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A SIMPLE ALGEBRA OF FIRST ORDER LOGIC

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1. *Introduction*[1] The idea of making algebra out of logic is not a new one. In the middle of the last century George Boole investigated a class of algebras, subsequently named Boolean algebras, which arose naturally as a way of algebraizing the propositional calculus. More recently there have appeared several algebraizations of the first-order predicate calculus, of which the most important are the polyadic algebras of Halmos [3], and the cylindric algebras of Tarski [5]. Each of these two approaches to algebraic logic has its relative merits, and presents conceptual difficulties which have proved to be a stumbling block for many an interested reader.

The purpose of this paper is to present a formulation of algebraic logic which is closely related to both polyadic and cylindric algebras and is, in a sense, intermediate between the two. The advantage of the system we are about to present is that it is based upon a small number of axioms which are extremely simple and well motivated. From a didactic point of view, this may be the most satisfactory way of introducing the student and non-specialist to the ideas and methods of algebraic logic. We will show precisely how our algebra is related to cylindric and polyadic algebras.

2. *Quantifier algebras* In this section we introduce a class of algebras to be called *quantifier algebras*,[2] which may be viewed as an algebraization of the first-order predicate calculus without equality. We begin by examining a special class of quantifier algebras, called quantifier algebras of formulas. The construction of these algebras has a metalogical character and extends the well-known method for constructing Boolean algebras from the propositional calculus.

Let Λ be a first-order language with a sequence $\langle v_\kappa \rangle_{\kappa<\alpha}$ of variables, and let θ be a theory of Λ. We let $\mathsf{Fm}^{(\Lambda)}$ designate the set of all the formulas of Λ, and $\mathsf{Fm}^{(\Lambda)}/\equiv_\theta$, the preceding set modulo the relation $F \equiv_\theta G$

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2. The term *quantifier algebra* has been used by several authors in different senses, all differing from the present one.

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iff $\theta \vdash F \Longleftrightarrow G$. We define Boolean operations on $\mathsf{Fm}^{(\Lambda)}/\equiv_\theta$ by: $(F/\equiv) + (G/\equiv) = F \vee G/\equiv$, $(F/\equiv)\cdot(G/\equiv) = F \wedge G/\equiv$, and $-(F/\equiv) = \neg F/\equiv$; 1 designates the class of all the theorems of θ, and 0 the class of all negations of theorems. It is known that $\langle \mathsf{Fm}^{(\Lambda)}/\equiv_\theta, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra. We define two more operations on $\mathsf{Fm}^{(\Lambda)}/\equiv_\theta$ as follows: $\mathsf{S}_\lambda^\kappa(F/\equiv)$ is the equivalence class of the formula which results from F by validly replacing each free occurrence of v_κ by v_λ; $\exists_\kappa(F/\equiv)$ is the class of the formula $(\exists v_\kappa)F$. Now, $\langle \mathsf{Fm}^{(\Lambda)}/\equiv_\theta, +, \cdot, -, 0, 1, \mathsf{S}_\lambda^\kappa, \exists_\kappa\rangle_{\kappa,\lambda<\alpha}$ is called the *quantifier algebra of formulas* associated with θ.

The foregoing discussion should help to motivate our general definition:

2.1 *Definition* By a quantifier algebra of degree α, briefly a $\mathbf{QA}_\alpha$, we mean a system $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{S}_\lambda^\kappa, \exists_\kappa\rangle_{\kappa,\lambda<\alpha}$ where $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra and $\mathsf{S}_\lambda^\kappa$ and $\exists_\kappa$ are unary operations having the following properties for all $x, y \in A$ and $\kappa, \lambda, \mu < \alpha$:

(q_1) $\mathsf{S}_\lambda^\kappa(-x) = -\mathsf{S}_\lambda^\kappa x$
(q_2) $\mathsf{S}_\lambda^\kappa(x + y) = \mathsf{S}_\lambda^\kappa x + \mathsf{S}_\lambda^\kappa y$
(q_3) $\mathsf{S}_\kappa^\kappa = \mathrm{id}$
(q_4) $\mathsf{S}_\lambda^\kappa \mathsf{S}_\kappa^\mu = \mathsf{S}_\lambda^\kappa \mathsf{S}_\lambda^\mu$
(q_5) $\exists_\kappa(x + y) = \exists_\kappa x + \exists_\kappa y$
(q_6) $x \leqslant \exists_\kappa x$
(q_7) $\mathsf{S}_\lambda^\kappa \exists_\kappa = \exists_\kappa$
(q_8) $\exists_\kappa \mathsf{S}_\lambda^\kappa = \mathsf{S}_\lambda^\kappa$ if $\kappa \neq \lambda$
(q_9) $\mathsf{S}_\lambda^\kappa \exists_\mu = \exists_\mu \mathsf{S}_\lambda^\kappa$ if $\mu \neq \kappa, \lambda$.

