552
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A SEMANTICAL PROOF OF THE UNDECIDABILITY OF THE MONADIC INTUITIONISTIC PREDICATE CALCULUS OF THE FIRST ORDER

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A constructive proof of the undecidability of the monadic intuitionistic predicate calculus of the first order was given in [3]. We shall denote this calculus with "**MIP**". The aim of this article is to give a proof of the undecidability of the **MIP**, using Kripke semantics. We shall show that the class of formulae of the **MIP** which contain only two predicate variables is undecidable. We shall denote this class by $\mathcal{L}$. It is well known that in the classical two-valued predicate calculus of the first order, the class of formulae which contain only one binary predicate variable is undecidable. We shall denote this class by $\mathcal{M}$. Let $\mathcal{K}$ be the class of all formulae of the intuitionistic predicate calculus of the first order, then we can obtain any formula $H \epsilon \mathbf{M}$ from $\mathcal{K}$, by simply interpreting classically every propositional functor and every quantifier occurring in H. Since $\mathcal{M}$ is undecidable, it follows that the class $\mathcal{K}'$ of all formulae of $\mathcal{K}$ with only one binary predicate variable is also undecidable. To prove the undecidability of $\mathcal{L}$ we shall assign, to every closed formula $H \epsilon \mathcal{K}'$, a formula $H^* \epsilon \mathcal{L}$ such that H is valid in $\mathcal{K}$ if and only if H^* is valid in **MIP**, then since $\mathcal{K}'$ is undecidable, it follows that $\mathcal{L}$ is also undecidable. The details of Kripke semantics will be assumed (see [1]).

The following definitions are given in [1], p. 94, namely:

(a) $\phi(A \wedge B, H) = \mathbf{T}$ iff $\phi(A, H) = (B, H) = \mathbf{T}$, otherwise $\phi(A \wedge B, H) = \mathbf{F}$,
(b) $\phi(A \vee B, H) = \mathbf{T}$ iff $\phi(A, H) = \mathbf{T}$ or $\phi(B, H) = \mathbf{T}$, otherwise $\phi(A \vee B, H) = \mathbf{F}$,
(c) $\phi(A \rightarrow B, H) = \mathbf{T}$ iff for all $H' \epsilon \mathcal{K}$ such that $H R H'$, $\phi(A, H) = \mathbf{F}$ or $\phi(B, H) = \mathbf{T}$, otherwise $\phi(A \rightarrow B) = \mathbf{F}$,
(d) $\phi(\sim A, H) = \mathbf{T}$ iff for all $H' \epsilon \mathcal{K}$ such that HRH', $\phi(A, H') = \mathbf{F}$, otherwise, $\phi(\sim A, H) = \mathbf{F}$.

In addition to the above definitions we shall add the following:

(e) $\phi(\sim\sim A, H) = \mathbf{T}$ iff there exist an $H' \epsilon \mathcal{K}$ such that $\phi(A, H') = \mathbf{T}$, otherwise, $\phi(\sim\sim A, H) = \mathbf{F}$.

The definition (e) is clearly consistent with the definitions (a)-(d).

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Now let $H \epsilon K$ be a closed formula of K containing only one dyadic predicate variable $R(x, y)$, we define the formula H^* to be the result of replacing $R(x, y)$ by $(\sim\sim(P(x) \wedge Q(y)))$ throughout H, where P and Q are given monadic predicate variables, so that $H^* \epsilon \mathcal{L}$ is a formula of the **MIP**. Since H^* is obtained from H by the above substitution, it follows that if H is valid then H^* is also valid; thus if H is provable in K then H^* is also provable in the **MIP**.

Conversely, we shall show that if H is not valid in the intuitionistic predicate calculus of the first order, then H is also not valid and hence not provable in the **MIP**. Suppose that H is not valid and hence not provable in the intuitionistic predicate calculus of the first order, then by the Löwenheim-Skolem theorem, H has a counter model ϕ_1 in the domain of positive integers; that is if we assign a certain set Γ of ordered pairs of positive integers, $\phi_1(H, H_k) = \mathbf{F}$ for a certain H_k. Using the model theory of [1], we shall show that in this case H^* also has a counter model in the domain of positive integers. Let K be a countable set of all mappings which assign positive integers to the individual variables x and y occurring in H and H^*; we index the elements of K on the positive integers. We define a relation R on K as follows:

Let H_i, H_j be any two elements of K, and let m, m' be the least positive integers which H_i and H_j respectively assign to the predicate variable P in H^*, such that $\langle m, i\rangle \epsilon \Gamma$ and $\langle m', j\rangle$ (if there is no such integer for H_i or H_j, we put m or $m' = 0$), then the relation R is defined by $H_i R\, H_j$ if $m \leqslant m'$ and $i \leqslant j$. The relation R is reflexive and transitive on K. We define a domain function ψ on K by $\psi(H) = \Gamma_H$ for every $H \epsilon K$ where Γ_H, is the set of pairs of positive integers which H assigns to the predicate variables P and Q in H^*. It follows from the definition of K that $\bigcup \psi(H) = I$ where I is the set of all positive integers, and that $\psi(H)$ contains at least one element for every $H \epsilon K$.

