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ANOTHER SYSTEM OF NATURAL DEDUCTION

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In the pages* that follow a system of natural deduction is described and shown to be adequate. Among the noteworthy features of the system are the perfect symmetry and intuitive plausibility of the restrictions that govern applications of the rules **UG** and **EI**. These features are made possible through the use of a precisely defined notion of arbitrariness. With one exception deductions run no longer than those of other commonly taught systems. The exception is the system found in the second edition of Quine [3].[1] But, it is perhaps to be expected that somewhat longer deductions are the price that must be paid to avoid Quine's devices of flagging and ordering.

We assume a system of sentences (well-formed formulas having no free occurrences of variables) built up in familiar ways from predicate and name letters together with apparatus for truth functions, existential quantification, and universal quantification. A *deduction* is to be understood as any finite sequence of ordered couples generated by rules that will shortly follow. But first, here are some needed definitions. Where $\langle \mu, A\rangle$ is the k'th term of a deduction $\mathfrak{D}$, $\langle \mu, A\rangle$ will be referred to as the *k'th line of* $\mathfrak{D}$, members of μ will be referred to as *premise numbers of* the k'th line of $\mathfrak{D}$, and A will be said to *occur as* or to be *written as* the k'th line of $\mathfrak{D}$. Where j is a premise number of line k, the sentence occurring as the j'th line of $\mathfrak{D}$ will be said to be a *premise of* the k'th line of $\mathfrak{D}$. And, finally, a name letter will be said to be *arbitrary for* the k'th line of $\mathfrak{D}$ if it occurs neither in any premise of that line nor in any earlier line obtained by **EI**.

In what follows 'n' and 'm' are restricted to name letters, $n/m\ B$ is the result of replacing each occurrence of m in B by an occurrence of n, and

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1. A variation of Quine's system can be found in Massey [1].

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$n/v\ B$ is the result of replacing each free occurrence of the variable v in B by an occurrence of n.

Rule of premise (**RP**) Any sentence may be written as the k'th line of a deduction with k as its only premise number.

Rule of truth function (**TF**) If $B_1, \ldots, B_i$ occur as lines $j_1, \ldots, j_i$ of a deduction and if $(B_1 \ \& \ \ldots \ \& \ B_i)$ truth-functionally implies the sentence C, then C may be written as a later line. The premise numbers of the new line are to be the premise numbers of lines $j_1, \ldots, j_i$.

Conditionalization (**Cd**) If C occurs as the j'th line of a deduction then the sentence $(B \supset C)$ may be written as a later line. The premise numbers of the new line are to be either the premise numbers of line j or, if B is a premise of line j and occurs as line i, the premise numbers of line j less i.

Universal instantiation (**UI**) If $(v)B$ occurs as the j'th line of a deduction, then $n/v\ B$ may be written as a later line. The premise numbers of the new line are to be the premise numbers of line j.

Existential generalization (**EG**) If $n/v\ B$ occurs as the j'th line of a deduction, then $(\exists v)B$ may be written as a later line. The premise numbers of the new line are to be the premise numbers of line j.

Universal generalization (**UG**) If $n/v\ B$ occurs as the j'th line of a deduction and if n does not occur in B, then $(v)B$ may be written as a later line provided that n is arbitrary for that later line. The premise numbers of the new line are to be the premise numbers of line j.

Existential instantiation (**EI**) If $(\exists v)B$ occurs as the j'th line of a deduction and if n does not occur in B, then $n/v\ B$ may be written as a later line provided that n is arbitrary for that later line. The premise numbers of the new line are to be the premise numbers of line j.

A deduction will be said to be *finished* if no name letter occurring in its last line has been introduced (anywhere) into the deduction by an application of **EI**. A sentence B will be said to be *derivable from* a set of sentences Δ ($\Delta \vdash B$ for short) if B occurs as the last line of some finished deduction and each premise of that line is a member of Δ.

