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PARADOX-FREE DEONTIC LOGICS

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One of the standard systems of deontic logic is the monadic system developed by G. H. von Wright.[1] I will begin by discussing a version of von Wright's system. This system has the following vocabulary:

(a) Propositional variables "p", "q", "r", etc.
(b) Truth functional connectives "$\sim$", "$.$", "$\vee$", "$\supset$", "$\equiv$"

for negation, conjunction, disjunction, material implication, material equivalence.

(c) Deontic operators: "P__" for "it is permitted to see to it that —" and "O__" for "it is obligatory to see to it that —".

The system has the following formation rules:

(a) An expression of the form P__ or O__ is well-formed when the place of "__" is taken by a well-formed expression of propositional logic.
(b) Truth functional compounds of well-formed expressions are well-formed. (Such compounds will be enclosed in parentheses for the sake of clarity; e.g., "Pp" but "$P(p \vee q)$".)

The system has the following axioms:

W1 All tautologies of propositional logic (with well-formed expressions of the deontic language substituted for the variables).
W2 $P(p \vee q) \equiv (Pp \vee Pq)$.
W3 $Pp \vee P \sim p$.

The system has the rules:

WR1 Substitution of well-formed expressions for propositional variables.
WR2 Detachment (*modus ponens*).

1. G. H. von Wright, "Deontic logic," *Mind*, vol. 60 (1951), pp. 1-15; also, *cf.* G. H. von Wright, *An Essay in Deontic Logic and the General Theory of Value*, North Holland Publishing Company, Amsterdam (1968).

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WR3 Replacement Rule for logical equivalent expressions (including those within the scope of a deontic operator).

The system has one definition:

$$\text{WDef } Op = def \sim P \sim p$$

and replacement of definiens by definiendum and vice versa is regarded as authorized by WR3.

By simple propositional logic manipulations of W2 we can get

T1 $O(p \cdot q) \equiv (Op \cdot Oq)$.

Within the von Wright system the following paradoxes[2] arise:

Ross' Paradox: From "Pp" derive "$P(p \vee q)$" by Addition and W2. Thus if it is permitted to mail a letter it is permitted to mail it or burn it. A strengthened form of Ross' paradox, (from "Op" derive "$O(p \vee q)$") arises when we show that "$(Op \vee Oq) \supset O(p \vee q)$" follows from W2 with the aid of W1.

Paradox of Derived Obligation: From $O \sim p$ derive $O(p \supset q)$ by Addition and the theorem mentioned above. Thus if I am obliged not to kill one man I am obliged to see to it that *if* I kill one man, I kill two more.

Since the theorem just mentioned is crucial it will be well to derive it from the axioms. This will also give us some feeling for derivations in a system like von Wright's. We will use a conditional proof for the sake of convenience though rules for indirect proofs are not part of von Wright's system.

*1.	Op	ACP
*2.	$O((p \vee q) \cdot p)$	1, WR3, $p \equiv ((p \vee q) \cdot p)$
*3.	$O(p \vee q) \cdot Op$	2, T1
*4.	$O(p \vee q)$	3, Simp
5.	$Op \supset O(p \vee q)$	1-4 RCP
*6.	Oq	ACP
*7.	$O((p \vee q) \cdot q)$	6, WR3, $q \equiv ((p \vee q) \cdot q)$
*8.	$O(p \vee q) \cdot Oq$	7, TI
*9.	$O(p \vee q)$	8, Simp
10.	$Oq \supset O(p \vee q)$	6-9 RCP
*11.	$Op \vee Oq$	ACP
*12.	$O(p \vee q)$	5, 10, 11 CD, Repetition
13.	$(Op \vee Oq) \supset O(p \vee q)$	11-13 RCP, Q.E.D.

Notice that by simple Transposition and Double Negation from this theorem which we will call T2, we can get another which we will call T3:

T3 $P(p \cdot q) \supset (Pp \cdot Pq)$

2. For a good general discussion of the paradoxes, see Dagfinn Føllesdal and Risto Hilpinen, "Deontic logic: an introduction" in Hilpinen (ed.), *Deontic Logic Introductory and Systematic Readings*, D. Reidel Publishing Company, Dordrecht (1971).

The set W2, T1, T2, and T3 constitutes a set of Deontic Operator Distribution rules analagous to the distribution rules of standard modal logic. In any system with such a set of distribution rules, Ross' Paradox and the Paradox of Derived Obligation are unavoidable.