Thus (H_1, K, R) together with ψ, as defined above, is a quantification model structure in the sense of [1]. Let ϕ_2 be a model on the above model structure (see [1]). Since P and Q are the only atomic formulae in H^*, and since H^* contains no free individual variables, we shall define the values of $\phi_2(P, H_n)$ and $\phi_2(Q, H_n)$ for every $H_n \epsilon K$ as follows:

$\phi_2(P, H_n) = \mathbf{T}$ iff H_n assigns a positive integer m to P such that $\langle m, n\rangle \epsilon \Gamma$, otherwise $\phi_2(P, H_n) = \mathbf{F}$.

$\phi_2(Q, H_n) = \mathbf{T}$ iff H_n assigns only the positive integer n to Q, otherwise $\phi_2(Q, H_n) = \mathbf{F}$.

We shall show that for any $H_n \epsilon K$, if x is assigned m and y is assigned n' by H_n, then $\phi_2(\sim\sim(P(x) \wedge Q(y)), H_n) = \mathbf{T}$ iff $n' = n$ and $\langle m, n\rangle \epsilon \Gamma$, otherwise, $\phi_2(\sim\sim(P(x) \wedge Q(y)), H_n) = \mathbf{F}$.

By definition (e), if x is assigned m and y is assigned n', then $\phi_2(\sim\sim(P(x) \wedge Q(y)), H_n) = \mathbf{T}$ iff there is an H_i with $H_n R H_i$ such that $\phi((P(x) \wedge Q(y)), H_i) = \mathbf{T}$. Hence, by the definitions of $\phi_2(P, H_n)$ and $\phi_2(Q, H_n)$, and the definition (a), if y is assigned n', then $\phi_2(Q(y), H_n) = \mathbf{T}$ iff $n' = n$; otherwise $\phi_2(Q(y), H_n) = \mathbf{F}$. Hence, if $\phi_2(\sim\sim(P(x) \wedge Q(y)), H_n) = \mathbf{T}$, then $n' = n$. But, if $\phi_2((P(x) \wedge Q(y)), H_n) = \mathbf{T}$ then $\phi_2(P(x), H_n) = \mathbf{T}$ and, therefore, by definition of $\phi_2(P(x), H_n)$, it follows that if $n' = n$ and $\langle m, n\rangle \in \Gamma$, then $\phi_2(\sim\sim(P(x) \wedge Q(y)), H_n) = \mathbf{T}$.

Conversely, let $n' = n$ and $\langle m, n\rangle \in \Gamma$, then, by definition, $\phi(P(x), H_n) = \mathbf{T}$ and $\phi(Q(y), H_n) = \mathbf{T}$, hence $\phi_2(P(x) \wedge Q(y)), H_n) = \mathbf{T}$ and with it $\phi_2(\sim\sim(P(x) \wedge Q(y)), H_n) = \mathbf{T}$, as required.

On the other hand, if m and n are assigned to the individual variables x and y respectively such that $\langle m, n\rangle \in \Gamma$, then $\phi_1(R(x, y), H_k) = \phi_2(R(x, y), H_n) = \mathbf{T}$, and since H^* is obtained from H by substituting $(\sim\sim(P(x) \wedge Q(y)))$ for $R(x, y)$ throughout H, it follows that $\phi_1(H_2, H_k) = \mathbf{T}$, if $\phi_2(H^*, H_n) = \mathbf{T}$, and since, by hypothesis, $\phi_1(H, H_k) = \mathbf{F}$, it follows that $\phi_2(H^*, H_n) = \mathbf{F}$.

Since the intuitionistic predicate calculus is complete relative to the system of modelling above, it follows that H is provable in K' if and only if H^* is provable in the **MIP**, and since K' is undecidable, the undecidability of the **MIP** follows.

REFERENCES

[1] Kripke, S. A., "Semantical analysis of intuitionistic logic I" in *Formal Systems and Recursive Functions*, Edited by J. N. Crossley and M. A. E. Dummet, North-Holland Publishing Company, Amsterdam (1965), pp. 92-130.

[2] Kripke, S. A., "The undecidability of monadic quantification theory," *Zeitschrift für Mathematische und Grundlagen der Mathematik*, Bd. 8 (1962), pp. 113-116.

[3] Maslov, S. J., G. E. Minc, and V. P. Opervkov, "Unsolvability in the constructive predicate calculus of certain classes of formulas containing only monadic predicate variables," *Soviet Mathematics*, Doklady (1965), Tom 163, No. 2, pp. 918-920.

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