Deductions may be pictured in the manner of Mates [2]. Here is a simple example illustrating each of the rules. Where B is any sentence in which 'a' does not occur:

$\{1\}$	(1)	$((\exists x)Fx \supset B)$	**RP**
$\{2\}$	(2)	Fa	**RP**
$\{2\}$	(3)	$(\exists x)Fx$	2 **EG**
$\{1, 2\}$	(4)	B	1, 3 **TF**
$\{1\}$	(5)	$(Fa \supset B)$	2, 4 **Cd**
$\{1\}$	(6)	$(x)(Fx \supset B)$	5 **UG**
$\{\ \}$	(7)	$(((\exists x)Fx \supset B) \supset (x)(Fx \supset B))$	1, 6 **Cd**
$\{8\}$	(8)	$(x)(Fx \supset B)$	**RP**

$\{9\}$	(9)	$(\exists x)Fx$	**RP**
$\{9\}$	(10)	Fa	9 **EI**
$\{8\}$	(11)	$(Fa \supset B)$	8 **UI**
$\{8, 9\}$	(12)	B	10, 11 **TF**
$\{8\}$	(13)	$((\exists x)Fx \supset B)$	9, 12 **Cd**
$\{\ \}$	(14)	$((x)(Fx \supset B) \supset ((\exists x)Fx \supset B))$	8, 13 **Cd**
$\{\ \}$	(15)	$((x)(Fx \supset B) \equiv ((\exists x)Fx \supset B))$	7, 14 **TF**

The system should be seen at its worst. To this end, one should look at deductions establishing the validity of '$(\exists x)(Fx \supset (x)Fx)$' and '$((x)(\exists y)Fxy \supset (x)(\exists z)Fxz)$'. These deductions can be considerably shortened through the use of two (derived) quantifier conversion rules.

QC1 If $\sim(\exists v)B$ occurs as the j'th line of a deduction, then $(v) \sim B$ may be written as a later line. The premise numbers of the new line are to be those of line j.

QC2 If $\sim(v)B$ occurs as the j'th line of a deduction, then $(\exists v) \sim B$ may be written as a later line. The premise numbers of the new line are to be those of line j.

Even so, the shortened deductions are not very pretty. It will perhaps have been noticed that the rules are so devised that only (closed) sentences can occur as lines of a deduction. The motivation behind this is solely that it simplifies formulation and helps to streamline certain portions of the metatheory. And this feature should not be regarded as essential to the system. If open sentences are allowed, then modifications are of course required. But these modifications are straightforward and unproblematic. For example, where t may be either a name letter or a variable, **UI** might be written[2]:

If $(v)B$ occurs as the j'th line of a deduction and if $t/v\ B$ is an instance of $(v)B$, then $t/v\ B$ may be written as a later line. The premise numbers of the later line are to be those of line j.

And, **UG** may be written:

If $t/v\ B$ occurs as the j'th line of a deduction and if $t/v\ B$ is a conservative instance of $(v)B$, then $(v)B$ may be written as a later line provided that t is arbitrary for that later line. The premise numbers of the new line are to be those of line j.

Those who have scruples concerning either the introduction of name letters by **EI** or their elimination by **UG** may conveniently restrict 't' to variables in the formulation of those rules.

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2. The notions of instance and conservative instance are from the second edition of Quine [3]. The reformulations of **UG** and **EI** require a slightly different understanding of arbitrariness: t is arbitrary for a line if t occurs free neither in any premise of that line nor in any earlier line obtained by **EI**. All occurrences of name letters are to be counted as free.

If the system has an identity predicate (say '='), then the rules can be supplemented with:

Self identity $n = n$ may be written as a line of a deduction with no premise numbers.

Indiscernibility of identicals If $n = m$ and B occur as the i'th and j'th lines of a deduction and if $n/m\ B = n/m\ C$, then C may be written as a later line. The premise numbers of the new line are to be those of lines i and j.

It should be noted that $n/m\ B = n/m\ C$ just in case C is exactly like B except for having an occurrence of m at zero or more places where B has an occurrence of n.

Let us say that a deduction is *sound in* a line if the sentence that occurs as that line is a consequence of the set of its premises. Then,

Lemma 1 *A deduction that makes no use of* **EI** *is sound in each of its lines.*

The proof is a straightforward induction appealing to semantic analogues of the various rules used in such a deduction. (Where '$\vDash$' is short for 'has as a consequence', the semantic analogue of **Cd** is the principle that if $\Delta \vDash C$ then $\Delta - \{B\} \vDash (B \supset C)$. The semantic analogue of **UG** is the principle that if $\Delta \vDash n/v\ B$ and if n occurs neither in B nor in any member of Δ, then $\Delta \vDash (v)B$. Etc.)