We assume throughout, that $\alpha \geqslant 2$.

The operations $\mathsf{S}_\lambda^\kappa$ are called *substitutions* and the operations $\exists_\kappa$ are called *quantifiers*. We observe that (q_1)-(q_4) are properties of substitutions, (q_5) and (q_6) are properties of quantifiers, and (q_7)-(q_9) are conditions which relate substitutions to quantifiers. The metalogical interpretation of these equations is obvious.

If $\mathfrak{A}$ is a quantifier algebra, as above, and $x \in A$, then the *dimension set* of x is the set

$$\Delta x = \{\kappa < \alpha : \exists_\kappa x \neq x\}.$$

In view of (q_7) and (q_8), Δx is also the set of all κ such that $\mathsf{S}_\lambda^\kappa x \neq x$, for any $\lambda \neq \kappa$. $\mathfrak{A}$ is said to be *locally finite* if it satisfies the condition

(Lf) *for every* $x \in A$, Δx *is a finite set.*

It is easy to show that every quantifier algebra of formulas is a quantifier algebra in the sense of Definition 2.1, and is, in fact, locally finite. Conversely, it is not hard to show that if $\mathfrak{A}$ is any locally finite quantifier algebra, there is a theory θ such that $\mathfrak{A}$ is (isomorphic with) the quantifier algebra of formulas associated with θ. Furthermore, adapting a result by Hoehnke [6], if θ_1 and θ_2 are first-order theories, then the quantifier algebras associated with θ_1 and θ_2, respectively, are isomorphic iff θ_1 and θ_2 are equivalent by definitions.

2.2 Lemma *If* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{S}^\kappa_\lambda, \exists_\kappa\rangle_{\kappa,\lambda<\alpha}$ *is a quantifier algebra, then the following statements hold for all* $x, y \in A$ *and all* $\kappa, \lambda, \mu < \alpha$:

(i) $\exists_\kappa 0 = 0$

(ii) $\exists_\kappa(x \cdot \exists_\kappa y) = \exists_\kappa x \cdot \exists_\kappa y$

(iii) $\exists_\kappa \exists_\lambda = \exists_\lambda \exists_\kappa$

(iv) $\mathsf{S}^\kappa_\lambda \mathsf{S}^\kappa_\mu = \mathsf{S}^\kappa_\mu$ *if* $\mu \neq \kappa$

(v) $\mathsf{S}^\kappa_\lambda \mathsf{S}^\mu_\nu = \mathsf{S}^\mu_\nu \mathsf{S}^\kappa_\lambda$ *if* $\kappa \neq \mu, \nu$ *and* $\mu \neq \lambda$

Proof: We derive, successively, the following statements:

(1) $x \leqslant y$ *implies* $\mathsf{S}^\kappa_\lambda x \leqslant \mathsf{S}^\kappa_\lambda y$.

(2) $x \leqslant y$ *implies* $\exists_\kappa x \leqslant \exists_\kappa y$.

(3) $\mathsf{S}^\kappa_\lambda x \leqslant \exists_\kappa x$.

(1) is an immediate consequence of (q_2); (2) is an immediate consequence of (q_5). Finally, (3) follows from (1), (q_6) and (q_7).

(4) $\exists_\kappa x$ *is the least element of* $\{y \in \text{range } \mathsf{S}^\kappa_\lambda : y \geqslant x\}$, *if* $\kappa \neq \lambda$.