Some have argued that such paradoxes are only apparent, or at least are tolerable when properly understood.[3] This is especially dubious, however, if we take expressions like "$O(p \supset q)$" as representing what we mean by saying things such as "Doing *A* obliges you to do *B*." We would certainly not want to admit, e.g., that if it is forbidden to kill one man, killing one man *obliges* us to kill two men. Yet this would be a consequence of the Paradox of Derived Implication and this paradox is unavoidable once we have von Wright's W2.

It has been suggested that the notion "Doing *A* obliges us to do *B*" could better be formalized by expressions of the form "$p \supset Oq$", e.g., "If I promise to do a certain action, then I am obliged to perform that action," or more formally, "If it is the case that I promise to do *A* then it is obligatory to see to it that I do *A*." This in itself brings out one oddity of von Wright's notation, for it is surely I that have the obligation: there is not some sort of abstract obligation for "one" (or "anyone") to see to it that my promise is carried out.

But aside from that, defining "derived obligation" in this way has odd consequences. Suppose that in a given case something of the form "$p \supset Oq$" was true, e.g., "If I promise to do *A* then it is obligatory that I perform *A*." But then by Transposition and use of WDef we would get first "$\sim Oq \supset \sim p$" and then "$P \sim q \supset \sim p$", e.g., "If it is permitted that I not perform *A* then I do not promise to do *A*." But this is nonsensical in many cases. It is certainly permissible for me not to have dinner with one of my friends on a given night. And it is quite plausible to say that if I promise to have dinner with him that night then I am obliged to have dinner with him that night. But put together these premises give an argument of the form

1.	$p \supset Oq$	Premise
2.	$P \sim q$	Premise
3.	$\sim Oq \supset \sim p$	1, Transp
4.	$\sim\sim P \sim q \supset \sim p$	3, WDef, WR3
5.	$P \sim q \supset \sim p$	4, DN
6.	$\sim p$	5, 2, M.P.

which would give the conclusion that I do not promise. But since I often promise to do things I am permitted not to do, premises of this kind are often true while the conclusion is false. Thus something is seriously wrong, probably the proposed explication of derived obligation. Since the form of "$p \supset Op$" resembles what are sometimes called "*de re*" modalities, I will call this the Paradox of *De Re* Obligation.

Thus paradoxes seem unavoidable in any system like von Wright's.

3. For example, Føllesdal and Hilpinen, *op. cit.*

This has led von Wright himself to develop a dyadic system which is not, however, without difficulties of its own.[4] In what follows, I will develop several monadic systems of deontic logic which are paradox-free and which avoid several other difficulties of systems like that of von Wright. The first system has the following characteristics:

Vocabulary:

(a) Individual variables "*m*", "*n*", "*o*", "m^1", "n^1", "o^1" . . . etc. which range over persons, propositional variables "*p*", "*q*", "*r*", "*s*", "p^1", "q^1", etc.
(b) Truth functional connectives "$\sim$", "$\cdot$", "$\vee$", "$\supset$", "$\equiv$".
(c) Deontic operators "*Pnp*" for "it is permitted for *n* to make it true that *p*" and "*Onp*" for "it is obligatory for *n* to make it true that *p*".

Formation Rules:

(a) An expression of the form "*Pnp*" or "*Onp*" is well-formed when the place of "*p*" is taken by a well-formed expression of propositional logic.
(b) Truth functional compounds of well-formed expressions are well-formed.

Rules: As in von Wright's system, plus any complete set of propositional logic rules. Our first system, which we will call S.O.1 has the axioms

A1 $Onp \equiv \sim Pn \sim p$
A2 $(Onp \cdot Onq) \supset On(p \cdot q)$
A3 $On(p \supset q) \supset (Onp \supset Onq)$

From Axiom 1 by substitution and simple propositional logic maneuvers one can derive:

T1.1 $\sim Onp \equiv Pn \sim p$
T1.2 $On \sim p \equiv \sim Pnp$
T1.3 $\sim On \sim p \equiv Pnp$

From A2 we can derive:

T2.1 $Pn(p \vee q) \supset (Pnp \vee Pnq)$
T2.2 $((p \supset Onq) \cdot (p \supset Onr)) \supset (p \supset On(q \cdot r))$

From A3:

T3.1 $On(p \supset q) \supset (\sim Pnq \supset \sim Pnp)$
T3.2 $On(p \supset q) \supset (Pnp \supset Pnq)$
T3.3 $On(p \supset q) \supset (\sim Onq \supset \sim Onp)$

However, one cannot derive the stronger distribution rules of von Wright's system, nor such principles as

4. *Cf.*, Føllesdal and Hilpinen, *op. cit.*, Section IX.

$$\sim(Onp \cdot On \sim p)$$
$$Onp \supset Pnp$$
$$On(p \vee \sim p), \text{ etc.}$$

