Notre Dame Journal of Formal Logic
Volume XVI, Number 3, July 1975
NDJFAM

IMPLICATION CONNECTIVES IN ORTHOMODULAR LATTICES

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1 *Introduction* It was pointed out as long ago as 1936 by Birkhoff and von Neumann [2], that the logic of empirically verifiable propositions about a quantum mechanical system is not classical. That is, quantal propositions do not tend to band together to form a Boolean algebra. The distinguishing feature of quantum mechanics, namely the existence of quantities which are not simultaneously measurable, led to an attack on the *distributive law* as the law of logic which is least tenable in quantum logic. Though Birkhoff and von Neumann argued in favor of the weaker *modular law*, subsequent researchers have rejected modularity in favor of the yet more general *orthomodular law*. Thus, the basic assumption of the quantum logic approach to quantum mechanics today is that the empirically verifiable propositions relevent to a given physical system form an orthomodular lattice (see Jauch [17], Varadarajan [36]). Some have objected that to demand a lattice is already too much. The appropriate structure for quantum logic then becomes an orthomodular partially ordered set (see Mackey [26], Pool [32]). However, we shall ignore this controversy here and maintain a lattice structure throughout.

The assumption of conventional quantum mechanics is that the logic of a physical system is modeled by the orthomodular lattice of all closed subspaces of a complex separable infinite dimensional Hilbert space, or, equivalently, by the lattice of all projection operators on the Hilbert space (see von Neumann [37], n.b. Projections as Propositions, p. 247 ff.). One of the purposes of the quantum logic approach to quantum mechanics has been to elucidate this somewhat ad hoc assumption. More recently, Foulis and Randall [9, 10, 33] have shown that orthomodular lattices lie at the heart of their "operational statistics". This should help to clarify the logical foundations of any empirical science.

Our paper begins with a brief description of quantum logic and the problem of introducing an implication in such a logic. Next, several candidates for such an implication are studied. Our point of view then becomes axiomatic as we abstract what we consider to be the essential features of an internal implication connective. With the help of a given

Received May 2, 1973

implication, the modal operators of "necessity" and "possibility" can be introduced. This is interesting since orthomodular lattices generalize Boolean algebras. As the latter model classical logic, it is natural to ask what logical analogue might correspond to the former. We shall present evidence that the analogue seems to be a system resembling the classical Lewis modal system S4. The culmination of this evidence is in the section of the paper where we prove a restricted deduction theorem of the type that is well known to hold in S4.

The authors would like to express sincerest thanks to J. Jay Zeman for many interesting and stimulating conversations. Indeed, to understand our focus and our axioms, the reader would do well to consult Zeman's paper, "Quantum logic with implication" [38]. The authors are grateful to Veronica K. Piziak for the many hours she spent typing several drafts of this paper.

2 *Implication in Quantum Logic* In this section, a brief sketch is given of the basic axioms for a quantum logic and the problem of introducing an implication connective in a quantum logic is discussed.

To begin, let $\mathcal{L}$ be a nonempty set of elements which may be referred to as "propositions". We postulate two binary connectives (i.e., binary operations) on $\mathcal{L}$ which model "conjunction" (the "and" connective symbolized by $\wedge$) and "disjunction" (the "or" connective symbolized by $\vee$), subject to the following axioms:

(A1) $a \vee b = b \vee a$ *for all* a, b *in* $\mathcal{L}$.
(A2) $a \wedge b = b \wedge a$ *for all* a, b *in* $\mathcal{L}$.
(A3) $a \vee (b \vee c) = (a \vee b) \vee c$ *for all* a, b, c *in* $\mathcal{L}$.
(A4) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *for all* a, b, c *in* $\mathcal{L}$.
(A5) $a \vee (a \wedge b) = a$ *for all* a, b *in* $\mathcal{L}$.
(A6) $a \wedge (a \vee b) = a$ *for all* a, b *in* $\mathcal{L}$.

In other words, we have demanded that the structure $\langle \mathcal{L}, \wedge, \vee \rangle$ be a lattice.

Next, we ask that there exist in $\mathcal{L}$ two distinguished elements, 0 and 1, such that

(A7) $0 \vee a = a$ *for all* a *in* $\mathcal{L}$.
(A8) $1 \wedge a = a$ *for all* a *in* $\mathcal{L}$.

We might interpret 0 as the "contradictory" proposition and any proposition equal to 0 could be called a "contradiction". Similarly, 1 could be interpreted as the "tautologous" proposition and any proposition equal to 1 could be called a "tautology".

At this stage it is convenient to define a binary relation $\leqslant$ on $\mathcal{L}$ as follows:

(P01) $a \leqslant b$ *if and only if* $a = a \wedge b$.

It is easy to see that (P01) is equivalent to saying $a \leqslant b$ if and only if $b = a \vee b$. Also it is easy to deduce that $\leqslant$ is a partial order relation on $\mathcal{L}$. That is, we have:

(P02) $a \leqslant a$ *for all* a *in* $\mathcal{L}$.
(P03) *if* $a \leqslant b$ *and* $b \leqslant a$ *then* $a = b$.
(P04) *if* $a \leqslant b$ *and* $b \leqslant c$ *then* $a \leqslant c$.
(P05) $0 \leqslant a \leqslant 1$ *for all* a *in* $\mathcal{L}$.

Negation is introduced as a unary operation $a \mapsto a'$ on $\mathcal{L}$ subject to the following very classical axioms:

(A9) $a = (a')'$ *for all* a *in* $\mathcal{L}$.
(A10) *if* $a \leqslant b$, *then* $b' \leqslant a'$ *for all* a, b *in* $\mathcal{L}$.
(A11) $a \wedge a' = 0$ *for all* a *in* $\mathcal{L}$.
(A12) $a \vee a' = 1$ *for all* a *in* $\mathcal{L}$.

In brief, a structure $\langle \mathcal{L}, \wedge, \vee, ', 0, 1\rangle$ satisfying (A1) through (A12) is called an *orthocomplemented lattice*. It is called a *Boolean algebra* when in addition to (A1) through (A12) we have the *distributive law*:

(A13) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *for all* a, b, c *in* $\mathcal{L}$.

This law may be weakened in several ways. The *modular law* is:

(A14) *if* $a \leqslant c$, *then* $(a \vee b) \wedge c = a \vee (b \wedge c)$.

If we replace (A13) by (A14), maintaining (A1)-(A12), we say our structure forms an *orthocomplemented modular lattice*. This is the structure originally advocated by Birkhoff and von Neumann. The weakening of (A13) that interests us most is the *orthomodular law*:

(A15) *if* $a \leqslant b$, *then* $b = a \vee (b \wedge a')$.

The structure $\langle \mathcal{L}, \wedge, \vee, ', 0, 1\rangle$ is called an *orthomodular lattice* when it satisfies (A1)-(A12) together with (A15). For the remainder of this section, we shall deal only with orthomodular lattices. It is clear that every Boolean lattice is an orthocomplemented modular lattice and every such is an orthomodular lattice.

In view of (P02)-(P05), it is tempting, indeed it is often done, to let the partial order play the role of implication in the logic. However, Zeman [38], among others, has objected to this, since implication is treated classically as a binary connective (see the truth table definition in any elementary book on logic). This implication should be on the same linguistic level as conjunction and disjunction. However, $\leqslant$ is a relation on $\mathcal{L}$ rather than a binary operation on $\mathcal{L}$. To insist on using $\leqslant$ to play the role of implication amounts to a violation of the sacrosanct distinction between object language and metalanguage. The relation $\leqslant$ should be viewed as a statement about "deducibility". One might read "$a \leqslant b$" as "b is deducible from a". That this is more than a matter of language should be clear in the sequel. With all this in mind, we now attack the problem of finding a binary connective in quantum logic that plays the role of the "if—then" connective of classical logic.

Logicians have, of course, studied the notion of implication extensively. Let $\supset: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ be a mapping. The question is "What properties should $\supset$ satisfy to justify calling it an implication connective?". In both classical and intuitionistic logic, the meaning of implication is expressed by the following condition (see Curry [4], p. 140 ff):

(CI) $\quad x \leqslant a \supset b$ *if and only if* $x \wedge a \leqslant b$.

We shall refer to the image of $\supset$ as the set of implications, an element of this set being referred to, naturally, as an implication. It is tempting to simply require the existence of such a "hook" on our orthomodular lattice. Then, by axiom, (CI) would be satisfied and we would be done with the problem of implication. That this cannot be done for quantum logic follows from a classic result of Skolem (see Birkhoff [1]).

2.1 Theorem (Skolem) *Any lattice with a binary connective satisfying* (CI) *is necessarily distributive. Moreover,* $a \supset b$ *is the maximal solution for* x *in the inequality* $x \wedge a \leqslant b$.

This theorem, according to Curry [4], is the main reason why logicians have studied, for the most part, only distributive lattices (when they have bothered to study lattices at all). Before we abandon (CI), let us try to gain some insight into the problem by considering the classical situation.

2.2 Theorem *Let* $\mathcal{L}$ *be a Boolean lattice with a binary connective* $\rightarrow$ *satisfying* (CI). *Then* $\rightarrow$ *is uniquely determined. In fact,* $a \rightarrow b = a' \vee b$ *for all* a, b *in* $\mathcal{L}$. *Also,* $a \rightarrow b$ *is the maximal solution for* x *in the inequality* $x \wedge a \leqslant b$. *Moreover,*

(1) $\quad a \leqslant b$ *if and only if* $a \rightarrow b = 1$
(2) $\quad a \wedge (a \rightarrow b) \leqslant b$
(3) $\quad a \rightarrow a = 1$ *for all* a
(4) $\quad 0 \rightarrow a = 1$ *for all* a
(5) $\quad a \rightarrow 0 = a'$ *for all* a
(6) $\quad 1 \rightarrow a = a$ *for all* a
(7) $\quad a \rightarrow 1 = 1$ *for all* a
(8) $\quad a' \rightarrow 0 = a$ *for all* a
(9) $\quad a \rightarrow a' = a'$ *for all* a
(10) $\quad a' \rightarrow a = a$ *for all* a
(11) $\quad$ *if* $a \wedge b = 0$, *then* $a \rightarrow b = a'$
(12) $\quad (a \vee a) \rightarrow a = 1$ *for all* a
(13) $\quad b \rightarrow (a \vee b) = 1$
(14) $\quad (a \vee b) \rightarrow (b \vee a) = 1$
(15) $\quad (b \rightarrow c) \rightarrow ((a \vee b) \rightarrow (a \vee c)) = 1$
(16) $\quad a \rightarrow b = b' \rightarrow a'$
(17) $\quad (a \vee b) \rightarrow b = b' \rightarrow a'$

Proof: The proofs here are standard. We shall only argue uniqueness. Put $x = a' \vee b$ in (CI). Then $x \wedge a = (a' \vee b) \wedge a = (a' \wedge a) \vee (b \wedge a) = b \wedge a \leqslant b$. Thus we conclude by (CI) that $a' \vee b \leqslant a \rightarrow b$. On the other hand, since $(a \rightarrow b) \leqslant (a \rightarrow b)$, we have $(a \rightarrow b) \wedge a \leqslant b$. Thus $(a \rightarrow b) \wedge a \wedge b' \leqslant b \wedge b' = 0$ so $(a \rightarrow b) \wedge (a \wedge b') = 0$. But $(a \rightarrow b) = (a \rightarrow b) \wedge 1 = (a \rightarrow b) \wedge ((a' \vee b) \vee (a' \vee b')') =$

$((a \to b) \wedge (a' \vee b)) \vee ((a \to b) \wedge (a \wedge b')) = (a \to b) \wedge (a' \vee b) \leq a' \vee b$. Thus $a \to b = a' \vee b$. We note in passing that **2.2** (12) through **2.2** (15) are the axioms usually called **PM** for the classical propositional calculus.