Indeed, by (q_7), $\exists_\kappa x \in$ range $\mathsf{S}^\kappa_\lambda$ and by (q_6), $\exists_\kappa x \geqslant x$. Note that if $y \in$ range $\mathsf{S}^\kappa_\lambda$, then for some $z \in A$, $y = \mathsf{S}^\kappa_\lambda z = \exists_\kappa \mathsf{S}^\kappa_\lambda z = \exists_\kappa y$. Thus, if $y \in$ range $\mathsf{S}^\kappa_\lambda$ and $y \geqslant x$, then by (2), $\exists_\kappa x \leqslant \exists_\kappa y = y$.

It follows from (4) and Halmos ([2], Theorem 5) that $\exists_\kappa$ is a quantifier (in the sense of Halmos) for each $\kappa < \alpha$, hence we have (i) and (ii). Next, using (q_7)-(q_9) repeatedly, we have, for any $\mu \neq \kappa, \lambda$,

$$\exists_\lambda \exists_\kappa \exists_\lambda x = \exists_\lambda \exists_\kappa \mathsf{S}^\lambda_\mu \exists_\lambda x = \exists_\lambda \mathsf{S}^\lambda_\mu \exists_\kappa \exists_\lambda x = \mathsf{S}^\lambda_\mu \exists_\kappa \exists_\lambda x = \exists_\kappa \mathsf{S}^\lambda_\mu \exists_\lambda x = \exists_\kappa \exists_\lambda x.$$

Now by (q_6) and (2), $\exists_\lambda \exists_\kappa x \leqslant \exists_\lambda \exists_\kappa \exists_\lambda x = \exists_\kappa \exists_\lambda x$; symmetrically, $\exists_\kappa \exists_\lambda x \leqslant \exists_\lambda \exists_\kappa x$, giving (iii). (iv) follows from (q_7) and (q_8); for if $\mu \neq \kappa$, then $\mathsf{S}^\kappa_\lambda \mathsf{S}^\kappa_\mu = \mathsf{S}^\kappa_\lambda \exists_\kappa \mathsf{S}^\kappa_\mu = \exists_\kappa \mathsf{S}^\kappa_\mu = \mathsf{S}^\kappa_\mu$. If we let $\mu = \lambda$ in (iv), we get

(5) $\mathsf{S}^\kappa_\lambda \mathsf{S}^\kappa_\lambda = \mathsf{S}^\kappa_\lambda$.

It remains only to prove (v); first, we need the following:

(6) *Let* f *and* g *be Boolean endomorphisms such that* $ff = f$ *and* $gg = g$; *if range* f *and kernel* f *are both closed under* g, *then* $fg = gf$.

From the hypotheses $gf(x) \in$ range f and $ff = f$, we conclude that $fgf(x) = gf(x)$. Now, $gf(x) \oplus fg(x) = fgf(x) \oplus fg(x) = fg(f(x) \oplus x)$; ($\oplus$ is the operation of symmetric difference). But clearly, $f(x) \oplus x \in$ kernel f, hence by hypothesis, $g(f(x) \oplus x) \in$ kernel f, and therefore $fg(f(x) \oplus x) = 0$. Thus, $gf(x) \oplus fg(x) = 0$, which proves (6).

(7) *If* $\mu \neq \kappa, \lambda, \nu$, *then range* S^μ_ν *is closed under* $\mathsf{S}^\kappa_\lambda$.

Indeed, if $\mu \neq \kappa, \lambda, \nu$ then $\mathsf{S}^\kappa_\lambda \mathsf{S}^\mu_\nu x = \mathsf{S}^\kappa_\lambda \exists_\mu \mathsf{S}^\mu_\nu x = \exists_\mu \mathsf{S}^\kappa_\lambda \mathsf{S}^\mu_\nu x \in$ range S^μ_ν.

(8) *If* $\kappa \neq \mu, \nu$ *then kernel* S^μ_ν *is closed under* $\mathsf{S}^\kappa_\lambda$.

Indeed, suppose $\mathsf{S}^\mu_\nu x = 0$; then $\exists_\kappa \mathsf{S}^\mu_\nu x = \exists_\kappa 0 = 0$ by **2.2** (i). Thus, by (q_9), $\mathsf{S}^\mu_\nu \exists_\kappa x = 0$. But by (3), $\mathsf{S}^\kappa_\lambda x \leqslant \exists_\kappa x$, so by (1), $\mathsf{S}^\mu_\nu \mathsf{S}^\kappa_\lambda x \leqslant \mathsf{S}^\mu_\nu \exists_\kappa x = 0$. It follows that $\mathsf{S}^\kappa_\lambda x \in$ kernel S^μ_ν. From (5), (6), (7) and (8), we immediately get (v). Q.E.D.