Thus, some would regard the system as too weak. However, let us explore this weak system a little. It does not permit either Ross' Paradox or the Paradox of Derived Obligation, since Theorem 1 of von Wright's system is not derivable in S.O.1. Axiom 1 connects the notions of obligation and permissibility in the same way as von Wright's definition, not making the connection a matter of definition. Axiom 2 provides weak distribution rules, and Axiom 3 provides plausible rules for derived obligation. Thus, if I am obliged to pay my secretary and obliged to treat my secretary politely, then I am obliged to pay my secretary and treat her politely. Also, if hiring a secretary obliges me to pay her, then if I am obliged to hire a secretary I am obliged to pay her, and if I am permitted to hire a secretary I am permitted to pay her. All this is sound ethics and good sense.

But, it might be objected, we lose some plausible rules of von Wright's system, such as (in our new notation)

$$On(p \cdot q) \supset (Onp \cdot Onq).$$

Surely, it might be argued, if I am obliged to do two things I am obliged to do each of them. Well, perhaps not always. If I am obliged to pay for some goods and carry them away, but fail for some reason to pay for them, I can hardly carry the goods away, claiming that I am keeping at least one of my obligations! Of course it might be claimed that something of the form "$O(p \cdot q)$" does not do justice to this situation. Still it is hard to find plausible substitutes.

Similarly, if I am permitted to go to the movies and permitted not to go to the movies, it follows from

$$(Pp \vee Pq) \supset P(p \vee q)$$

that I am permitted to either go or not go to the movies, which has the form "$Pn(p \vee \sim p)$". And it may be doubted whether it makes sense to speak of being permitted or obliged to "make true" a tautology.

These objections are not decisive—they do not amount to "paradoxes"—but they do suggest that we might have some reasons for distrusting the stronger distribution rules even if they lead to no outright paradox. More serious is the objection that without such rules as

$$\sim(Onp \cdot On \sim p)$$

our notion of obligation is seriously distorted. However, the fact is that at least apparently obligations *can* conflict, and some writers on eithics, especially W. D. Ross,[5] have wished to talk about *prima facie* obligations which *can* conflict. Thus perhaps S.O.1 is the appropriate system for *prima facie* obligations.

5. *The Right and The Good*, Oxford University Press, Oxford (1930).

We can get a stronger system, S.O.2, by adding as an axiom

A4 $\sim(Onp \cdot On \sim p)$

From this can be proved

T4.1 $Pnp \vee Pn \sim p$

and also

T4.2 $Onp \supset Pnp$

This axiom thus gives us a "square of opposition" for deontic operators; "Onp" and "$On \sim p$" are contraries, "Pnp" and "$Pn \sim p$" subcontraries, and "Pnp" and "$Pn \sim p$" are subalterns of "Onp" and "$On \sim p$". Of course Axiom 1 ensures that "Onp" and "$Pn \sim p$" are contradictories, as are "$On \sim p$" and "Pnp". Thus S.O.2 may be the appropriate system for "strict obligation", if we suppose that strict obligations will have these relationships.

We can strengthen our system still further by adding

A5 $On(p \vee q) \supset (Onp \vee Onq)$

giving us S.O.3. But S.O.3 contains Ross' Paradox and the Paradox of Derived Obligation. A still stronger system contains

A6 $On(p \vee \sim p)$

giving us S.O.4. But S.O.4 has some very odd consequences, as we shall see.

S.O.3, as can be seen, is essentially von Wright's system, and S.O.4 is the system sometimes called "the standard system".[6] Our paradox-free systems, then, are S.O.1 and S.O.2.

However, it might be claimed that S.O.1 is not paradox-free, since we can derive in it the Paradox of *De Re* Obligation. However, this is a paradox only if we try to formalize statements of "derived obligation" using expressions of the form "$p \supset Onq$". However, I do not see how this could ever be plausibly done, since a simple transposition will give us statements in which statements about obligation and permissibility imply statements of fact, i.e., expressions of the form "$\sim Oq \supset \sim p$", "$Pn \sim q \supset \sim p$", etc. This surely is unacceptable on almost any theory of ethics. Thus, although we can write certain statements of the form "$p \supset Onq$", no such statements may be defensible.

There is a supposed paradox created by Chisholm,[7] which uses the following premises:

(1) It ought to be the case that a certain man go to the assistance of his neighbors.

6. By Føllesdal and Hilpinen, *op. cit.*, Section V.

7. Roderick Chisholm, "Contrary-to-duty imperatives and deontic logic," *Analysis*, vol. 24 (1963), pp. 33-36; *cf.*, Føllesdal and Hilpinen, *op. cit.*, Section VIII.