Theorem **2.2** suggests that we should at least consider the connective $\to$ on any orthomodular lattice (indeed, on any orthocomplemented lattice) defined by the formula $a \to b = a' \vee b$. We then ask if this is a suitable implication connective for quantum logic. In particular, we need to know what properties of $\to$ carry over from the Boolean case and how (CI) is weakened.

Before we state the next theorem, we recall some definitions. Let $\mathcal{L}$ be an orthomodular lattice. If a, b are in $\mathcal{L}$ and $a \leq b'$, we say a is *orthogonal* to b and write $a \perp b$. Note $\perp$ is a symmetric relation on $\mathcal{L}$. When $a \perp b$, the logical interpretation is that a and b are "mutually exclusive" propositions in the sense that from each is deducible the negation of the other. We say a is *compatible* with b, in symbols $a\mathrm{C}b$, when $a \wedge b = a \wedge (b \vee a')$. The conventional interpretation of two propositions being compatible is that they are simultaneously testable. It is a fact that C is a symmetric relation [1], p. 53. One fact we shall use is that $a\mathrm{C}b$ if and only if $a = (a \wedge b) \vee (a \wedge b')$. The essential result for performing computations in orthomodular lattices is the Foulis-Holland Theorem [1], p. 53: this fundamental theorem states that every distributive law holds among any three elements as long as one is compatible with the other two. In the sequel, we use this result almost without comment. The *center* of $\mathcal{L}$, in symbols $\mathrm{C}(\mathcal{L})$, is the set of all elements of $\mathcal{L}$ which are compatible with all elements of $\mathcal{L}$. For notational convenience, we use "iff" for the phrase "if and only if". We next state the properties of $\to$ which remain valid in any orthomodular lattice.

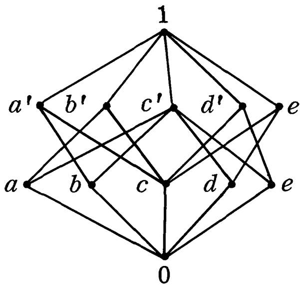
2.3 Theorem *Let $\mathcal{L}$ be an orthomodular lattice. For a, b in $\mathcal{L}$ define $a \to b = a' \vee b$. Then*

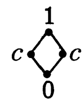
(1) $a \to a = 1$ *for all* a
(2) $0 \to a = 1$ *for all* a
(3) $a \to 0 = a'$ *for all* a
(4) $1 \to a = a$ *for all* a
(5) $a \to 1 = 1$ *for all* a
(6) $a' \to 0 = a$ *for all* a
(7) $a \to a' = a'$ *for all* a
(8) $a' \to a = a$ *for all* a
(9) *if* $a \leq b$, *then* $a \to b = 1$
(10) *if* $a \perp b$, *then* $a \to b = a'$ *and conversely*
(11) $a\mathrm{C}b$ *iff* $a \wedge (a \to b) \leq b$. *Moreover, in this case* $a \wedge (a \to b) = a \wedge b$
(12) *if* $a \perp b$, *then* $a \wedge (a \to b) = 0$
(13) $x \to a = a$ *iff* $x' \leq a$
(14) $(a \vee b) \to b \leq a \to b$
(15) $a \to b = b$ *iff* $a' \leq b$

The proofs of these statements are easy and left to the reader. Of more interest is the following example.

2.4 Example The following is the Hasse diagram of a non-Boolean orthomodular lattice.

$\mathcal{L}_{12}$ =



Routine computation shows that $\mathbf{C}(\mathcal{L}_{12}) = \{0, 1, c, c'\} =$  . We compute $\rightarrow$ explicitly for $\mathcal{L}_{12}$.

$\rightarrow$	0	a	b	c	d	e	a'	b'	c'	d'	e'	1
0	1	1	1	1	1	1	1	1	1	1	1	1
a	a'	1	a'	a'	1	1	a'	1	1	1	1	1
b	b'	b'	1	b'	1	1	1	b'	1	1	1	1
c	c'	c'	c'	1	c'	c'	1	1	c'	1	1	1
d	d'	1	1	d'	1	d'	1	1	1	d'	1	1
e	e'	1	1	e'	e'	1	1	1	1	1	e'	1
a'	a	a	c'	b'	c'	c'	1	b'	c'	1	1	1
b'	b	c'	b	a'	c'	c'	a'	1	c'	1	1	1
c'	c	b'	a'	c	e'	d'	a'	b'	1	d'	e'	1
d'	d	c'	c'	e'	d	c'	1	1	c'	1	e'	1
e'	e	c'	c'	d'	c'	e	1	1	c'	d'	1	1
1	0	a	b	c	d	e	a'	b'	c'	d'	e'	1

There are several interesting things to note from this example. First we note every element of $\mathcal{L}_{12}$ is an implication. More importantly, the converse of Theorem **2.3** (9) fails in general. We see $a \rightarrow d = 1$ but $a \nleq d$. Thus 1 appears in the table in at least two ways. There are other pathologies that could be pointed out. However, most of them are due to the failure of a very important rule of inference, namely Modus Ponens: $a \wedge (a \rightarrow b) \leqslant b$. Note for example in $\mathcal{L}_{12}$, $a \wedge (a \rightarrow d) = a \wedge 1 = a$ but $a \nleq d$. It is principally for this reason that we reject $\rightarrow$ as a suitable implication in quantum logic.

We now proceed to give our first candidate for a suitable implication for quantum logic. To motivate our condition replacing (CI) we note the following theorem.

2.5 Theorem *Let $\mathcal{L}$ be an orthomodular lattice. Then $\mathcal{L}$ is Boolean iff $x \wedge a = (x \vee a') \wedge a$ for all x, a in $\mathcal{L}$.*

Proof: If $\mathcal{L}$ is Boolean, we can distribute and get $(x \vee a') \wedge a = (x \wedge a) \vee (a' \wedge a) =$

$(x \wedge a) \vee 0 = x \wedge a$. Conversely, the condition says all pairs of elements of $\mathcal{L}$ are compatible. The global distributive law now follows from the Foulis-Holland Theorem (see Birkoff, [1]).

In view of Theorem **2.5**, we could replace $x \wedge a$ in (CI) by $(x \vee a') \wedge a$ in the classical case and not change the meaning of (CI) at all. Our generalization of (CI) is

(QI) $x \leqslant (a \supset b)$ *if and only if* $(x \vee a') \wedge a \leqslant b$.

The next theorem shows that $\supset$ is well behaved on any orthomodular lattice.

2.6 Theorem *Let $\mathcal{L}$ be an orthomodular lattice with a binary connective $\supset$ satisfying* (QI). *Then $\supset$ is uniquely determined. In fact, $a \supset b = a' \vee (a \wedge b)$ for all a, b in $\mathcal{L}$. Moreover,*

(1) $a \wedge (a \supset b) \leqslant b$
(2) $a \leqslant b$ *iff* $a \supset b = 1$
(3) $a \supset a = 1$ *for all* a
(4) $0 \supset a = 1$ *for all* a
(5) $a \supset 0 = a'$ *for all* a
(6) $a \supset 1 = 1$ *for all* a
(7) $1 \supset a = a$ *for all* a
(8) $a' \supset 0 = a$ *for all* a
(9) $a \supset a' = a'$ *for all* a
(10) $a' \supset a = a$ *for all* a
(11) $a \supset b \leqslant a \rightarrow b$ *for all* a, b
(12) $(a \supset b) \mathrm{C} (a \rightarrow b)$
(13) *if* $a \wedge b = 0$, *then* $a \supset b = a'$
(14) *if* $a \perp b$, *then* $a \supset b = a'$
(15) *if* $a \mathrm{C} b$, *then* $a \supset b = a \rightarrow b$
(16) *if* $a \perp b$, *then* $a \supset b = a \rightarrow b$
(17) $a \wedge (a \supset b) = a \wedge b$
(18) $((a \supset b) \vee a') \wedge a \leqslant b$
(19) $(a \vee b) \supset b \leqslant a \rightarrow b$
(20) $a \supset (b \supset c) = (a \wedge b) \supset c$ *iff* $a \mathrm{C} b$
(21) *if* $a \leqslant b$, *then* $c \supset a \leqslant c \supset b$
(22) *if* $a \leqslant b$, *then* $a \supset (b \supset c) = a \supset c$
(23) $a \leqslant (b \supset (b \supset a')')$
(24) $a' \leqslant a \supset b$
(25) $a \mathrm{C} b$ *iff* $b \leqslant a \supset b$
(26) $a \mathrm{C} (a \supset b)$
(27) $a \supset (a \wedge b) = a \supset b$
(28) $(a \supset b) \wedge (a \supset c) = a \supset (b \wedge c)$
(29) $(a \supset b) \supset (a \supset c) = (a \wedge b) \supset c$
(30) $(a \supset b) \supset (a \supset (b \wedge c)) = (a \supset b) \supset (a \supset c) = (a \wedge b) \supset c$

Proof: Let $x = a' \vee (a \wedge b)$. Then $(x \vee a') \wedge a = ((a' \vee (a \wedge b)) \vee a') \wedge a = ((a' \vee (a \wedge b)) \wedge a) \vee (a' \wedge a) = (a' \vee (a \wedge b)) \wedge a = (a' \wedge a) \vee ((a \wedge b) \wedge a) = a \wedge b \leqslant b$. We have used the Foulis-Holland Theorem in the above calculation. Thus by (QI), $a' \vee (a \wedge b) \leqslant a \Supset b$. Also, $a \Supset b \leqslant a \Supset b$, so by (QI), $((a \Supset b) \vee a') \wedge a \leqslant b$. Then $((a \Supset b) \vee a') \wedge a \leqslant b \wedge a$ so $(((a \Supset b) \vee a') \wedge a) \vee a' \leqslant a' \vee (a \wedge b)$. But $(((a \Supset b) \vee a') \wedge a) \vee a' = ((a \Supset b) \vee a' \vee a') \wedge (a \vee a') = (a \Supset b) \vee a'$. So $a \Supset b \leqslant (a \Supset b) \vee a' \leqslant a' \vee (a \wedge b)$.