(q_6), together with **2.2** (i) and (ii), show that the operations $\exists_\kappa$ are quantifiers in the sense of Halmos [2] and of Henkin, Monk and Tarski [5].

3. *Quantifier algebras and polyadic algebras* An extended notion of substitutions and quantifiers is used in polyadic algebras. Roughly speaking, (from the metalogical point of view), instead of merely quantifying over a single variable, one may quantify over an arbitrary set of variables; similarly, the simultaneous substitution of arbitrarily many variables is permitted.

In the definition which follows, I is taken to be an arbitrary set. I^I designates the set of all functions from I into I; if $\tau \in I^I$ then, in the metalogical interpretation, S_τ may be regarded as the operation of simultaneously replacing each variable v_κ by $v_{\tau(\kappa)}$. For each $\tau \in I^I$ and $J \subseteq I$, $\tau|J$ designates the restriction of τ to J.

3.1 Definition: An I-polyadic algebra is a system $\langle A, +, \cdot, -, 0, 1, \mathsf{S}_\tau, \exists_J\rangle_{\tau\in I^I, J\subseteq I}$ where $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra and S_τ and $\exists_J$ are unary operations which satisfy the following conditions for all $x, y \in A$, $\sigma, \tau \in I^I$ and $J, K \subseteq I$:

(p_1) $\mathsf{S}_\tau(x + y) = \mathsf{S}_\tau x + \mathsf{S}_\tau y$

(p_2) $\mathsf{S}_\tau(-x) = -\mathsf{S}_\tau x$

(p_3) $\mathsf{S}_\sigma \mathsf{S}_\tau = \mathsf{S}_{\sigma\tau}$

(p_4) $\mathsf{S}_{\mathrm{id}} = \mathrm{id}$

(p_5) $\exists_J 0 = 0$

(p_6) $x \leqslant \exists_J x$

(p_7) $\exists_J(x \cdot \exists_J y) = \exists_J x \cdot \exists_J y$

(p_8) $\exists_\emptyset = \mathrm{id}$

(p_9) $\exists_{J\cup K} = \exists_J \exists_K$

(p_{10}) $\mathsf{S}_\sigma \exists_J = \mathsf{S}_\tau \exists_J$ if $\sigma|I - J = \tau|I - J$

(p_{11}) $\exists_J \mathsf{S}_\tau = \mathsf{S}_\tau \exists_{\tau^{-1}(J)}$ if $\tau|\tau^{-1}(J)$ is injective.

It is customary to write $\exists_\kappa$ for $\exists_{\{\kappa\}}$. A mapping $\tau \in I^I$ such that $\tau(\kappa) = \lambda$ and $\tau(\mu) = \mu$ for every $\mu \neq \kappa$ is called a *replacement* and is noted by (κ/λ). We write $\mathsf{S}^\kappa_\lambda$ for $\mathsf{S}_{(\kappa/\lambda)}$.

The connections between quantifier algebras and polyadic algebras are easy to describe. Note that we may always identify I with some ordinal α. Now, every α-polyadic algebra is a quantifier algebra of degree α; more precisely, if $\langle A, +, \cdot, -, 0, 1, \mathsf{S}_\tau, \exists_J\rangle_{\tau\in\alpha^\alpha, J\subset\alpha}$ is a polyadic algebra, then $\langle A, +, \cdot, -, 0, 1, \mathsf{S}^\kappa_\lambda, \exists_\kappa\rangle_{\kappa,\lambda<\alpha}$ is a quantifier algebra. (Another way of saying this is: a polyadic algebra becomes a quantifier algebra by removing some of its operations.)

The converse is true for locally finite quantifier algebras of infinite degree. Indeed, if $\langle A, +, \cdot, -, 0, 1, \mathsf{S}^\kappa_\lambda, \exists_\kappa\rangle_{\kappa,\lambda<\alpha}$ is such an algebra, it is possible to adjoin extended quantifiers and substitutions as follows: if $x \in A$ and $J \subseteq \alpha$, we define $\exists_J x$ by

(3.2) $\exists_J x = \exists_{\kappa_1} \ldots \exists_{\kappa_n} x$, where $\{\kappa_1, \ldots, \kappa_n\} = J \cap \Delta x$.