Lemma 2 *If B occurs as the premiseless last line of a finished deduction, then B is valid.*

Proof: Suppose that B occurs as the premiseless last line of a finished deduction $\mathfrak{D}$. Construct a new deduction as follows. Immediately preceding each line of $\mathfrak{D}$ obtained by **EI** insert the **EI**-conditional[3] of that line. The new line is to be justified by **RP**. Then any line obtained by an application of **EI** in $\mathfrak{D}$ can be obtained in the new construction by an application of **TF**. It should be clear that with suitable renumbering this new construction can be converted into a deduction that has B occurring as its last line, makes no use of **EI**, and has only the newly introduced **EI**-conditionals as premises of its last line. (Care should be taken to note that applications of **UG** in $\mathfrak{D}$ continue to be correct in the new construction.) Where those **EI**-conditionals are (in order of introduction) $C_1, \ldots, C_{k-1}, C_k$, it can be concluded by Lemma 1 that:

$$\{C_1, \ldots, C_{k-1}, C_k\} \vDash B.$$

So,

$$\{C_1, \ldots, C_{k-1}\} \vDash (C_k \supset B).$$

3. The terminology is due to Quine [3]. Much else in the proof is due to the second edition of that work. The **EI**-conditional of a line $\langle \mu, C\rangle$ obtained from a line $\langle \mu, B\rangle$ by **EI** is the conditional $(B \supset C)$.

Let C_k be the **EI**-conditional $((\exists v)A \supset n/v\ A)$. Then the last claim can be written:

$$\{C_1, \ldots, C_{k-1}\} \vDash (((\exists v)A \supset n/v\ A) \supset B).$$

Since v can occur free in neither $(\exists v)A$ nor B, this is to say that

$$\{C_1, \ldots, C_{k-1}\} \vDash n/v(((\exists v)A \supset A) \supset B).$$

So, noting that the restrictions on **EI** together with the fact that $\mathfrak{D}$ is finished guarantee that n occurs neither in $(((\exists v)A \supset A) \supset B)$ nor in any member of $\{C_1, \ldots, C_{k-1}\}$, it can be concluded that

$$\{C_1, \ldots, C_{k-1}\} \vDash (v)(((\exists v)A \supset A) \supset B).$$

But v cannot occur free in B. So,

$$\{C_1, \ldots, C_{k-1}\} \vDash ((\exists v)((\exists v)A \supset A) \supset B).$$

But $(\exists v)((\exists v)A \supset A)$ is valid. So,

$$\{C_1, \ldots, C_{k-1}\} \vDash B.$$

Repeating this line of thought $k - 1$ times will show that $\{\ \} \vDash B$, i.e., that B is valid.

Soundness theorem *If* $\Delta \vdash B$, *then* $\Delta \vDash B$.

Proof: By hypothesis B occurs as the last line of a finished deduction having only members of Δ as premises. Let $A_1, \ldots, A_k$ be a complete list of those premises. Then, by k applications of **Cd**,

$$(A_1 \supset \ldots \supset (A_k \supset B) \ldots)$$

can be obtained as the premiseless last line of a finished deduction. So by Lemma 2 it can be concluded that the equivalent

$$((A_1 \ \& \ \ldots \ \& \ A_k) \supset B)$$

is valid. Thus $\{A_1, \ldots, A_k\} \vDash B$. So $\Delta \vDash B$.

It can also be proved that:

Completeness theorem *If* $\Delta \vDash B$, *then* $\Delta \vdash B$.

The result is most easily obtained by noting that the output of the system is not exceeded by that of comparable systems already known to be complete. In this connection Mates [2] is recommended.

REFERENCES

[1] Massey, G., *Understanding Symbolic Logic*, Harper and Row, New York (1970).

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[3] Quine, W. V., *Methods of Logic*, 3rd edition, Holt, Rinehart and Winston, Inc., New York (1972).

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