(1) Using Foulis-Holland, $a \wedge (a \Supset b) = a \wedge (a' \vee (a \wedge b)) = (a \wedge a') \vee (a \wedge (a \wedge b)) = a \wedge b \leqslant b$.
(2) If $a \leqslant b$, then $a \Supset b = a' \vee (a \wedge b) = a' \vee a = 1$. If $a \Supset b = 1$, then $a = a \wedge 1 = a \wedge (a \Supset b) \leqslant b$.
(3) through (11) are all one line proofs and will be omitted. (12) follows from (11) and the fact that in an orthomodular lattice, comparable elements are compatible.

For the sake of space, the remaining proofs will be left to the reader.

We shall refer to $\Supset$ as the "Sasaki implication connective" or, more briefly, as the "Sasaki hook".

2.7 Example

$\Supset$	0	a	b	c	d	e	a'	b'	c'	d'	e'	1
0	1	1	1	1	1	1	1	1	1	1	1	1
a	a'	1	a'	a'	a'	a'	a'	1	1	a'	a'	1
b	b'	b'	1	b'	b'	b'	1	b'	1	b'	b'	1
c	c'	c'	c'	1	c'	c'	1	1	c'	1	1	1
d	d'	d'	d'	d'	1	d'	d'	d'	1	d'	1	1
e	e'	e'	e'	e'	e'	1	e'	e'	1	1	e'	1
a'	a	a	c'	b'	a	a	1	b'	c'	b'	b'	1
b'	b	c'	b	a'	b	b	a'	1	c'	a'	a'	1
c'	c	b'	a'	c	e'	d'	a'	b'	1	d'	e'	1
d'	d	d	d	e'	d	c'	e'	e'	c'	1	e'	1
e'	e	e	e	d'	c'	e	d'	d'	c'	d'	1	1
1	0	a	b	c	d	e	a'	b'	c'	d'	e'	1

Note again that every element of $\mathcal{L}_{12}$ is an implication. The reader would do well to compare this table with our previous one. We note that Theorem **2.6** (15) says that $\Supset$ and $\rightarrow$ agree on compatible pairs of elements. Also we see from (**2.6**) (1) and (**2.6**) (2) that $\Supset$ retains our minimal demands for an implication connective. However, there are non-classical features that $\Supset$ exhibits. For example, the law of contraposition fails. That is, it is not universally true that $a \Supset b = b' \Supset a'$. A glance at the above example shows that $a \Supset d = a'$ while $d' \Supset a' = e'$. Finch [7] has shown that $\Supset$ satisfies contraposition iff $\mathcal{L}$ is Boolean. It is also interesting to note, as Finch points out, that $\wedge$ and $\vee$ are expressible in terms of $\Supset$. Namely, $a \vee b = a' \Supset (a' \Supset b')'$ and $a \wedge b = (a' \Supset (a \Supset b)')'$.

Also, a strong version of transitivity fails, namely, $(x \Supset y) \wedge (y \Supset z) \leqslant (x \Supset z)$. To see this, take $x = d$, $y = 1$, $z = a'$ in the above example; then $x \Supset y = d \Supset 1 = 1$ and $y \Supset z = 1 \Supset a' = a'$ and $x \Supset z = d \Supset a' = d'$, but $(x \Supset y) \wedge$

$(y \Supset z) = 1 \wedge a' = a' \nleq (x \Supset z) = d'$. It might be desirable that these laws of logic fail in quantum logic. This will need to be decided on empirical grounds. However, we can still ask if it is possible to have an implication connective which is consistent with the tenets of quantum logic, and which satisfies the law of contraposition. Our next candidate shows this is indeed possible.

2.8 Definition Let $\mathcal{L}$ be an orthomodular lattice. For a, b in $\mathcal{L}$ define

$$a \unrhd b = (a \wedge b) \vee (a' \wedge b) \vee (a' \wedge b')$$

2.9 Theorem *Let $\mathcal{L}$ be an orthomodular lattice. Then*

(1) $(a \unrhd b) \wedge a \leqslant b$
(2) $a \leqslant b$ *iff* $a \unrhd b = 1$
(3) $a \unrhd a = 1$ *for all* a
(4) $a \unrhd 0 = a'$ *for all* a
(5) $1 \unrhd a = a$ *for all* a
(6) $a \unrhd b = b' \unrhd a'$
(7) $a \unrhd a' = a'$ *for all* a
(8) $a' \unrhd a = a$ *for all* a
(9) *if* $a \mathrm{C} b$, *then* $a \unrhd b = a \Supset b = a \rightarrow b$
(10) $a \mathbin{'\!\unrhd} b \leqslant a \unrhd b \leqslant a \rightarrow b$
(11) $(a \unrhd b) \mathrm{C} a$ *and* $(a \unrhd b) \mathrm{C} b$
(12) $a \unrhd b = (a \Supset b) \wedge (b' \Supset a')$

Proof:

(1) $(a \unrhd b) \wedge a = a \wedge [(a \wedge b) \vee (a' \wedge b) \vee (a' \wedge b')] = (a \wedge a \wedge b) \vee (a \wedge a' \wedge b) \vee (a \wedge a' \wedge b') = a \wedge b \leqslant b$, using lots of compatibility.
(2) If $a \leqslant b$ then $a \wedge b = a$ and $b' \leqslant a'$ so $a' \wedge b' = b'$. Thus $a \unrhd b = a \vee (a' \wedge b) \vee b' = (a \vee b') \vee (a' \wedge b) = (a \vee b') \vee (a' \wedge b) = (a \vee b') \vee (a \vee b')' = 1$. Conversely if $a \unrhd b = 1$, then $a = a \wedge 1 = a \wedge (a \unrhd b) \leqslant b$.
(3) $a \unrhd a = (a \wedge a) \vee (a' \wedge a) \vee (a' \wedge a') = a \vee 0 \vee a' = 1$
(4) $a \unrhd 0 = (a \wedge 0) \vee (a' \wedge 0) \vee (a' \wedge 1) = a'$
(5) $1 \unrhd a = (1 \wedge a) \vee (0 \wedge a) \vee (0 \wedge a') = a \vee 0 \vee 0 = a$
(6) $b' \unrhd a' = (b' \wedge a') \vee (b'' \wedge a') \vee (b'' \wedge a'') = (b' \wedge a) \vee (b \wedge a') \vee (b \wedge a) = a \unrhd b$
(7) $a \unrhd a' = (a \wedge a') \vee (a' \wedge a') \vee (a' \wedge a'') = 0 \vee a' \vee 0 = a'$
(8) $a' \unrhd a = a' \unrhd (a')' = a'' = a$
(9) Clear since then $a \unrhd b = (a \wedge b) \vee ((a' \wedge b) \vee (a' \wedge b')) = (a \wedge b) \vee b' = (a \vee a') \wedge (b \vee a') = 1 \wedge (b \vee a') = a \rightarrow b$
(10) Clear
(11) Clear
(12) $(a \Supset b) \wedge (b' \Supset a') = (a' \vee (a \wedge b)) \wedge (b \vee (b' \wedge a')) = ((a' \vee (a \wedge b)) \wedge b) \vee ((a' \vee (a \wedge b)) \wedge (b' \wedge a')) = (b \wedge a') \vee (a \wedge b) \vee (b' \wedge a') \vee (a \wedge b) \wedge (b' \wedge a') = (a \wedge b) \vee (a' \wedge b) \vee (a' \wedge b') = a \unrhd b$.

2.10 Example Once again we compute on $\mathcal{L}_{12}$.

$\unrhd$	0	a	b	c	d	e	a'	b'	c'	d'	e'	1
0	1	1	1	1	1	1	1	1	1	1	1	1
a	a'	1	a'	a'	c	c	a'	1	1	c	c	1
b	b'	b'	1	b'	c	c	1	b'	1	c	c	1
c	c'	c'	c'	1	c'	c'	1	1	c'	1	1	1
d	d'	c	c	d'	1	d'	c	c	1	d'	1	1
e	e'	c	c	e'	e'	1	c	c	1	1	e'	1
a'	a	a	c'	b'	0	0	1	b'	c'	c	c	1
b'	b	c'	b	a'	0	0	a'	1	c'	c	c	1
c'	c	b'	a'	c	e'	d'	a'	b'	1	d'	e'	1
d'	d	0	0	e'	d	c'	c	c	c'	1	e'	1
e'	e	0	0	d'	c'	e	c	c'	c'	d'	1	1
1	0	a	b	c	d	e	a'	b'	c'	d'	e'	1

Note that once again, strong transitivity fails. Let $x = d$, $y = 1$, and $z = a$. Then $(x \unrhd y) \wedge (y \unrhd z) = 1 \wedge a = a \not\leqslant (x \unrhd z) = c$.

Is it possible to have an implication connective which is consistent with the tenets of quantum logic and which satisfies both the law of contraposition and the strong transitivity? Once again, we can give a positive answer.

For the remainder of this section, assume $\mathcal{L}$ is a complete orthomodular lattice. Recall a complete lattice is one in which arbitrary meets and joins exist.

2.11 Definition Let $\mathsf{C}(\mathcal{L})$ denote the center of the complete orthomodular lattice $\mathcal{L}$. For a and b in $\mathcal{L}$, define $a \Supset b = \bigvee \{c \in \mathsf{C}(\mathcal{L}) \mid c \wedge a \leqslant b\}$.

