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A GENERALISED PROPOSITIONAL CALCULUS

PETER JABLON

1 *Introduction* Any propositional variable ambiguously denotes a statement, but it is also the case that any pair of *different* propositional variables are orthodoxly taken always to ambiguously denote statements *completely independently of each other*. In this paper we examine some of the consequences of suppressing the convention that they do so completely independently of each other. Of these, perhaps the most interesting is the undecidability of the two-valued propositional calculus which results when the propositional variables of the classical two-valued propositional calculus are replaced by a more general kind of variable which, although more general, still ranges over statements.

2 *Independence of variables* In this paper, the term "propositional variable" shall refer to any variable that ranges over statements and satisfies the following conditions. (1) To each pair of propositional variables there corresponds a (unique) mutual truth-table. This truth-table has 1, 2, 3 or 4 rows and 2 columns. The rows are all distinct. Each entry is a 1 or a 0, where 1 represents truth and 0 represents falsity. (2) Given any 2 propositional variables, consider the pairs of statements whose first member is one of the statements which the first propositional variable ambiguously denotes and whose second member is one of the statements which the second propositional variable ambiguously denotes at the same time. For each such pair of statements, the first statement must have a truth-value which occurs in the first column of the mutual truth-table for the 2 variables, and the second statement must have a truth-value which occurs in the second column and in a row containing, in its first column, the truth-value of the first statement.

The variables which are here referred to as "propositional variables" include all the propositional variables of the classical propositional calculus. In fact, the latter form a proper subset—that of those variables any pair of which always has a mutual truth-table made up of 4 rows. The following definition is therefore meaningful.

Definition Two propositional variables will be said to be *independent* if, and only if, their mutual truth-table has 4 rows.

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We shall refer to pairs of propositional variables which are not independent of each other as *dependent* (on each other).

Thus if p and q are propositional variables which are dependent on each other, there will be pairs of statements which the pair of variables consisting of p and q ambiguously denotes but which will be such that the connection between their truth-values is an intensional and not (orthodoxly) a formal one.

Example If p and q have the truth-table

p	q
1	1
0	1
0	0

then the pair of statements "Joan takes an aspirin" and "Joan has a drink of water" would (respectively) be one pair of statements which p and q together simultaneously denote *if* it should happen that in fact Joan always takes aspirins with water (though not otherwise).

Some elementary properties of the (binary) dependence relation include the following:

(a) it is symmetric;
(b) it is not transitive;
(c) p is dependent on p, as well as on $\sim p$ and on $q \vee \sim q$;
(d) the independence relation is not transitive.

Although each pair of propositional variables has a determinate truth-table associated with it, it shall be completely ambiguous (as far as the notation for propositional variables is concerned) *which* truth-table their mutual truth-table may be, and therefore whether or not they are independent of each other.

3 *A generalized propositional calculus* When the classical propositional variables of the propositional calculus are replaced by the "propositional variables" of this paper, the number of tautologies is greatly increased. For example, if p and q have the truth-table

p	q
1	1
0	1
0	0

then $p \supset q$ is a tautology. We shall be using the convention that it shall be notationally ambiguous which pairs of propositional variables are independent of each other and which are not.

Example There exists at least one pair of propositional variables which yield a tautology when substituted for X and Y in $X \supset Y$, but by convention it will not be possible to specify which would be a pair of this sort and which would not, as not all formulas of the form $X \supset Y$ will be tautologies. On the other hand, however, all formulas of the form $X \vee (Y \vee \sim Y)$, for instance, will be tautologies.

Consider, in the interest of precision, any one of the deductively equivalent systems commonly referred to as the classical propositional calculus, and call it **CPC**. Let **GPC** ("generalized propositional calculus") be the system which is derived from **CPC** by suppressing the convention of uniform substitutivity of letters representing propositional variables, and which has, in addition, *all* propositional variables (an uncountable number).

Theorem 1 *Let* ϕ *map the theory* **GPC** *into* $\{0, 1\}$ *and be defined as follows:*
If X *is a tautology then* $\phi X = 1$;
if X *is not a tautology then* $\phi X = 0$.
Then **GPC** *is incomplete relative to the interpretation* ϕ.

Proof: Consider the propositional variable which has a truth-table—relative to any other propositional variable—that contains solely 1's in its column—that is, which is itself a tautology. Then its image under ϕ is 1, but it is unprovable in **GPC** since the wff p is unprovable in **CPC**.

Theorem 2 **GPC** *is undecidable.*

Proof: It suffices to show that the cardinality of the truth-functional relations between pairs of propositional variables is uncountable, since it then follows that it is impossible to finitely axiomatize **GPC**. But this cardinality is uncountable, as can be demonstrated in the following way. Given any set C of propositional variables, there always exists a propositional variable p' not in C which is independent of every propositional variable in C that is independent of at least one element of C. Thus, letting $C_1 = \{\ \}$, take $C = C_1$, and then let $p_1' = p'$. Then take $C = C_2 =_{Df} C_1 \cup \{p_1'\}$, and let $p_2' = p'$ for $C = C_2$; and in general, take $C = C_{i+1} =_{Df} C_i \cup \{p_i'\}$, for $i = 1, 2, 3, \ldots$. Every element of the resulting set $\{p_i'\}$ is independent of every other, and therefore the totality of the truth-functional relations between pairs of elements of the set $\{p_i'\}$ is uncountable. Hence the theorem.

Australian National University
Canberra, Australia