Furthermore, it has been established in [3] that if $\tau \in \alpha^\alpha$ and J is a finite subset of α, then $\tau|J = \tau_1 \ldots \tau_n|J$ for some replacements $\tau_1, \ldots, \tau_n$. Now, we define $\mathsf{S}_\tau x$ by

(3.3) $\mathsf{S}_\tau x = \mathsf{S}^{\kappa_1}_{\lambda_1} \ldots \mathsf{S}^{\kappa_n}_{\lambda_n} x$, where $\tau|\Delta x = (\kappa_1/\lambda_1) \ldots (\kappa_n/\lambda_n)|\Delta x$.

B. Galler has proved in [1] that if $\alpha \geqslant \omega$ and $\langle A, +, \cdot, -, 0, 1, \mathsf{S}^{\kappa}_{\lambda}, \exists_{\kappa}\rangle_{\kappa,\lambda<\alpha}$ is an algebra satisfying (q_1)-(q_9) and (Lf), (together with **2.2** (i)-(v) which follow from (q_1)-(q_9)), and if operations $\exists_J$ and S_τ are introduced by (3.2) and (3.3) respectively, then $\langle A, +, \cdot, -, 0, 1, \mathsf{S}_\tau, \exists_J\rangle_{\tau\in\alpha^\alpha, J\subseteq\alpha}$ is a polyadic algebra. We may paraphrase this as follows: if we adjoin operations S_τ and $\exists_J$ to a locally finite quantifier algebra of infinite degree by means of (3.2) and (3.3), we get a polyadic algebra.

4. *Quantifier algebras with equality and cylindric algebras* By adjoining certain distinguished elements to quantifier algebras, we get an algebraization of the first-order predicate calculus with equality. In the metalogical interpretation, the distinguished element $\mathsf{e}_{\kappa\lambda}$ is taken to be the equivalence class of the formula $v_\kappa = v_\lambda$.

4.1 Definition: A quantifier algebra with equality is an algebra $\langle A, +, \cdot, -, 0, 1, \mathsf{S}^{\kappa}_{\lambda}, \exists_{\kappa}, \mathsf{e}_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ such that $\langle A, +, \cdot, -, 0, 1, \mathsf{S}^{\kappa}_{\lambda}, \exists_{\kappa}\rangle_{\kappa,\lambda<\alpha}$ is a quantifier algebra and $\mathsf{e}_{\kappa\lambda}$ are distinguished elements which satisfy

(q_{10}) $\mathsf{S}^{\kappa}_{\lambda}\mathsf{e}_{\kappa\lambda} = 1$

(q_{11}) $x\cdot\mathsf{e}_{\kappa\lambda} \leqslant \mathsf{S}^{\kappa}_{\lambda}x.$

We present, next, a few properties of quantifier algebras with equality. We let min X designate the least element of X if X is an ordered set having one.

4.2 Lemma: *If* $\langle A, +, \cdot, -, 0, 1, \mathsf{S}^{\kappa}_{\lambda}, \exists_{\kappa}, \mathsf{e}_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ *is a quantifier algebra with equality, the following conditions hold for all* $x\in A$ *and* $\kappa, \lambda, \mu < \alpha$:

(i) $x\cdot\mathsf{e}_{\kappa\lambda} = \mathsf{S}^{\kappa}_{\lambda}x\cdot\mathsf{e}_{\kappa\lambda}$

(ii) $\exists_\kappa\mathsf{e}_{\kappa\lambda} = 1$

(iii) $\mathsf{S}^{\kappa}_{\lambda}x = \exists_\kappa(x\cdot\mathsf{e}_{\kappa\lambda})$ if $\kappa\neq\lambda$

(iv) $\mathsf{e}_{\kappa\lambda} = \min\{x : \mathsf{S}^{\kappa}_{\lambda}x = 1\}$

(v) $\mathsf{e}_{\kappa\kappa} = 1$

(vi) $\mathsf{e}_{\kappa\lambda} = \mathsf{e}_{\lambda\kappa}$

(vii) $\mathsf{S}^{\kappa}_{\nu}\mathsf{e}_{\lambda\mu} = \mathsf{e}_{\lambda\mu}$ if $\kappa\neq\lambda, \mu$

(viii) $\mathsf{S}^{\kappa}_{\mu}\mathsf{e}_{\kappa\lambda} = \mathsf{e}_{\mu\lambda}$ if $\kappa\neq\lambda, \mu$

(ix) $\exists_\kappa(\mathsf{e}_{\lambda\kappa}\cdot\mathsf{e}_{\kappa\mu}) = \mathsf{e}_{\lambda\mu}$ if $\kappa\neq\lambda, \mu$.