2.12 Theorem *Let $\mathcal{L}$ be a complete orthomodular lattice. Then*

(1) $a \Supset a = 1$ *for all* a
(2) $0 \Supset a = 1$ *for all* a
(3) $a \Supset 1 = 1$ *for all* a
(4) $1 \Supset 0 = 0$
(5) $a \Supset b \in \mathsf{C}(\mathcal{L})$ *for all* a, b *in* $\mathcal{L}$
(6) $a \wedge (a \Supset b) \leqslant b$
(7) $a \wedge (a \Supset b) \leqslant a \wedge b$
(8) $a \leqslant b$ *iff* $a \Supset b = 1$
(9) *For* $c \in \mathsf{C}(\mathcal{L})$, $c \leqslant a \Supset b$ *iff* $c \wedge a \leqslant b$
(10) $a \Supset b \leqslant a \unrhd b \leqslant a \rightarrow b$ *for all* a, b
(11) $(a \Supset b) \wedge (b \Supset c) \leqslant (a \Supset c)$ *for all* a, b, c
(12) $a \Supset (a \Supset b) \leqslant a \Supset b$
(13) $a \leqslant (b \Supset b)$
(14) *if* $b \leqslant c$, *then* $a \Supset b \leqslant a \Supset c$
(15) *if* $c \leqslant a$, *then* $a \Supset b \leqslant c \Supset b$
(16) $(a \Supset b) \wedge (a \Supset c) \leqslant a \Supset (b \wedge c)$
(17) $(a \Supset b) \leqslant c \Supset (a \Supset b)$ *for all* c

(18) $a \supseteq b = b' \supseteq a'$
(19) $(a \supseteq a') \leqslant a'$
(20) $(a \supseteq a') \wedge a = 0$
(21) *if* $c \in \mathrm{C}(\mathcal{L})$, *then* $c \supseteq 0 = c'$

Proof:

(1) through (4) are clear since 0 and 1 are always in the center of an orthomodular lattice.
(5) follows from the fact that the center of an orthomodular lattice is subcomplete, that is, the join of central elements is central.
(6) Here we use another fact about the center, namely a certain distributive law [32], p. 86. $a \wedge (a \supseteq b) = a \wedge \bigvee\{c \in \mathrm{C}(\mathcal{L}) / c \wedge a \leqslant b\} = \bigvee\{c \wedge a \mid c \in \mathrm{C}(\mathcal{L}), c \wedge a \leqslant b\} \leqslant b$.
(7) is clear from (6).
(8) If $a \leqslant b$, then $1 \wedge a \leqslant b$ and $1 \in \mathrm{C}(\mathcal{L})$ so $a \supseteq b = 1$. On the other hand, if $a \supseteq b = 1$, then $b \geqslant a \wedge (a \supseteq b) = a \wedge 1 = a$.
(9) If $c \in \mathrm{C}(\mathcal{L})$ and $c \wedge a \leqslant b$, then $c \leqslant a \supseteq b$. Conversely, if $c \leqslant a \supseteq b$, then $c \wedge a \leqslant (a \supseteq b) \wedge a \leqslant b$.
(10) $(a \supseteq b) \wedge (a \supseteq b)' = (a \supseteq b) \wedge (a' \vee (a \wedge b))' = (a \supseteq b) \wedge (a \wedge (a \wedge b)') = ((a \supseteq b) \wedge a) \wedge (a \wedge b)' \leqslant (a \wedge b) \wedge (a \wedge b)' = 0$. Now, $a \supseteq b$ is central and disjoint from $(a \supseteq b)'$. Hence $a \supseteq b$ is orthogonal to $(a \supseteq b)'$. That is $a \supseteq b \leqslant (a \supseteq b)''$.
(11) Note $(a \supseteq b) \wedge (b \supseteq c)$ is central. Now $(a \supseteq b) \wedge (b \supseteq c) \wedge a = (a \supseteq b) \wedge a \wedge (b \supseteq c) \leqslant b \wedge (b \supseteq c) \leqslant c$. So by (9), $(a \supseteq b) \wedge (b \supseteq c) \leqslant a \supseteq c$.
(12) $a \supseteq (a \supseteq b) \wedge a \leqslant (a \supseteq b) \wedge a \leqslant b$. So by (9), $a \supseteq (a \supseteq b) \leqslant (a \supseteq b)$.
(13) is clear since $b \supseteq b = 1$.
(14) Let $b \leqslant c$. Then $(a \supseteq b) \wedge a \leqslant b \leqslant c$. So by (9), $a \supseteq b \leqslant a \supseteq c$.
(15) Let $c \leqslant a$. Then $(a \supseteq b) \wedge c \leqslant (a \supseteq b) \wedge a \leqslant b$. So by (9), $a \supseteq b \leqslant c \supseteq b$.
(16) $(a \supseteq b) \wedge (a \supseteq c) \wedge a = (a \supseteq b) \wedge a \wedge ((a \supseteq c) \wedge a) \leqslant b \wedge c$. So by (9), $(a \supseteq b) \wedge (a \supseteq c) \leqslant a \supseteq (b \wedge c)$.
(17) $(a \supseteq b) \wedge c \leqslant a \supseteq b$. So by (9), $(a \supseteq b) \leqslant c \supseteq (a \supseteq b)$.
(18) It suffices to show $a \supseteq b \leqslant b' \supseteq a'$. Let $c \in \mathrm{C}(\mathcal{L})$ with $c \wedge a \leqslant b$. Then $b' \leqslant (c \wedge a)' = c' \vee a'$. Then $c \wedge b' \leqslant c \wedge (c' \vee a') = (c \wedge c') \vee (c \wedge a') = c \wedge a' \leqslant a'$. That is $c \wedge b' \leqslant a'$. Thus $\{c \in \mathrm{C}(\mathcal{L}) \mid c \wedge a \leqslant b\} \subseteq \{c \in \mathrm{C}(\mathcal{L}) \mid c \wedge b' \leqslant a'\}$ and so $\bigvee\{c \in \mathrm{C}(\mathcal{L}) \mid c \wedge a \leqslant b\} \leqslant \bigvee\{c \in \mathrm{C}(\mathcal{L}) \mid c \wedge b' \leqslant a'\}$.
(19) $a \supseteq a' = \bigvee\{c \in \mathrm{C}(\mathcal{L}) \mid c \wedge a \leqslant a'\} = \bigvee\{c \in \mathrm{C}(\mathcal{L}) \mid c \wedge a = 0\} = \bigvee\{c \in \mathrm{C}(\mathcal{L}) \mid c \leqslant a'\} \leqslant a'$.
(20) is clear from (19).
(21) $c \supseteq 0 = \bigvee\{z \in \mathrm{C}(\mathcal{L}) \mid c \wedge z \leqslant 0\} = \bigvee\{z \in \mathrm{C}(\mathcal{L}) \mid z \leqslant c'\} \leqslant c'$.

Again we make an explicit computation of our canonical example $\mathcal{L}_{12}$.

2.13 Example

$\Supset$	0	a	b	c	d	e	a'	b'	c'	d'	e'	1
0	1	1	1	1	1	1	1	1	1	1	1	1
a	c	1	c	c	c	c	c	1	1	c	c	1
b	c	c	1	c	c	c	1	c	1	c	c	0
c	c'	c'	c'	1	c'	c'	1	1	c'	1	1	1
d	c	c	c	c	1	c	c	c	1	c	1	1
e	c	c	c	c	c	1	c	c	1	1	c	1
a'	0	0	c'	c	0	0	1	c	c'	c	c	1
b'	0	c'	0	c	0	0	c	1	c'	c	c	1
c'	c	c	c	c	c	c	c	c	1	c	c	0
d'	0	0	0	c	0	c'	c	c	c'	1	c	1
e'	0	0	0	c	c'	c'	c	c	c'	c	1	1
1	0	0	0	c	0	0	c	c	c'	c	c	1

Here we note that the set of implications is exactly equal to the center of $\mathcal{L}_{12}$. We remark further that whereas $a \Supset b$ depends only on a and b, $a \Supset b$ takes into account a more global view of the structure of the whole logic. Also note that $\Supset$ satisfies the axioms given by Zeman for implications in quantum logic, and our example **2.10** differs from his in [36]. Zeman's example can be recaptured by replacing $\mathsf{C}(\mathcal{L})$ in definition (2.8) by $\mathsf{C} = \{0, c, 1\}$. Thus we see the possibility of defining many more implication connectives analogous to $\Supset$ by joining over other subsets of $\mathsf{C}(\mathcal{L})$. This motivates an axiomatic treatment of implication connectives which is the subject of the next section.

3 *Deductive Lattices* In this section, we give an axiomatic treatment of certain kinds of implication connectives on general lattices. Indeed the reader will note that our arguments will work even for a meet semilattice. Later we will focus on the case where the lattice is orthomodular. We begin by writing down what we consider the minimal properties a mapping $\supset$ should satisfy to be considered as a candidate for an implication connective.

3.1 Definition A *weakly deductive lattice* is a pair $\langle \mathcal{L}, \supset \rangle$ where $\mathcal{L}$ is a lattice and $\supset: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is a mapping such that

$$(\supset 1) \text{ if } a \leqslant b, \text{ then } a \supset b = 1 \qquad \text{(Exportation)}$$

and

$$(\supset 2)\ a \wedge (a \supset b) \leqslant b \qquad \text{(Modus Ponens)}$$

3.2 Theorem *Let* $\langle \mathcal{L}, \supset \rangle$ *be a weakly deductive lattice. Then*

(1) $0 \supset b = 1$ *for all* b
(2) $a \supset a = 1$ *for all* a
(3) $a \supset 1 = 1$ *for all* a
(4) $a \wedge (a \supset b) \leqslant a \wedge b$
(5) $a \leqslant b$ *iff* $a \supset b = 1$
(6) $a \supset (a \supset b) \leqslant a \wedge b$
(7) $(a \supset b) \wedge (b \supset c) \wedge a \leqslant c$

(8) $(a \supset b) \wedge (a \supset c) \wedge a \leqslant b \wedge c$
(9) *if* $b \leqslant c$, *then* $(a \supset b) \wedge a \leqslant c$
(10) *if* $c \leqslant a$, *then* $(a \supset b) \wedge c \leqslant b$
(11) $(a \supset b) \wedge (a \wedge c) \leqslant a \wedge b \wedge c$
(12) *if* $c \wedge b = 0$, *then* $(a \supset b) \wedge (a \wedge c) = 0$
(13) *if* $c \leqslant a \supset b$, *then* $c \wedge a \leqslant b$
(14) $a \leqslant (b \supset b)$ *for all* b

Proof:

(1) through (4) are clear from the definition.
(5) The if part is ($\supset$1). Suppose $a \supset b = 1$. Then $a = a \wedge 1 = a \wedge (a \supset b) \leqslant b$ so $a \leqslant b$.
(6) $(a \supset (a \supset b)) \wedge a \leqslant a \wedge (a \supset b) \leqslant b$.
(7) $(a \supset b) \wedge (b \supset c) \wedge a = ((a \supset b) \wedge a) \wedge ((b \supset c)) \leqslant b \wedge (b \supset c) \leqslant c$.
(8) $(a \supset b) \wedge (a \supset c) \wedge a = ((a \supset b) \wedge a) \wedge ((a \supset c) \wedge a) \leqslant b \wedge c$.
(9) If $b \leqslant c$, then $(a \supset b) \wedge a \leqslant b \leqslant c$.
(10) If $c \leqslant a$, then $(a \supset b) \wedge c \leqslant (a \supset b) \wedge a \leqslant b$.
(11) $(a \supset b) \wedge (a \wedge c) \leqslant a \wedge b \wedge c$.
(12) is clear from (11).
(13) If $c \leqslant a \supset b$, then $c \wedge a \leqslant (a \supset b) \wedge a \leqslant b$.
(14) Clear since $(b \supset b) = 1$ for all b.

