebra $\mathfrak{A}$ the supremum of a subset may be equal to the unit $\mathsf{U}_{\mathfrak{A}}$ of $\mathfrak{A}$ without necessitating that $\mathsf{U}_{\mathfrak{A}}$ be an element of that subset. In other words, it is possible that formula $(\exists x)P(x)$ be true in model (2) without necessitating that any of the instances of $P(x)$ be true. This is perhaps one of the most unintuitive aspects of a model such as (2) whose entries b_{ij} are elements of a complete Boolean algebra other than the two-element Boolean algebra $\{0, 1\}$. Indeed in a model where the evaluation of formulas is performed over the two-element Boolean algebra a statement such as "there exists an x such that $P(x)$" is true if and only if "there exists an instance, say, $P(a)$ of formula $P(x)$ such that $P(a)$ is true". However, as mentioned above, in a model where the evaluation of formulas is performed over a complete Boolean algebra other than the two-element Boolean algebra a statement such as "there exists an x such that $P(x)$" may be true *without necessitating* that a single instance $P(a)$ of formula $P(x)$ be true.

In connection with the above, let us observe that if a Boolean algebra $\mathfrak{A}$ has more than two elements then it always has a subset whose supremum is the unit of $\mathfrak{A}$ and such that the unit of $\mathfrak{A}$ is not an element of that subset. For instance, in the case of four-element Boolean algebra (whose elements we represent by dyadic sequences of length 2) we have

$$\sup\{(0, 1),\ (1, 0)\} = (1, 1)$$

Next, we prove that in a model such as (2),

(9) $$(a_j \in a_i) \rightarrow (a_h \in a_k) \text{ if and only if } b_{ij} \leqslant b_{kh}$$

To show (9), assume $(a_j \in a_i) \rightarrow (a_h \in a_k)$. But then from (5) and the fourth equality in (6), we obtain

$$b_{ij} b_{kh} + b_{ij} + \cup_{\mathfrak{A}} = \cup_{\mathfrak{A}} \text{ which implies } b_{ij} b_{kh} = b_{ij}$$

and which, in view of (1), implies $b_{ij} \leqslant b_{kh}$. The converse is proved by reversing the steps.

We introduce the "subset" predicate symbol "$\subseteq$" by:

(10) $$x \subseteq y \textit{ if and only if } (\forall z)((z \in x) \to (z \in y))$$

From the sixth equality in (6), Remark 1, (9) and (10) it follows that in a model such as (1),

(11) $$a_i \subseteq a_k \textit{ if and only if } b_{ij} \leqslant b_{kj} \textit{ for every } j$$

Denoting the elements of an eight-element Boolean algebra by dyadic sequences of length 3, we give the following matrix illustrating examples of (11).

(12)

$\ni$	a_1	a_2	a_3	a_4
a_1	(0, 0, 1)	(1, 1, 1)	(0, 0, 0)	(1, 1, 0)
a_2	(0, 0, 1)	(0, 0, 1)	(0, 0, 0)	(0, 0, 0)
a_3	(0, 0, 0)	(1, 1, 0)	(0, 0, 0)	(1, 1, 0)
a_4	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)

Clearly, in the above, in view of (11) and (7), we have:

(13) $$a_1 \subseteq a_1,\ a_2 \subseteq a_1,\ a_3 \subseteq a_1,\ a_4 \subseteq a_1$$

However, in model (12) none of $a_2 \subseteq a_3$, $a_3 \subseteq a_2$, $a_1 \subseteq a_2$ is true.

Remark 2 It can easily be verified that in model (12) the set a_1 has one and only one element, namely, a_2 and, in fact, a_1 is a singleton (see discussion on page 561). This is because in model (12) the logical value of the formula

$$(\forall x)((x \in a_1) \leftrightarrow (x = a_2))$$

which is equivalent to

$$(\forall x)((x \in a_1) \leftrightarrow (\forall y)((y \in x) \leftrightarrow (y \in a_2)))$$

is equal to (1, 1, 1). But this means that in model (12) the set a_1 is singleton a_2 (since every element of a_1 is equal to a_2). It is also easy to verify that in model (12) the axiom of Extensionality is valid (see discussion on page 563). In other words, in model (12) equal sets are elements of the same sets. Nevertheless, as (13) shows, a_1 has *four* distinct subsets in model (12). Clearly, more than two-element Boolean valued models can readily be constructed where the axiom of Extensionality is valid and *where a singleton has any finite* (>0), *or infinite number of subsets*. We observe and emphasize that this situation (whose analog is a key circumstance in the proof of the independence of the Continuum Hypothesis from the axioms of **ZF**) cannot occur in two-valued models. In other words, in a familiar (i.e., intuitive) two-element Boolean valued model, in which the axiom of Extensionality is also valid, a singleton cannot possibly have more than two subsets. Thus, in many cases, for the proof of consistency of statements which seem to be unintuitive, *more than two-element Boolean valued models seem to be useful.*