(2) It ought to be that if he does go he tell them he is coming.
(3) If he does not go he ought not to tell them he is coming.
(4) He does not go.

The premises are formalized in the general form:

(1′) Onp
(2′) $On(p \supset q)$
(3′) $\sim p \supset On \sim q$
(4′) $\sim p$

It can be seen that (4′) and (3′) imply "$On \sim q$" by *modus ponens* whereas (1′) and (2′) imply "Onq" by our A3. Using T4.2 we can derive "Pnq" from "Onq," use A1 to replace "Pnq" by "$\sim On \sim q$", and get an explicit contradiction: "$On \sim q \cdot \sim On \sim q$" (this retraces Chisholm's reasoning; of course A6 rules out "$Onp \cdot On \sim p$" more directly).

But this is only a "paradox" because of the formalization of (3) by (3′). But (3′) is totally implausible. By simple transposition and operator exchange (3′) is equivalent to "$Pnq \supset p$," in this case "if he is permitted to tell, he goes." But that is plainly not what is intended by the natural language (3). Plainly, (3) should be formalized by

(3″) $On(\sim p \supset \sim q)$

and no paradox arises.

For the same reason I would reject Hintikka's suggestion[8] that we formalize *prima facie* obligation by expressions of the form "$On(p \supset q)$" and strict obligation by expressions of the form "$p \supset Onp$". Expressions of that form misbehave too seriously to be usable as formalizations of strict obligation, and with the proper axioms expressions of the form "$On(p \supset q)$" do very well as formalizations of cases where we wish to say that doing A obliges us to do B.

It may be of interest to know that there is a tabular method for these weak modal systems, based on the four-valued tables devised by W. T. Parry and discussed by C. I. Lewis in Appendix II of his book *Symbolic Logic*.[9] Using the same tables for "$\sim$", "$\cdot$", "$\vee$", "$\supset$", "$\equiv$" as for any system of modal logic we can get S.O.1 by using the tables

p	$\sim p$	Pnp	Onp
1	4	2	2
2	3	2	3
3	2	2	3
4	1	3	3

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

with 1 as a designated value. Thus A.1 is shown to be acceptable by the "value table"

8. Jaako Hintikka, "Some main problems of deontic logic," in Hilpinen *op. cit.*

9. C. I. Lewis and C. H. Langford, *Symbolic Logic*, Dover Publications, New York (1959).

Onp	$\equiv$	$\sim Pnp$
2 1	1	2341
3 2	1	3232
3 3	1	3223
3 4	1	3214

and, e.g., A.6 is shown not to be part of S.O.1 by the table

$\sim$	$(Onp$		$\cdot$	On	$\sim$	$p)$
1	2	1	4	3	4	1
2	3	2	3	3	3	2
2	3	3	3	3	2	3
1	3	4	4	2	1	4

As this table suggests, S.O.2 is distinguished from S.O.1 by allowing 2 as a designated value, but only under certain circumstances. For S.O.2 designated values are 1 and 2, but any final column which contains 2's must contain an equal number of 1's, and two 2's must be adjacent.

S.O.3 allows mixed final columns of 1's and 2's without this restriction, but not final columns with only 2's, and S.O.4 allows 2 as a designated value with no restrictions. Characterizing the systems in this way is untidy but effective. For S.O.1, S.O.2, and S.O.3, the oddities due to the four-valued characters of the tables pointed out by Dugundji[10] and Prior[11] do not arise, but in S.O.4 the Dugundji-like statement $(On(p \equiv q) \vee On(q \equiv r)) \vee (On(r \equiv s) \vee On(s \equiv t))$ receives designated values, as well as a statement like the one noted by Prior: $(Pn \sim p \cdot Pn \sim q) \supset (On(p \supset q) \vee On(p \supset \sim q))$. Since these are plainly unacceptable, the four-valued tables must be regarded only as disproof procedures[12] for S.O.4, but so far as I can see there is no objection to using the four-valued tables as a decision procedure for S.O.1, S.O.2, and S.O.3.

Let me close with a modest proposal for deontic logic; that we try to get along with the weak systems S.O.1 and S.O.2 and see whether this really hinders us in proving or disproving anything of real ethical interest. Deontic logicians have spent a good deal of time sharpening their tools; let us see what this relatively modest set of equipment will enable us to cut.

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10. James Dugundji, "Note on a property of matrices for Lewis and Langford's calculi of propositions," *The Journal of Symbolic Logic*, vol. 5 (1940), pp. 150-151.

11. A. N. Prior, *Time and Modality*, Oxford University Press, Oxford (1957), pp. 16-17.

12. *Cf.*, my "Four-valued tables for modal logic," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 505-511.