3.3 Proposition *If* $\supset_1$ *and* $\supset_2$ *make* $\mathcal{L}$ *weakly deductive then so does* $\supset_3$ *where* $a \supset_3 b = (a \supset_1 b) \wedge (a \supset_2 b)$ *for all* a, b.

If arbitrary meets exist this proposition generalizes in the obvious way.

We shall see that ($\supset$1) and ($\supset$2) are not enough to prove the deduction theorem we are after. So we move to a definition suggested to us by Zeman and considerations from modal logic.

3.4 Definition An *n-deductive lattice* is a pair $\langle \mathcal{L}, \supset \rangle$ where $\mathcal{L}$ is a lattice and $\supset$: $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is a mapping such that

($\supset$1) if $a \leqslant b$, then $a \supset b = 1$
($\supset$2)$'$ if c can be written as a meet of n implications, then

$$c \wedge a \leqslant b \text{ if and only if } c \leqslant (a \supset b)$$

We say $\langle \mathcal{L}, \supset \rangle$ is *completely deductive* iff it is n-deductive for all cardinals n.

Note that ($\supset$2)$'$ is a weakening of (CI) in the sense that (CI) is applicable only for distinguished elements c, namely, when c is a meet of implications. There is still a kernel of distributivity in the axiom as we shall presently bring out.

3.5 Theorem *Let* n *be a natural number. If* $\langle \mathcal{L}, \supset \rangle$ *is* n*-deductive, then* $\langle \mathcal{L}, \supset \rangle$ *is* k*-deductive for all* $k \leqslant n$.

Proof: The proof is clear since 1 is an implication.

3.6 Theorem *Let $\langle \mathcal{L}, \supset\rangle$ be a 1-deductive lattice. Then $\langle \mathcal{L}, \supset\rangle$ is weakly deductive. Moreover*

(1) $(a \supset b) \wedge a \leqslant b \wedge a$
(2) $a \leqslant b$ *iff* $a \supset b = 1$
(3) $a \supset b \leqslant (c \supset (a \supset b))$ *for all c; so that if x is an implication,* $x \leqslant c \supset x$ *for all c*
(4) $a \supset (b \supset c) \leqslant (a \wedge b) \supset c$
(5) $a \supset (a \supset b) = a \supset b$; *so repeated antecedents can be absorbed*
(6) *if* $b \leqslant c$, *then* $a \supset b \leqslant a \supset c$; *thus the map* $x \mapsto a \supset x$ *for fixed a is isotone*
(7) *if* $c \leqslant a$, *then* $a \supset b \leqslant c \supset b$; *thus the map* $y \mapsto y \supset b$ *for fixed b is antitone*

Proof: To show $\langle \mathcal{L}, \supset\rangle$ is weakly deductive, we need only argue modus ponens. But $a \supset b \leqslant a \supset b$ and $a \supset b$ is an implication so by $(\supset 2)'$, $(a \supset b) \wedge a \leqslant b$. Now (1) and (2) are clear.

(3) $(a \supset b) \wedge c \leqslant (a \supset b)$ so by $(\supset 2)'$, $a \supset b \leqslant c \supset (a \supset b)$.
(4) $(a \supset (b \supset c)) \wedge (a \wedge b) \leqslant (b \supset c) \wedge b \leqslant c$ so (4) follows by $(\supset 2)'$.
(5) Take $c = a$ in (3) and get $a \supset b \leqslant a \supset (a \supset b)$. Also take $a = b$ and $b = c$ in (4) to get $a \supset (a \supset b) \leqslant a \supset b$.
(6) follows from (3.2) (9) and $(\supset 2)'$.
(7) follows from (3.2) (10) and $(\supset 2)'$.

3.7 Theorem *Let $\langle \mathcal{L}, \supset\rangle$ be 1-deductive. If a and c are implications, then for any b in $\mathcal{L}$ we have $(a \wedge b) \vee (c \wedge b) = (a \vee c) \wedge b$. In particular, if the set of implications in a 1-deductive lattice form a sublattice, this sublattice must be distributive.*

Proof: Let a and c be implications. Let $b \in \mathcal{L}$ and let $r = (a \wedge b) \vee (c \wedge b)$. Clearly $r \leqslant b \wedge (a \vee c)$. Now $a \wedge b \leqslant r$ so since a is an implication, $a \leqslant b \supset r$. Similarly $c \wedge b \leqslant r$ so $c \leqslant b \supset r$. Thus $a \vee c \leqslant b \supset r$. By **3.2** (13) we get $(a \vee c) \wedge b \leqslant r$. Thus $r = b \wedge (a \vee c)$ and we are done.

3.8 Theorem *Let $\langle \mathcal{L}, \supset\rangle$ be a 2-deductive lattice. Then*

(1) $(a \supset b) \wedge (b \supset c) \leqslant (a \supset c)$ (strong transitivity)
(2) $(a \supset b) \wedge (a \supset c) = a \supset (b \wedge c)$
(3) $a \supset b = a \supset (a \wedge b)$
(4) $a \supset (b \supset c) \leqslant (a \supset b) \supset (a \supset c)$
(5) $a \supset b \leqslant (b \supset c) \supset (a \supset c)$
(6) $a \supset (a \supset b) = a \supset b$
(7) $a \leqslant (b \supset b)$

Proof:

(1) $((a \supset b) \wedge (b \supset c)) \wedge a = ((a \supset b) \wedge a) \wedge (b \supset c) \leqslant b \wedge (b \supset c) \leqslant c$. Now use $(\supset 2)'$.
(2) $((a \supset b) \wedge (a \supset c)) \wedge a = ((a \supset b) \wedge a) \wedge ((a \supset c) \wedge a) \leqslant b \wedge c$ so $(a \supset b) \wedge (a \supset c) \leqslant a \supset (b \wedge c)$. But $b \wedge c \leqslant b$ so $a \supset (b \wedge c) \leqslant a \supset b$ and $b \wedge c \leqslant c$ so $a \supset (b \wedge c) \leqslant a \supset c$. Thus $a \supset (b \wedge c) \leqslant (a \supset b) \wedge (a \supset c)$.

(3) Take $c = a$ in (2) and get $a \supset b = a \supset (a \wedge b)$.
(4) $(a \supset (b \supset c) \wedge (a \supset b)) \wedge a = ((a \supset (b \supset c)) \wedge a) \wedge (a \supset b) \leqslant (b \supset c) \wedge (a \supset b) \leqslant a \supset c$, so $a \supset (b \supset c) \wedge (a \supset b) \leqslant a \supset (a \supset c) = a \supset c$. Again, $a \supset (b \supset c) \leqslant (a \supset b) \supset (a \supset c)$.
(5) through (7) were obtained previously.

In view of **3.8**, we see that in a 2-deductive lattice, the axioms for implication in the Lewis modal system S4 are satisfied.

3.9 Swap Lemma *Let* $\langle \mathcal{L}, \supset \rangle$ *be a* 2-*deductive lattice. Then*

(1) *if* b *is an implication, then* $a \supset (b \supset c) \leqslant b \supset (a \supset c)$
(2) *if* a *and* b *are implications, then* $a \supset (b \supset c) = b \supset (a \supset c)$

Proof:

(1) Let b be an implication. By **3.8** (4) we have $a \supset (b \supset c) \leqslant (a \supset b) \supset (a \supset c)$. By **3.6** (3), $b \leqslant a \supset b$ so by **3.6** (6), we get $(a \supset b) \supset (a \supset c) \leqslant b \supset (a \supset c)$.
(2) Use (1) and get $a \supset (b \supset c) \leqslant b \supset (a \supset c) \leqslant a \supset (b \supset c)$.

3.10 Proposition *Let* $\langle \mathcal{L}, \supset \rangle$ *be a* 2-*deductive lattice. Then*

(1) *if* b *is an implication, then* $(a \supset b) \supset (a \supset c) \leqslant b \supset (a \supset c)$
(2) *if* a *and* b *are implications, then* $(a \supset b) \supset (a \supset c) = a \supset (b \supset c)$
(3) *if* a *is an implication, then* $a \leqslant (a \supset b) \supset b$
(4) $a \supset b \leqslant ((a \supset b) \supset b) \supset b$ *for all* a, b
(5) *if* a *is an implication, then* $a \supset b = ((a \supset b) \supset b) \supset b$
(6) *if* a *is an implication, then* $1 \supset a = a$

Proof:

(1) By **3.6** (3), we have $b \leqslant (a \supset b)$ so $((a \supset b) \supset (a \supset c)) \wedge b \wedge a \leqslant ((a \supset b) \supset (a \supset c)) \wedge (a \supset b) \wedge a \leqslant (a \supset c) \wedge a \leqslant c$. Thus $(a \supset b) \supset (a \supset c) \wedge b \leqslant a \supset c$ and so $(a \supset b) \supset (a \supset c) \leqslant b \supset (a \supset c)$.
(2) Using (1) we have $a \supset (b \supset c) \leqslant (a \supset b) \supset (a \supset c) \leqslant b \supset (a \supset c) = a \supset (b \supset c)$.
(3) Let a be an implication. Then $a \wedge (a \supset b) \leqslant b$ so by $(\supset 2)'$ we get $a \leqslant (a \supset b) \supset b$.
(4) is clear.
(5) Let a be an implication. By (3), $a \leqslant (a \supset b) \supset b$. By **3.6** (6), $((a \supset b) \supset b) \supset b \leqslant a \supset b$. The other inequality is (4) above.
(6) Let a be an implication. By modus ponens, $1 \wedge (1 \supset a) \leqslant a$ so $1 \supset a \leqslant a$ always. By **3.6** (3) with $b = 1$, we get $a \leqslant (1 \supset a)$.