As expected, we introduce the "equality" predicate symbol "=" by:

(14) $$x = y \text{ if and only if } (x \subseteq y) \wedge (y \subseteq x)$$

Hence, in a model such as (1),

(15) $$a_i = a_k \text{ if and only if } b_{ij} = b_{kj} \text{ for every } j$$

As usual, in any set-theory a set n is called an *empty* set if and only if

(16) $$(\forall x)(\sim(x \in n))$$

From (5), the sixth and first equalities in (6) and (16) it follows that in a model such as (1) the set a_n is an empty set if and only if

$$\inf\{(b_{n1} + \mathsf{U}_{\mathfrak{A}}), (b_{n2} + \mathsf{U}_{\mathfrak{A}}), (b_{n3} + \mathsf{U}_{\mathfrak{A}}), \ldots\} = \mathsf{U}_{\mathfrak{A}}$$

which, in view of Remark 1, implies

(17) $$b_{nj} = \mathsf{U}_{\mathfrak{A}} + \mathsf{U}_{\mathfrak{A}} = 0_{\mathfrak{A}} \text{ for every } j$$

where $0_{\mathfrak{A}}$ is the zero of the Boolean algebra under consideration. From (15) and (17) it follows that any set-theory has at most one empty set. Clearly, in the example given by (12), the set a_4 is the empty set. In any set-theory a set s is called *nonempty* if and only if

(18) $$(\exists x)(x \in s)$$

From (5), the seventh equality in (6) and (18) it follows that in a model such as (1) the set a_s is a nonempty set if and only if

(19) $$\sup\{b_{s1}, b_{s2}, b_{s3}, \ldots\} = \mathsf{U}_{\mathfrak{A}}$$

Remark 3 Let us consider the following matrix

$\ni$	v_1	v_2	v_3	v_4
v_1	(0, 0, 1)	(1, 1, 1)	(0, 0, 0)	(1, 1, 0)
v_2	(0, 0, 1)	(1, 1, 0)	(0, 0, 0)	(0, 0, 0)
. .	. . . .			

Let us examine in the above matrix the row corresponding to the set v_2. In view of (17), we see that v_2 is not an empty set. However, it is also obvious that none of the sets v_1, v_2, v_3, v_4 is an element of v_2. Nevertheless, from (19) and (12) it follows that

$$(\exists x)(x \in v_2) = \sup\{(0, 0, 1), (1, 1, 0), (0, 0, 0), (0, 0, 0)\} = (1, 1, 1)$$

and, therefore, in the above model the statement "there exists an x such that $x \in v_2$" is true. This is an instance (referred to in Remark 1) of the unintuitive situation which may arise in connection with more than two-element Boolean valued models. For example, in the above model none of $v_1 \in v_3$, $v_2 \in v_3$, $v_3 \in v_3$, $v_4 \in v_3$ is true. Hence, intuitively, one expects that v_2 is an empty set. However, this is not the case, since the statement "there exists an x such that $x \in v_2$" is true in the above model. We observe that in all of this there is no logical inconsistency. The seeming awkwardness of the situation results from our intuitive interpretation of a statement such as "there exists . . .".

Finally, we consider our Example.

Example *Let* S^* *be a set-theory whose individual constants are* a, b, c, d, e, n *and whose axioms are:*

(i) *The only elements of* b *are* a *and* c. *Thus,* b *is a doubleton.*
(ii) *The powerset of* b *is* e.
(iii) n *is an empty set.*
(iv) a, b, c, d, e, n *are pairwise distinct.*
(v) *The axiom of Extensionality* (i.e., equal sets are elements of the same sets).

Proposition 1 *The set-theory* S^* *is consistent.*

Proof: The consistency of S^* follows from the fact that the following two-element Boolean valued matrix is a model for S^*, i.e., axioms (i) to (v) are true with respect to matrix (20).

(20)

$\ni$	a	b	c	d	e	n
a	0	0	1	0	0	0
b	1	0	1	0	0	0
c	1	0	0	0	0	0
d	0	1	0	1	0	0
e	1	1	1	0	0	1
n	0	0	0	0	0	0

It is easy to verify that axioms (i) to (v) are true in the intuitive (i.e., two-element Boolean valued) model (20).