Proof: The proof of (i) is due to Halmos [4]: by (q_{11}), $\mathsf{S}^{\kappa}_{\lambda}x\cdot\mathsf{e}_{\kappa\lambda}\cdot -x \leqslant \mathsf{S}^{\kappa}_{\lambda}x\cdot\mathsf{S}^{\kappa}_{\lambda}(-x) = \mathsf{S}^{\kappa}_{\lambda}(x\cdot -x) = 0$. Thus, $\mathsf{S}^{\kappa}_{\lambda}x\cdot\mathsf{e}_{\kappa\lambda}\leqslant x$; combining this with (q_{11}) yields (i). We have seen, in (3) of the proof of **2.2**, that $\mathsf{S}^{\kappa}_{\lambda}x\leqslant\exists_\kappa x$; from this fact and (q_{10}) we immediately get (ii).

$$\begin{aligned}\text{Next, } \exists_\kappa(x\cdot\mathsf{e}_{\kappa\lambda}) &= \exists_\kappa(\mathsf{S}^{\kappa}_{\lambda}x\cdot\mathsf{e}_{\kappa\lambda}) && \text{by (i)}\\ &= \exists_\kappa(\exists_\kappa\mathsf{S}^{\kappa}_{\lambda}x\cdot\mathsf{e}_{\kappa\lambda}) && \text{by } (q_8)\\ &= \mathsf{S}^{\kappa}_{\lambda}x\cdot\exists_\kappa\mathsf{e}_{\kappa\lambda} && \text{by \textbf{2.2} (ii) and } (q_8)\\ &= \mathsf{S}^{\kappa}_{\lambda}x && \text{by (ii).}\end{aligned}$$

This proves (iii); (iv) follows immediately from (q_{10}) and (i), and (v) is an immediate consequence of (iv). Now, by (q_3) and (q_4), $\mathsf{S}^{\kappa}_{\lambda}\mathsf{S}^{\lambda}_{\kappa} = \mathsf{S}^{\kappa}_{\lambda}$; thus, $\mathsf{S}^{\kappa}_{\lambda}\mathsf{e}_{\lambda\kappa} = \mathsf{S}^{\kappa}_{\lambda}\mathsf{S}^{\lambda}_{\kappa}\mathsf{e}_{\lambda\kappa} = 1$, so by (iv), $\mathsf{e}_{\kappa\lambda}\leqslant\mathsf{e}_{\lambda\kappa}$; symmetrically, $\mathsf{e}_{\lambda\kappa}\leqslant\mathsf{e}_{\kappa\lambda}$, which proves (vi). To prove (vii), we note that by (q_9), $\mathsf{S}^{\lambda}_{\mu}\exists_\kappa(-\mathsf{e}_{\lambda\mu}) = \exists_\kappa\mathsf{S}^{\lambda}_{\mu}(-\mathsf{e}_{\lambda\mu}) = 0$, that is, $\mathsf{S}^{\lambda}_{\mu}[-\exists_\kappa(-\mathsf{e}_{\lambda\mu})] = 1$; thus by (iv), $\mathsf{e}_{\lambda\mu}\leqslant -\exists_\kappa(-\mathsf{e}_{\lambda\mu})$, that is, $\exists_\kappa(-\mathsf{e}_{\lambda\mu})\leqslant -\mathsf{e}_{\lambda\mu}$. Combining this with (q_6) gives $\exists_\kappa(-\mathsf{e}_{\lambda\mu}) = -\mathsf{e}_{\lambda\mu}$; thus by (q_7), $-\mathsf{S}^{\kappa}_{\nu}\mathsf{e}_{\lambda\mu} = \mathsf{S}^{\kappa}_{\nu}\exists_\kappa(-\mathsf{e}_{\lambda\mu}) = \exists_\kappa(-\mathsf{e}_{\lambda\mu}) = -\mathsf{e}_{\lambda\mu}$, that is, $\mathsf{S}^{\kappa}_{\nu}\mathsf{e}_{\lambda\mu} = \mathsf{e}_{\lambda\mu}$.