We remark in passing that the implications in a 2-deductive lattice form an implicative model [28, 29]. If the set of implications forms a sublattice, then this sublattice is a Brouwerian lattice.

4 *Semantics in Deductive Lattices—An* S4 *Deduction Theorem* We take our point of view of semantics from that expressed by Halmos [13] in his approach to algebraic logic. Throughout this section, let $\langle \mathcal{L}, \supset \rangle$ denote a 2-deductive lattice.

4.1 Definition A $\supset$-*filter* is a subset $\mathcal{F}$ of $\mathcal{L}$ such that

(1) $1 \in \mathcal{F}$ (Law of Tautology)
(2) if $a \in \mathcal{F}$ and $a \supset b \in \mathcal{F}$, then $b \in \mathcal{F}$ (Law of Detachment)

Let S be a subset of $\mathcal{L}$. Define $\overline{S} = \bigcap\{\mathcal{F} \mid \mathcal{F}$ is a $\supset$-filter and $\mathcal{F} \supseteq S\}$. As notation, we write $S \vdash w$ iff $w \in \overline{S}$. If $\mathcal{F}$ is a $\supset$-filter on $\mathcal{L}$, then the pair $\langle \mathcal{L}, \mathcal{F}\rangle$ might be called an *applied logic*. It is *consistent* iff $0 \notin \mathcal{F}$.

4.2 Lemma *Let $\mathcal{F}$ be a $\supset$-filter on $\mathcal{L}$. If $a \in \mathcal{F}$ and $a \leqslant b$, then $b \in \mathcal{F}$.*

Proof: Since $a \leqslant b$, we have by ($\supset$1) that $a \supset b = 1$. Thus $a \supset b = 1 \in \mathcal{F}$ and $a \in \mathcal{F}$, so by Detachment, $b \in \mathcal{F}$.

4.3 Lemma *Let S be a subset of $\mathcal{L}$.. Then $\overline{S}$ is a $\supset$-filter. Indeed, it is the smallest $\supset$-filter in $\mathcal{L}$ which contains S.*

Proof: The proof is easy and is omitted.

4.4 Lemma *Let $S \subseteq \mathcal{L}$ with $1 \in S$. Suppose there exist $a_1, a_2, \ldots, a_n$ in S such that $a_1 \supset (\ldots (a_{n-2} \supset (a_{n-1} \supset (a_n \supset w)) \ldots) = 1$ where w is some element of $\mathcal{L}$. Then $S \vdash w$.*

Proof: Suppose the condition holds. Then $a_1 \in S \subseteq \overline{S}$ and $a_1 \supset (a_2 \supset \ldots) = 1 \in S$. So by Detachment, $a_2 \supset (a_3 \supset \ldots) \in \overline{S}$, since $\overline{S}$ is a $\supset$-filter. But now $a_2 \in S \subseteq \overline{S}$, so again by Detachment, $a_3 \supset (a_4 \supset \ldots) \in \overline{S}$. Continuing in this manner, we finally arrive at $a_{n-1} \in S \subseteq \overline{S}$ and $a_{n-1} \supset (a_n \supset w) \in \overline{S}$, so $a_n \supset w \in \overline{S}$. Now $a_n \in S \subseteq \overline{S}$ and $a_n \supset w \in \overline{S}$ so by Detachment $w \in \overline{S}$, that is $S \vdash w$.

4.5 Lemma *Let a_1 and a_2 be implications. Then*

$$(a_1 \supset a_2) \supset (w \supset x) \leqslant a_1 \supset ((a_2 \supset w) \supset (a_2 \supset x)).$$

Proof: $a_2 \leqslant (a_1 \supset a_2)$ by **3.11** (7) so $(a_1 \supset a_2) \supset (w \supset x) \leqslant a_2 \supset (w \supset x)$ by **3.11** (6). But $a_2 \supset (w \supset x) \leqslant (a_2 \supset w) \supset (a_2 \supset x)$. Then $(a_2 \supset w) \supset (a_2 \supset x) \leqslant a_1 \supset ((a_2 \supset w) \supset (a_2 \supset x))$ again by **3.11** (7). The lemma is now clear.

4.6 Theorem *Let S be a set of implications in $\mathcal{L}$ with $1 \in S$. Let w be any element of $\mathcal{L}$. Then $S \vdash w$ iff there exist $a_1, a_2, \ldots, a_n$ in S with $a_1 \supset (a_2 \supset \ldots \supset (a_{n-1} \supset (a_n \supset w)) \ldots) = 1$.*

Proof: That the right side implies the left is lemma **4.4**. We need to prove the converse. Let

$T = \{w \in \mathcal{L} \mid$ *there exist* $a_1, \ldots, a_n$ *in* S *with* $a_1 \supset (a_2 \supset \ldots \supset (a_n \supset w) \ldots) = 1\}$.

It suffices to show T is a filter which contains S. For any $a \in S$, $a \supset a = 1$ so $S \subseteq T$ and $1 \in T$. Let $w \in T$ and $w \supset x \in T$. Show $x \in T$. There exist $a_1, \ldots, a_n \in S$ such that $a_1 \supset (a_2 \supset \ldots \supset (a_n \supset w) \ldots) = 1$ and there exist $c_1, \ldots, c_k$ in S such that $c_1 \supset (c_2 \supset \ldots \supset (w \supset x) \ldots) = 1$. As the notation becomes somewhat cumbersome, we shall consider a special case which is still general enough to make all contingencies arise. Assume $n = 3$ and $k = 2$. Then we have $a_1 \supset (a_2 \supset (a_3 \supset w)) = 1$ and $c_1 \supset (c_2 \supset (w \supset x)) = 1$. Now $1 = ((a_1 \supset a_2) \supset a_3) \supset 1 = ((a_1 \supset a_2) \supset a_3) \supset (c_1 \supset (c_2 \supset (w \supset x))) = c_1 \supset (((a_1 \supset a_2) \supset a_3) \supset (c_2 \supset (w \supset x)))$ by the Swap Lemma.

The idea of the proof is to swap out the c's, and then use Lemma **4.5** repeatedly. Thus $c_1 \supset (((a_1 \supset a_2) \supset a_3) \supset (c_2 \supset (w \supset x))) \leqslant c_1 \supset (c_2 \supset (((a_1 \supset a_2) \supset a_3) \supset (w \supset x))) \leqslant c_1 \supset (c_2 \supset ((a_1 \supset a_2) \supset ((a_3 \supset w) \supset (a_3 \supset x)))) \leqslant c_1 \supset (c_2 \supset (a_1 \supset ((a_2 \supset (a_3 \supset w)) \supset (a_2 \supset (a_3 \supset x))))) \leqslant c_1 \supset (c_2 \supset ((a_1 \supset (a_2 \supset (a_3 \supset w))) \supset (a_1 \supset (a_2 \supset (a_3 \supset x))))) = c_1 \supset (c_2 \supset (1 \supset (a_1 \supset (a_2 \supset (a_3 \supset x))))) = c_1 \supset (c_2 \supset (a_1 \supset (a_2 \supset (a_3 \supset x))))$. Thus x is in T as was to be shown.

4.7 Corollary (S4 Deduction Theorem) *Let S be a set of implications with $1 \in S$. Then $S \cup \{a\} \vdash b$ iff $S \vdash (a \supset b)$.*

Proof: Suppose $S \vdash (a \supset b)$. Show $S \cup \{a\} \vdash b$. We have $a \supset b \in \overline{S}$. We want $b \in \overline{S \cup \{a\}}$. But $\overline{S} \subseteq \overline{S \cup \{a\}}$ so $a \supset b \in \overline{S \cup \{a\}}$. By Detachment, $b \in \overline{S \cup \{a\}}$.

Conversely, let $S \cup \{a\} \vdash b$. Show $S \vdash (a \supset b)$. Let $b \in \overline{S \cup \{a\}}$. Show $a \supset b \in \overline{S}$. We must find $a_1, \ldots, a_n$ in S such that $(a_1 \supset \ldots \supset (a_n \supset (a \supset b)) \ldots) = 1$. Now there do exist $c_1, \ldots, c_k$ in $S \cup \{a\}$ such that $(c_1 \supset \ldots \supset (c_k \supset b) \ldots) = 1$. Once again we simplify the situation to avoid notational problems. Let $k = 3$.

Case 1: No c is equal to a. Then $c_1 \supset (c_2 \supset (c_3 \supset b)) = 1$ and $c_1 \supset (c_2 \supset (c_3 \supset (a \supset b))) \geqslant c_1 \supset (c_2 \supset (a \supset (c_3 \supset b))) \geqslant c_1 \supset (a \supset (c_2 \supset (c_3 \supset b))) \geqslant a \supset (c_1 \supset (c_2 \supset (c_3 \supset b))) = a \supset 1 = 1$. Thus $a \supset b \in \overline{S}$.

Case 2: Some c is equal to a. Say $c_2 = a$. We have $1 = c_1 \supset (c_2 \supset (c_3 \supset b)) = c_1 \supset (a \supset (c_3 \supset b)) \leqslant c_1 \supset (c_3 \supset (a \supset b))$. Now the idea is to swap all a's down to the right. By **3.9** (1), the quantity will at worst get bigger. If you should collide with another a while doing this, use **3.6** (4) to absorb the iterated antecedent. Once again we conclude that $a \supset b \in \overline{S}$.

5 *Deductive Orthomodular Lattices* In this section we return to the context of orthomodular lattices to see what more can be said about implication connectives in this more specialized environment.

5.1 Definition A weakly deductive orthomodular lattice is a weakly deductive lattice $\langle \mathcal{L}, \supset \rangle$ where $\mathcal{L}$ is an orthomodular lattice.