Proposition 2 *The statement:*

(21) *The powerset* e *of the doubleton* b *has five elements* a, b, c, d, n,

is consistent with axioms (i) *to* (v) *of* S^*.

Proof: We prove Proposition 2 in a four-element Boolean valued model. Since our proof involves an unintuitive model, we first rewrite the axioms as formulas of set-theory where "$\subseteq$" and "=" are given respectively by (10) and (14). Accordingly, axioms (i) to (v) become:

(i*) $(\forall x)((x \in \mathsf{b}) \leftrightarrow ((x = \mathsf{a}) \vee (x = \mathsf{c})))$.
(ii*) $(\forall x)((x \in \mathsf{e}) \leftrightarrow (x \subseteq \mathsf{b}))$.
(iii*) $(\forall x)(\sim (x \in \mathsf{n}))$.
(iv*) a, b, c, d, e, n *are pairwise distinct.*
(v*) $(\forall x)(\forall y)(\forall z)(((x = y) \wedge (x \in z)) \rightarrow (y \in z))$.

Now, representing each element of the four-element Boolean algebra by a dyadic sequence of length 2, we propose the following four-valued Boolean model for axioms (i*) to (v*) of S^*.

(22)

∋	a	b	c	d	e	n
a	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(0, 0)	(0, 0)
b	(1, 1)	(1, 0)	(1, 1)	(1, 0)	(0, 0)	(0, 1)
c	(1, 0)	(1, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
d	(1, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 0)	(0, 1)
e	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(0, 0)	(1, 1)
n	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)

Clearly, axioms (iii*) and (iv*) are true in (22) by virtue of (17) and (15), respectively. To prove that axiom (i*) is true in (22), we have to show that in (22),

(23) $$(a \in b) \leftrightarrow ((a = a) \vee (a = c))$$
(24) $$(b \in b) \leftrightarrow ((b = a) \vee (b = c))$$
(25) $$(c \in b) \leftrightarrow ((c = a) \vee (c = c))$$
(26) $$(d \in b) \leftrightarrow ((d = a) \vee (d = c))$$
(27) $$(e \in b) \leftrightarrow ((e = a) \vee (e = c))$$
(28) $$(n \in b) \leftrightarrow ((n = a) \vee (n = c))$$

Examination of (22) shows that (23) and (25) are obviously true. To prove that (24) is true in (22), in view of the third and fifth equalities in (6), we have to show:

(29) $$|b \in b| = \sup\{|b = a|,\ |b = c|\}$$

and in order to prove (29), in view of (14), (10) and sixth and fifth equalities in (6), we have to show that

(30) $$\begin{aligned} |b \in b| = \sup\{&\inf\{(|a \in b| + |a \in a| + (1, 1)),\ (|b \in b| + |b \in a| + (1, 1)),\\ &(|c \in b| + |c \in a| + (1, 1)),\ (|d \in b| + |d \in a| + (1, 1)),\\ &(|e \in b| + |e \in a| + (1, 1)),\ (|n \in b| + |n \in a| + (1, 1))\},\\ &\inf\{(|a \in b| + |a \in c| + (1, 1)),\ (|b \in b| + |b \in c| + (1, 1)),\\ &(|c \in b| + |c \in c| + (1, 1)),\ (|d \in b| + |d \in c| + (1, 1)),\\ &(|e \in b| + |e \in c| + (1, 1)),\ (|n \in b| + |n \in c| + (1, 1))\}\} \end{aligned}$$

which, in view of (22), amounts to proving

(31) $$\begin{aligned}(1, 0) = \sup\{&\inf\{(1, 0),\ (1, 1),\ (1, 0),\ (1, 1),\ (1, 1),\ (1, 0)\},\\ &\inf\{(1, 0),\ (1, 1),\ (0, 0),\ (0, 1),\ (1, 1),\ (1, 0)\}\}\end{aligned}$$

or to proving

(32) $$(1, 0) = \sup\{(1, 0),\ (0, 0)\}$$

But the above equality, in view of (7), is obviously true. Thus, (24) is true in model (22). To prove that (26) is true in (27), we have to show:

(33) $$|d \in b| = \sup\{|d = a|,\ |d = c|\}$$

by performing for (33) the analogs of the three steps (30), (31), (32) which we performed for (29). The last two steps amount to proving

$$\begin{aligned}(1, 0) = \sup\{&\inf\{(1, 0),\ (1, 1),\ (0, 0),\ (0, 1),\ (1, 1),\ (1, 0)\},\\ &\inf\{(1, 0),\ (1, 1),\ (1, 0),\ (1, 1),\ (1, 1),\ (1, 0)\}\}\end{aligned}$$

or to proving

$$(1, 0) = \sup\{(0, 0), (1, 0)\}$$

But the above equality, in view of (7), is obviously true. Thus, (26) is true in model (22). Again the truth of (27) in (22) is established by observing that

$$|\mathbf{e} \in \mathbf{b}| = \sup\{|\mathbf{e} = \mathbf{a}|, |\mathbf{e} = \mathbf{c}|\}$$

is true in (22), because

$$(0, 0) = \sup\{\inf\{(1, 0), (1, 0), (1, 0), (1, 0), (1, 1), (0, 0)\}, \inf\{(1, 0), (1, 0), (0, 0), (0, 0), (1, 1), (0, 0)\}\}$$

or, because

$$(0, 0) = \sup\{(0, 0), (0, 0)\}$$

Similarly, the truth of (28) in (22) is established by observing that

$$|\mathbf{n} \in \mathbf{b}| = \sup\{|\mathbf{n} = \mathbf{a}|, |\mathbf{n} = \mathbf{c}|\}$$

is true in (22), because

$$(0, 1) = \sup\{\inf\{(0, 1), (0, 1), (0, 1), (0, 1), (1, 1), (1, 1)\}, \inf\{(0, 1), (0, 1), (1, 1), (1, 1), (1, 1), (1, 1)\}\}$$

or, because

$$(0, 1) = \sup\{(0, 0), (0, 1)\}$$

Thus, axiom (i*) is true in (22). Next, we prove that axiom (ii*) is true in (22). To this end we have to show that in (22),

(34) $(\mathbf{a} \in \mathbf{e}) \leftrightarrow (\mathbf{a} \subseteq \mathbf{b})$

(35) $(\mathbf{b} \in \mathbf{e}) \leftrightarrow (\mathbf{b} \subseteq \mathbf{b})$

(36) $(\mathbf{c} \in \mathbf{e}) \leftrightarrow (\mathbf{c} \subseteq \mathbf{b})$

(37) $(\mathbf{d} \in \mathbf{e}) \leftrightarrow (\mathbf{d} \subseteq \mathbf{b})$

(38) $(\mathbf{e} \in \mathbf{e}) \leftrightarrow (\mathbf{e} \subseteq \mathbf{b})$

(39) $(\mathbf{n} \in \mathbf{e}) \leftrightarrow (\mathbf{n} \subseteq \mathbf{b})$

A mere inspection of matrix (22) shows that (34) to (39), expect for (38), are obviously true in (22). To prove that (38) is also true in (22), in view of (10) and the fourth, fifth, and sixth equalities in (6), we have to show:

$$\begin{aligned}|\mathbf{e} \in \mathbf{e}| = \inf\{&(|\mathbf{a} \in \mathbf{e}| \cdot |\mathbf{a} \in \mathbf{b}| + |\mathbf{a} \in \mathbf{e}| + (1, 1)), (|\mathbf{b} \in \mathbf{e}| \cdot |\mathbf{b} \in \mathbf{b}| + |\mathbf{b} \in \mathbf{e}| + (1, 1)),\\ &(|\mathbf{c} \in \mathbf{e}| \cdot |\mathbf{c} \in \mathbf{b}| + |\mathbf{c} \in \mathbf{e}| + (1, 1)), (|\mathbf{d} \in \mathbf{e}| \cdot |\mathbf{d} \in \mathbf{b}| + |\mathbf{d} \in \mathbf{e}| + (1, 1)),\\ &(|\mathbf{e} \in \mathbf{e}| \cdot |\mathbf{e} \in \mathbf{b}| + |\mathbf{e} \in \mathbf{e}| + (1, 1)), (|\mathbf{n} \in \mathbf{e}| \cdot |\mathbf{n} \in \mathbf{b}| + |\mathbf{n} \in \mathbf{e}| + (1, 1))\}\end{aligned}$$

which, in view of (22), amounts to proving

$$(0, 0) = \inf\{(1, 1), (1, 0), (1, 1), (1, 0), (1, 1), (0, 1)\}$$

which is obviously true. Hence, $\mathbf{e}$ is the powerset of $\mathbf{b}$.

Finally, we show that axiom (v*) is true in (22). But this follows from the fact that in (22) for every pair of equal first (second) coordinate-rows the corresponding first (second) coordinate-columns are equal. For instance in (22), the first coordinate-rows of **a** and **b** are equal; so are the first coordinate-columns of **a** and **b**. Again, in (22) the second coordinate-rows of **b** and **d** are equal; so are the second coordinate-columns of **b** and **d**. Thus, indeed, (22) is a model for **S***. Now, let us observe that in (22) we have

$$|\mathbf{a} \in \mathbf{e}| = |\mathbf{b} \in \mathbf{e}| = |\mathbf{c} \in \mathbf{e}| = |\mathbf{d} \in \mathbf{e}| = |\mathbf{n} \in \mathbf{e}| = (1, 1)$$

Consequently, from (8) it follows that statement (21) is consistent with the axioms of **S***. Hence, Proposition 2 is proved.