To prove (viii), we note that by (q_4) and (q_{10}), $S^\lambda_\mu(S^\kappa_\mu e_{\kappa\lambda}) = S^\lambda_\mu S^\kappa_\lambda e_{\kappa\lambda} = 1$, hence by (iv), $e_{\lambda\mu} \leqslant S^\kappa_\mu e_{\kappa\lambda}$; symmetrically, $e_{\kappa\lambda} \leqslant S^\mu_\kappa e_{\lambda\mu}$. Thus, $S^\kappa_\mu e_{\kappa\lambda} \leqslant S^\kappa_\mu S^\mu_\kappa e_{\lambda\mu} = S^\kappa_\mu e_{\lambda\mu} = e_{\lambda\mu}$, where the last step follows by (vii). Finally, (ix) is an immediate consequence of (iii) and (viii). Q.E.D.

4.3 Definition: By a cylindric algebra of degree α we mean a system $\langle A, +, \cdot, -, 0, 1, \exists_\kappa, e_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ such that $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra, and $\exists_\kappa$ and $e_{\kappa\lambda}$ satisfy the following conditions for all $x \in A$ and $\kappa, \lambda < \alpha$:

(C_1) $\exists_\kappa 0 = 0$

(C_2) $x \leqslant \exists_\kappa x$

(C_3) $\exists_\kappa(x \cdot \exists_\kappa y) = \exists_\kappa x \cdot \exists_\kappa y$

(C_4) $\exists_\kappa \exists_\lambda = \exists_\lambda \exists_\kappa$

(C_5) $e_{\kappa\kappa} = 1$

(C_6) $e_{\lambda\mu} = \exists_\kappa(e_{\lambda\kappa} \cdot e_{\kappa\mu})$ if $\kappa \neq \lambda, \mu$

(C_7) $\exists_\kappa(e_{\kappa\lambda} \cdot x) \cdot \exists_\kappa(e_{\kappa\lambda} \cdot -x) = 0$ if $\kappa \neq \lambda$.

Cylindric algebras are equivalent to quantifier algebras with equality. Let us state this result precisely: if $\langle A, +, \cdot, -, 0, 1, \exists_\kappa, e_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ is a cylindric algebra, and if operations S^κ_λ are defined by

(4.4) $S^\kappa_\lambda x = x$ *if* $\kappa = \lambda$; $S^\kappa_\lambda x = \exists_\kappa(x \cdot e_{\kappa\lambda})$ *if* $\kappa \neq \lambda$,

then $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda, \exists_\kappa, e_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ is a quantifier algebra with equality. Indeed, (q_1)-(q_{11}) are all theorems of the theory of cylindric algebras; their proofs may be found in Chapter 1 of [5]. Conversely, if $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda, \exists_\kappa, e_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ is a quantifier algebra with equality, then $\langle A, +, \cdot, -, 0, 1, \exists_\kappa, e_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ is a cylindric algebra. Indeed, this clearly follows from Lemma **4.2**.

REFERENCES

[1] Galler, B., "Cylindric and polyadic algebras," *Proceedings of the American Mathematical Society*, vol. 8 (1957), pp. 176-183.

[2] Halmos, P., "Algebraic logic, I. Monadic Boolean algebras," *Compositio Mathematica*, vol. 12 (1956), pp. 217-249.

[3] Halmos, P., "Algebraic logic, II. Homogenous locally finite polyadic Boolean algebras of infinite degree," *Fundamenta Mathematicae*, vol. 43 (1956), pp. 255-325.

[4] Halmos, P., "Algebraic logic, IV. Equality in polyadic algebras," *Transactions of the American Mathematical Society*, vol. 86 (1957), pp. 1-27.

[5] Henkin, L., D. Monk, and A. Tarski, *Cylindric Algebras*, North-Holland, Amsterdam (1971).

[6] Hoehnke, H. J., "Zur Strukturgleichheit axiomatischer Klassen," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 12 (1966), pp. 69-83.

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