5.2 Theorem *Let $\mathcal{L}$ be a weakly deductive orthomodular lattice. Then in addition to the properties of* **3.2**, *we have*

(1) *if $b \perp c$, then $(a \supset b) \wedge (a \wedge c) = 0$*
(2) $(a \supset b) \wedge (a \wedge b') = (a \supset b) \wedge (a \rightarrow b)' = 0$
(3) $(a \supset b) \wedge (a \wedge (a \wedge b)') = (a \supset b) \wedge (a \Supset b)' = 0$
(4) $(a \supset b) \,\mathrm{C}\, (a \rightarrow b)$ *iff* $a \supset b \leqslant a \rightarrow b$
(5) $(a \supset b) \,\mathrm{C}\, (a \Supset b)$ *iff* $a \supset b \leqslant a \Supset b$
(6) *if $(a \supset b) \,\mathrm{C}\, (a \Supset b)$, then $(a \supset b) \,\mathrm{C}\, (a \rightarrow b)$*

Proof:

(1) If $b \perp c$, then $b \wedge c = 0$ so use **3.2** (12).
(2) is clear since $b \perp b'$.
(3) is clear with $c = (a \wedge b)'$ in (1).
(4) If $a \supset b \leqslant a' \vee b$, then, being comparable, they are compatible. Conversely, suppose $(a \supset b) \mathrm{C} (a' \vee b)$. Then $(a \supset b) = ((a \supset b) \wedge (a' \vee b)) \vee ((a \supset b) \wedge$

$(a' \vee b)') = ((a \supset b) \wedge (a' \vee b)) \vee ((a \supset b) \wedge (a \wedge b')) = ((a \supset b) \wedge (a' \vee b)) \vee 0 = (a \supset b) \wedge (a' \vee b) \leqslant a' \vee b$.
(5) If $a \supset b \leqslant a' \vee (a \wedge b)$, then $(a \supset b)\,\mathrm{C}\,(a' \vee (a \wedge b))$. Conversely, suppose $(a \supset b)\mathrm{C}(a' \vee (a \wedge b))$. Then $(a \supset b) = ((a \supset b) \wedge (a' \vee (a \wedge b)) \vee ((a \supset b) \wedge (a' \vee (a \wedge b))') = ((a \supset b) \wedge (a' \vee (a \wedge b)) \vee ((a \supset b) \wedge (a \wedge (a \wedge b)')) = ((a \supset b) \wedge (a' \vee (a \wedge b)) \vee 0 = (a \supset b) \wedge (a' \vee (a \wedge b)) \leqslant a' \vee (a \wedge b)$.
(6) If $(a \supset b)\mathrm{C}(a' \vee (a \wedge b))$ then $a \supset b \leqslant a' \vee (a \wedge b) \leqslant a' \vee b$ so that $(a \supset b)\mathrm{C}(a' \vee b)$.

5.3 Lemma *Let $\mathcal{L}$ be an orthomodular lattice with x, a, b in $\mathcal{L}$. If $x\mathrm{C}a$, then $x \wedge a \leqslant b$ implies $x \wedge b \leqslant a' \Supset b$.*

Proof: Suppose $x\mathrm{C}a$ and $x \wedge a \leqslant b$. Then $x = (x \wedge a) \vee (x \wedge a')$ so $x \wedge b = ((x \wedge a) \vee (x \wedge a')) \wedge b = (x \wedge a \wedge b) \vee (x \wedge a' \wedge b) = (x \wedge a) \vee (x \wedge (a' \wedge b)) \leqslant a \vee (a' \wedge b)$.

5.4 Theorem *Let $\langle \mathcal{L}, \supset \rangle$ be a weakly deductive orthomodular lattice. If $(a \supset b)\mathrm{C}a$, then $(a \supset b) \wedge b \leqslant a' \Supset b$.*

Proof: Let $x = a \supset b$ in **5.3**. Then $x\mathrm{C}a$ and $x \wedge a = (a \supset b) \wedge a \leqslant b$. So **5.3** applies and so $(a \supset b) \wedge b \leqslant (a' \wedge b) \vee a$.

5.5 Corollary *Let $\langle \mathcal{L}, \supset \rangle$ be a weakly deductive orthomodular lattice. If $a \supset b \in \mathrm{C}(\mathcal{L})$ for all a, b, then*

(1) *$a \supset b \leqslant a \Supset b \leqslant a \rightarrow b$ for all a, b*
(2) *$(a \supset b) \wedge b \leqslant a' \Supset b$*

5.6 Examples

(1) "trivial hook" D:

$$\text{Define } a \mathrel{\mathrm{D}} b = \begin{cases} 1 \text{ if } a \leqslant b \\ 0 \text{ if otherwise} \end{cases}$$

Then $\langle \mathcal{L}, \mathrm{D} \rangle$ is a weakly deductive orthomodular lattice for any orthomodular lattice $\mathcal{L}$.

(2) "Sasaki hook" $\Supset$:

Recall $a \Supset b = a' \vee (a \wedge b)$. From properties developed in a previous section, we see that $\langle \mathcal{L}, \Supset \rangle$ is a weakly deductive orthomodular lattice for any orthomodular lattice $\mathcal{L}$.

(3) "classical hook" $\rightarrow$:

Recall $a \rightarrow b = a' \vee b$. The next theorem shows that $\rightarrow$ is never weakly deductive in a quantum logic.

(4) "Kotas-Kalmbach hook" $\dot{\mathrm{D}}$:

Recall $a \mathrel{\dot{\mathrm{D}}} b = (a \wedge b) \vee (a' \wedge b) \vee (a' \wedge b')$. From properties developed in a previous section, we see that $\langle \mathcal{L}, \dot{\mathrm{D}} \rangle$ is a weakly deductive orthomodular lattice for any orthomodular lattice $\mathcal{L}$.

We remark that even a Boolean algebra can be a weakly deductive orthomodular lattice with several different "hooks".

5.7 Theorem *Let $\mathcal{L}$ be an orthomodular lattice. Then $\langle \mathcal{L}, \to \rangle$ is weakly deductive iff $\mathcal{L}$ is Boolean.*

Proof: Suppose $\mathcal{L}$ is Boolean. Then $a \wedge (a \to b) = a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b) = a \wedge b \leqslant b$ so ($\supset$2) holds. Next, let $a \leqslant b$. Then $a \vee a' \leqslant b \vee a' = a \to b$ so $1 \leqslant a \to b$. Hence $a \to b = 1$ and we have ($\supset$1) also.

Next suppose $\langle \mathcal{L}, \to \rangle$ is weakly deductive. Then by **3.2** (4) we get $(a \to b) \wedge a \leqslant b$ so $(a' \vee b) \wedge a \leqslant b$ whence $(a' \vee b) \wedge a \leqslant a \wedge b$. Also $a \wedge b \leqslant a \wedge (b \vee a')$ so $a \wedge b = a \wedge (a' \vee b)$. This means $a\mathrm{C}b$ and we have that every pair of elements in $\mathcal{L}$ is compatible. Thus $\mathcal{L}$ is Boolean.

5.8 Definition An n-deductive orthomodular lattice is an n-deductive lattice $\langle \mathcal{L}, \supset \rangle$ where $\mathcal{L}$ is an orthomodular lattice.

5.9 Proposition *Let $\mathcal{L}$ be a 1-deductive orthomodular lattice. Then*

(1) *if $(a \supset b)\mathrm{C}a$, then $a \supset b \leqslant b \supset (a' \Supset b)$*
(2) *if c is an implication and $c\mathrm{C}a$, then*

$$c \leqslant a \supset b \text{ implies } c \leqslant b' \supset a'$$

Proof:

(1) Let $(a \supset b)\mathrm{C}a$. Then by **5.4**, $(a \supset b) \wedge b \leqslant (a' \vee b) \vee a$ so by ($\supset$2)$'$, $a \supset b \leqslant b \supset (a' \wedge b) \vee a$.
(2) Let c be an implication and suppose $c\mathrm{C}a$. Suppose $c \leqslant a \supset b$. Then $c \wedge a \leqslant b$ so $b' \leqslant (c \wedge a)' = c' \vee a'$. Then $c \wedge b' \leqslant (c' \vee a') \wedge c = (c' \wedge c) \vee (a' \wedge c) = a' \wedge c \leqslant a'$. Thus $c \leqslant b' \supset a'$.

5.10 Proposition *$\langle \mathcal{L}, \Supset \rangle$ is a 1-deductive orthomodular lattice iff $\mathcal{L}$ is Boolean.*

Proof: Suppose $\langle \mathcal{L}, \Supset \rangle$ is 1-deductive. Recall $a \Supset b = a' \vee (a \wedge b)$. Note $1 \Supset b = b$. By **3.6** (3), $a \leqslant b \Supset a$ for all a, b. Thus by (QI), $(a \vee b') \wedge b \leqslant a$ so $a\mathrm{C}b$ for all a, b. Thus $\mathcal{L}$ is Boolean. Conversely, if $\mathcal{L}$ is Boolean, $\Supset = \to$ and we are in the classical situation.

5.11 Theorem *Let $\mathcal{L}$ be a 1-deductive orthomodular lattice. Then*

(1) *let c be an implication. Then $c\mathrm{C}b$ iff $c \leqslant (b \vee c') \supset b$*
(2) *let c be an implication. If $b' \supset (b' \wedge c) \leqslant (b \vee c') \supset b$, then $c\mathrm{C}b$*
(3) *if $a \supset b = b' \supset a'$ for all a, b in $\mathcal{L}$, then $a \supset b \in \mathrm{C}(\mathcal{L})$ for all a, b in $\mathcal{L}$*

Proof:

(1) Let c be an implication and let $c\mathrm{C}b$. Then $c \wedge (b \vee c') = (c \wedge b) \vee (c \wedge c') = c \wedge b \leqslant b$. So by ($\supset$2)$'$, $c \leqslant (b \vee c') \supset b$. Let $c \leqslant (b \vee c') \supset b$. Then $c \wedge (b \vee c') = b \wedge c$. Thus $b \wedge c = c \wedge (b \vee c')$ which means $c\mathrm{C}b$.
(2) Let c be an implication. Let $b' \supset b' \wedge c \leqslant b \vee c' \supset b$. But $c \wedge b' \leqslant c \wedge b'$ and c an implication implies $c \leqslant b' \supset (c \wedge b')$. Thus $c \leqslant (b \vee c') \supset b$ which says $c\mathrm{C}b$.
(3) If $a \supset b = b' \supset a'$ for all a, b, then $b' \supset (b' \wedge c) = (b \vee c') \supset b$ for all b and c. Put $c = a \supset b$ and take $b = x$. Then c is an implication so using (2), we get $(a \supset b)\mathrm{C}x$. But x is arbitrary so $a \supset b \in \mathrm{C}(\mathcal{L})$ for all a, b.