Remark 4 In connection with intuitive models (i.e., two-element Boolean valued models) it is a common practice to represent a set *s* as a pair of braces inside of which all the sets of the domain of the model which are elements of *s* are inserted. We may generalize this representation in connection with unintuitive models. For instance, in case of the four-element Boolean valued model (22), we may represent each set as a pair of braces with *three* compartments corresponding respectively to the logical values (1, 1), (1, 0), (0, 1), where in each compartment the appropriate sets of the domain of the model are inserted. Accordingly, we may represent the sets **a**, **b**, **c**, **d**, **e**, **n** of model (22) as follows:

$$\mathbf{a} = \{\,\| \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \|\,\}, \qquad \mathbf{b} = \{\mathbf{a}, \mathbf{c} \,\|\, \mathbf{b}, \mathbf{d} \,\|\, \mathbf{n}\},$$
$$\mathbf{c} = \{\,\| \mathbf{a}, \mathbf{b} \|\,\}, \qquad \mathbf{d} = \{\mathbf{a} \,\|\, \mathbf{b} \,\|\, \mathbf{c}, \mathbf{n}\},$$
$$\mathbf{e} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{n} \,\|\,\|\,\}, \qquad \mathbf{n} = \{\,\|\,\|\,\}.$$

We may call the above "*unintuitive representation of sets*" (or, perhaps, representation of *unintuitive sets*), and, we may observe that the above representation gives some insight as to the various properties of sets.

Proposition 3 *The statement:*

(40) *The powerset* **e** *of the doubleton* **b** *has four elements* **a**, **b**, **c**, **n**,

is consistent with axioms (i) *to* (v) *of* **S***.

Proof: The proof of Proposition 3 follows readily from the intuitive model (20). Indeed, it is easy to verify that in (20) axioms (i) to (v) of **S*** are true and that **e** is the powerset of **b** and **a**, **b**, **c**, **d** are the only elements of **e**. Consequently, statement (40) is consistent with the axioms of **S***.

Proposition 4 *The statement:*

(41) *The powerset* **e** *of the doubleton* **b** *has three elements* **b**, **c**, **n**,

is consistent with axioms (i) *to* (v) *of* **S***.

Proof: For the proof of Proposition 4, we consider the following intuitive model for **S***:

(42)

∋	a	b	c	d	e	n
a	0	1	0	0	0	0
b	1	0	1	0	0	0
c	1	0	0	0	0	0
d	0	1	0	1	0	0
e	0	1	1	0	0	1
n	0	0	0	0	0	0

It is easy to verify that in the intuitive model (42) axioms (i) to (v) of S* are true and that e is the powerset of b and b, c, n are the only elements of e. Consequently, statement (41) is consistent with the axioms of S*.

Combining Propositions 1 to 4 we have:

Proposition 5 *Consider the set-theory* S* *whose individual constants are* a, b, c, d, e, n *and whose axioms are given by* (i) *to* (v). *Then* S* *is consistent and each of the following statements is consistent with the axioms of* S*:

(21) *The powerset* e *of the doubleton* b *has five elements* a, b, c, d, n.
(40) *The powerset* e *of the doubleton* b *has four elements* a, b, c, n.
(41) *The powerset* e *of the doubleton* b *has three elements* b, c, n.

The reader is advised to observe an analogy between Proposition 5 and the following:

Proposition *Assuming that the Zermelo-Fraenkel set-theory* ZF *is consistent, let* $P(\aleph_0)$ *denote the powerset of the set* $\aleph_0$ *of all natural numbers of* ZF. *Then, for instance, each of the following statements is consistent with the axioms of* ZF:

(21*) $P(\aleph_0)$ *is equipollent to* $\aleph_5$.
(40*) $P(\aleph_0)$ *is equipollent to* $\aleph_4$.
(41*) $P(\aleph_0)$ *is equipollent to* $\aleph_3$.

REFERENCES

[1] Rosser, J. B., *Simplified Independence Proofs*, Academic Press, New York (1969).

[2] Scott, D., and Solovay, R., "Boolean-valued models for set-theory," to appear in *Proceedings of the American Mathematical Society*.

Iowa State University
Ames, Iowa