5.12 Theorem *Let $\langle \mathcal{L}, \supset \rangle$ be a 2-deductive orthomodular lattice with $a \supset b \leqslant b' \supset a'$ for all a, b in $\mathcal{L}$. Then*

(1) $a \supset b = b' \supset a'$
(2) $(a \supset c) \wedge (b \supset c) = (a \vee b) \supset c$
(3) $a \supset b = (a \vee b) \supset b = b' \supset a'$
(4) $(a \supset c) \wedge (b \supset c) \wedge (a \vee b) \leqslant c$
(5) $(a \supset b) \wedge (a \vee b) \leqslant b$
(6) $(a \supset a') \leqslant a'$
(7) $a \supset b \in \mathbf{C}(\mathcal{L})$ *for all* a, b
(8) $a \supset_t b \leqslant a \supset b \leqslant a \supset_s b \leqslant a \rightarrow b$
(9) $a \supset b \leqslant b \supset (a' \supset_s b)$

Proof:

(1) $a \supset b \leqslant b' \supset a' \leqslant a'' \supset b'' = a \supset b$ so $a \supset b = b' \supset a'$.
(2) $(a \supset c) \wedge (b \supset c) = (c' \supset a') \wedge (c' \supset b') \leqslant c' \supset (a' \wedge b') = (a' \wedge b')' \supset c = (a \vee b) \supset c$.
(3) Let $b = c$ in (2).
(4) is equivalent to (2) via $(\supset 2)'$.
(5) Take $b = c$ in (4).
(6) Set $b = a'$ in (5).
(7) follows from Theorem **5.11** (3) and (1) above.
(8) follows from Corollary **5.5** (1).
(9) follows from Proposition **5.9** (1).

5.13 Examples Let $\mathcal{L}$ be a complete orthomodular lattice. Let $C \subseteq \mathbf{C}(\mathcal{L})$ be such that $1 \in C$. Define $a \supset b$ by $a \supset b = \bigvee\{c \in C \mid c \wedge a \leqslant b\}$.

$(\supset 1)$: Let $a \leqslant b$. Then $1 \in C$ and $1 \wedge a = a \leqslant b$ so $a \supset b \geqslant 1$.

$(\supset 2)$: $a \wedge (a \supset b) = a \wedge \bigvee\{c \in C \mid c \wedge a \leqslant b\} = \bigvee\{a \wedge c \mid c \in C,\ c \wedge a \leqslant b\} \leqslant b$.

Hence, $\langle \mathcal{L}, \supset \rangle$ is a weakly deductive orthomodular lattice.

Now suppose $C \subseteq \mathbf{C}(\mathcal{L})$, $1 \in C$ and C is closed under the formation of arbitrary infima and suprema. Then $\langle \mathcal{L}, \supset \rangle$ is a completely deductive orthomodular lattice. For, suppose c is a meet of implications. Each implication in a supremum of elements from C, so is in C. Also, the meet of implications is a meet of elements of C, so is in C. Thus $c \in C$. Suppose $c \wedge a \leqslant b$. Since $c \in C$, $c \leqslant a \supset b$. Conversely, if $c \leqslant a \supset b$, then $a \wedge c \leqslant a \wedge (a \supset b) \leqslant b$. So we have $(\supset 2)'$. We note C is exactly the set of implications since C is closed under joins and since if $c \in C$, then $1 \supset c = c$.

Next, we show the law of contraposition. Let $c \in C$ with $c \wedge a \leqslant b$. We claim $c \wedge b' \leqslant a'$. But $c \in C$ and $c \wedge a \leqslant b$ gives $b' \leqslant (c \wedge a)' = c' \vee a'$ so $b' \wedge c \leqslant (c' \vee a') \wedge c = (c' \wedge c) \vee (a' \wedge c) = a' \wedge c \leqslant a'$. Thus $\{c \in C \mid c \wedge a \leqslant b\} \subseteq \{c \in C \mid c \wedge b' \leqslant a'\}$. Examples of subsets C satisfying all the above are $C = \{1, 0\}$ and $C = \mathbf{C}(\mathcal{L})$.

For the remainder of the paper, let $\langle \mathcal{L}, \supset\rangle$ be a 2-deductive orthomodular lattice. We finish our discussion by gathering more evidence that orthomodular lattices are intimately related to modal logic. Modal logic concerns itself with the concepts of "necessity" and "possibility". We take the view that to each proposition is associated another proposition asserting its necessity and another asserting its possibility. First we recall a definition.

5.14 Definition A *closure operator* on a lattice $\mathcal{L}$ is a mapping φ: $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that

(1) φ is isotone (i.e., $a \leqslant b$ implies $\varphi(a) \leqslant \varphi(b)$)
(2) $\varphi = \varphi^2$
(3) $a \leqslant \varphi(a)$ for all a

We say φ is *normalized* provided $\varphi(0) = 0$ whenever $\mathcal{L}$ has an order zero 0. Closure operators are discussed, for example, in [1] and [16].

5.15 Definition Define $\mathsf{L}a = 1 \supset a$ ("necessarily a") and $\mathsf{M}a = (\mathsf{L}a')'$ ("possibly a").

5.16 Theorem

(1) $(\mathsf{M}a)' = \mathsf{L}a'$
(2) $\mathsf{M}a' = (\mathsf{L}a)'$
(3) $\mathsf{L}a = (\mathsf{M}a')'$
(4) $\mathsf{LL}a = (\mathsf{MM}a')$
(5) $\mathsf{LL}a' = (\mathsf{MM}a)'$
(6) $\mathsf{MM}a' = (\mathsf{LL}a)'$
(7) $\mathsf{LM}a' = (\mathsf{ML}a)'$
(8) $\mathsf{ML}a' = (\mathsf{LM}a)'$

Proof: The proof of **5.16** is easy and is omitted.

5.17 Theorem

(1) $\mathsf{L}1 = 1$
(2) $\mathsf{L}a \leqslant a$ *for all* a
(3) $\mathsf{L}0 = 0$
(4) $\mathsf{LL}a = \mathsf{L}a$ *for all* a
(5) *if* $a \leqslant b$, *then* $\mathsf{L}a \leqslant \mathsf{L}b$
(6) $\mathsf{L}a \wedge \mathsf{L}b = \mathsf{L}(a \wedge b)$
(7) $\mathsf{L}a \vee \mathsf{L}b \leqslant \mathsf{L}(a \vee b)$
(8) $\mathsf{L}a \leqslant b \supset \mathsf{L}a$ *for all* b
(9) $\mathsf{L}(a \supset_s b) = a \supset b$
(10) $\mathsf{L}(a \supset b) = a \supset b$ *for all* a, b
(11) *The set of implications is exactly the set of fixed points of* L
(12) $\mathsf{L}(a \supset b) \leqslant \mathsf{L}a \supset \mathsf{L}b$

Proof:

(1) $\mathsf{L}1 = 1 \supset 1 = 1$.
(2) $\mathsf{L}a = 1 \wedge \mathsf{L}a = 1 \wedge (1 \supset a) \leqslant a$.

(3) By (2), $\mathrm{L}0 \leqslant 0$ so equality must hold.
(4) $\mathrm{LL}a = 1 \supset \mathrm{L}a = 1 \supset (1 \supset a) = 1 \supset a = \mathrm{L}a$.
(5) If $a \leqslant b$, then $1 \supset a \leqslant 1 \supset b$ so $\mathrm{L}a \leqslant \mathrm{L}b$.
(6) $a \wedge b \leqslant a$ so $\mathrm{L}(a \wedge b) \leqslant \mathrm{L}(a)$. Also $a \wedge b \leqslant b$ so $\mathrm{L}(a \wedge b) \leqslant \mathrm{L}b$. Thus $\mathrm{L}(a \wedge b) \leqslant \mathrm{L}a \wedge \mathrm{L}b$. But $\mathrm{L}(a \wedge b) = 1 \supset (a \wedge b) = (1 \supset a) \wedge (1 \supset b) = \mathrm{L}a \wedge \mathrm{L}b$.
(7) $a \leqslant a \vee b$ so $\mathrm{L}a \leqslant \mathrm{L}(a \vee b)$ and $b \leqslant a \vee b$ so $\mathrm{L}b \leqslant \mathrm{L}(a \vee b)$. Hence $\mathrm{L}a \vee \mathrm{L}b \leqslant \mathrm{L}(a \vee b)$.
(8) $\mathrm{L}a = 1 \supset a \leqslant b \supset (1 \supset a)$ for all b.
(9) through (12) are left to the reader.

Next we have a dual theorem to Theorem **5.17**.

5.18 Theorem

(1) $\mathrm{M}0 = 0$
(2) $a \leqslant \mathrm{M}a$ *for all* a
(3) $\mathrm{M}1 = 1$
(4) $\mathrm{MM}a = \mathrm{M}a$ *for all* a
(5) *if* $a \leqslant b$, *then* $\mathrm{M}a \leqslant \mathrm{M}b$
(6) $\mathrm{M}(a \vee b) = \mathrm{M}a \vee \mathrm{M}b$
(7) $\mathrm{M}(a \wedge b) \leqslant \mathrm{M}a \wedge \mathrm{M}b$
(8) $(\mathrm{M}(a \vee b))' = (\mathrm{M}a)' \wedge (\mathrm{M}b)'$
(9) $(b \supset \mathrm{L}a')' \leqslant \mathrm{M}a$
(10) $\mathrm{M}(a \supset b)' = (a \supset b)'$ *for all* a, b
(11) *the set of fixed points of* M *is the set of negations of implications*
(12) M *is a normalized closure operator on* $\mathcal{L}$

Proof: The proofs are easy using the definition of M and the properties of L derived in (5.17). They will be omitted.

5.19 Example Let $\mathcal{L}$ be any complete orthomodular lattice. Recall that $a \supseteq b = \bigvee\{c \in \mathrm{C}(\mathcal{L}) \mid c \wedge a \leqslant b\}$. Then $\langle \mathcal{L}, \supseteq \rangle$ is a completely deductive orthomodular lattice. Then $\mathrm{L}b = 1 \supseteq b = \bigvee\{c \in \mathrm{C}(\mathcal{L}) \mid c \wedge 1 \leqslant b\} = \bigvee\{c \in \mathrm{C}(\mathcal{L}) \mid c \leqslant b\}$ which is the largest central element under b, the "central kernel of b". Similarly, $\mathrm{M}b = (\mathrm{L}b')' = \left(\bigvee\{c \in \mathrm{C}(\mathcal{L}) \mid c \leqslant b'\}\right)' = \bigwedge\{c' \in \mathrm{C}(\mathcal{L}) \mid c \leqslant b'\} = \bigwedge\{c' \in \mathrm{C}(\mathcal{L}) \mid b \leqslant c'\} = \bigwedge\{c \in \mathrm{C}(\mathcal{L}) \mid b \leqslant c\}$, the central cover of b. This makes contact with a well studied object in the dimension theory of orthomodular lattices. One need only consult the work of Loomis, Maeda, and Janowitz to see the importance of the central cover, *cf.* [25, 27, 16].

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