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RADO'S THEOREM AND SOLVABILITY OF SYSTEMS OF EQUATIONS

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In this paper we consider finite or infinite systems of equations each in finitely many unknowns where each unknown ranges over a finite domain. We prove that such a system has a solution if and only if every finite subsystem has a solution. Moreover, we introduce the notion of an expanding system of equations and its partial solution and we give a necessary and sufficient condition for the existence of a partial solution of such a system of equations. Furthermore, we prove that Rado's theorem [1] is equivalent to the statement that if each equation of an expanding system of equations has a solution then the system has a partial solution.

In what follows we consider infinitely many (not necessarily denumerably many) unknowns (variables) $x_1, x_2, \ldots, x_j, \ldots$ ranging respectively over nonempty finite domains $D_1, D_2, \ldots, D_j, \ldots$. Moreover, by a function we mean a function of finitely many unknowns (variables). Hence, a function in the unknowns $x_i, \ldots, x_k$ is a mapping from $D_i \times \ldots \times D_k$. We do not impose any restriction (except for being nonempty) on the range of a function since that is not needed for our purpose.

From a given function we construct equations in the usual way. Thus, if

(1) $F_i(\ldots, x_j, \ldots)$

is a function then the configuration

(2) $F_i(\ldots, x_j, \ldots) = c_i$

is an equation, where c_i is an element of the range of the function given in (1). The notion of a solution of an equation as well as that of a system of equations is self-explanatory.

In the sequel, we let V denote a nonempty index set for the unknowns and we consider equations indexed by a nonempty set E. Although we make no restrictions (except for being nonzero) on the cardinalities of sets V and E, we would like to emphasize that each equation has finitely many unknowns and each unknown ranges over a nonempty finite domain.

Motivated by notation (2), we prove the following theorem.

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Theorem 1. *Consider a (finite or infinite) system of equations*

(3) $F_i(\ldots, x_j, \ldots) = c_i$ *with* $i \in E$

where each equation has finitely many unknowns and each unknown ranges over a nonempty finite domain. Let every finite subsystem of system (3) *have a solution. Then system* (3) *has a solution.*

Proof. Let us call a nonempty family $(x_j = s_j)_{j \in A}$ with $s_j \in D_j$ an *assignment* of system (3) if and only if upon replacing x_j by s_j (for every $j \in A$) in (3), the resulting system is such that every finite subsystem of it has a solution.

We first show that the set S of all the assignments of system (3) is nonempty. Let us consider an unknown x_u where without loss of generality we may take $D_u = \{p, q, r\}$. Assume on the contrary that the set S of all the assignments of system (3) is empty. Thus, upon replacing x_u by p or by q or by r, there correspond resulting finite subsystems $(F_i = c_i)_{i \in P}$ and $(F_i = c_i)_{i \in Q}$ and $(F_i = c_i)_{i \in R}$ such that each has no solution. But then the finite subsystem of (3) given by

$$(F_i(\ldots, x_j, \ldots) = c_i)_{i \in (P \cup Q \cup R)}$$

has no solution which contradicts the hypothesis of the theorem. Thus, our assumption is false and S is nonempty. Next, we observe that if we partial order S in an obvious manner we see that (in view of the fact that each equation has finitely many unknowns) every simply ordered subset of S has a least upper bound and therefore by Zorn's lemma, system (3) has a maximal assignment $(x_j = s_j)_{j \in M}$. Clearly, to complete the proof of the theorem, i.e., to show that system (3) has a solution, it is sufficient to prove that $M = V$ where, as mentioned above, V is the index set for the unknowns appearing in system (3). Assume on the contrary that $V \neq M$. Thus there exists an index u such that $u \in V$ and $u \notin M$. But then since $(x_j = s_j)_{j \in M}$ is an assignment of system (3) we see that upon replacing x_j by s_j (for every $j \in M$) in (3), the resulting system is such that every finite subsystem of it has a solution. Consequently (as in the earlier part of the proof) it follows that the resulting system under consideration has an assignment. However, since x_u is an unknown which does not appear in the resulting system under consideration, we see that $x_u = s_u$ for some $s_u \in D_u$ is an assignment of the resulting system under consideration. But then $(x_j = s_j)_{j \in (M \cup \{u\})}$ is an assignment of system (3) which contradicts the maximality of $(x_j = s_j)_{j \in M}$. Hence our assumption is false and system (3) has a solution, as desired.

Corollary 1. *Consider a (finite or infinite) system of equations*

$$F_i(\ldots, x_j, \ldots) = c_i \text{ with } i \in E$$

where each equation has finitely many unknowns and each unknown ranges over a nonempty finite domain. Then the system has a solution if and only if every finite subsystem has a solution.

Proof. Clearly, if the system has a solution then every finite subsystem has a solution. The converse follows from Theorem 1.

Let us observe that Theorem 1 does not imply the solvability of system (3) in cases where each individual equation of system (3) has a solution. In fact, it is easy to show that, in general, the solvability of each individual equation of a system of equations does not necessarily imply the solvability of the entire system. However, we show below that the solvability of each individual equation of an *expanding system* of equations implies the existence of a *partial solution* of the system.

As before, let V be the index set for the unknowns. A nonempty system of equations

(4) $G_i(\ldots, x_j, \ldots) = c_i$ *with* $i \in H$

is called *expanding* if and only if for every nonempty finite subset F of V there exists an equation of system (4) such that F is a subset of the set of all the indices of the unknowns which appear in that equation. In other words, given any finite number of unknowns $x_i, \ldots, x_k$, there exists an equation of system (4) in the unknowns $x_i, \ldots, x_k$ and possibly in some additional unknowns.

We call a family

(5) $(x = d_j)_{j \in V}$ *with* $d_j \in D_j$

a *partial solution* of system (4) if and only if for every nonempty finite subfamily $(x_j = d_j)_{j \in F}$ there exists an equation (with L a finite subset of V)

(6) $G_i(\ldots, x_j, \ldots) = c_i$ *with* $j \in (F \cup L)$

of system (4) such that

(7) $(x_j = d_j)_{j \in F}$ *and* $(x_j = h_j)_{j \in L}$

is a solution of equation (6) with $d_j \in D_j$ and $h_j \in D_j$. In other words, (5) is a partial solution of system (4) if and only if for every finite family of equalities $x_i = d_i, \ldots, x_k = d_k$ there exists an equation (6) of system (4) such that (6) is in the unknowns $x_i, \ldots, x_k$ and possibly in some additional unknowns $x_v, \ldots, x_w$ and such that $x_i = d_i, \ldots, x_k = d_k$ is a part of a complete solution of (6). We call $(x_j = d_j)_{j \in F}$ a *partial solution of equation* (6).

Motivated by the above, we prove the following theorem.

Theorem 2. *Consider a (finite or infinite) expanding system of equations*

(4) $G_i(\ldots, x_j, \ldots) = c_i$ *with* $i \in H$

where each equation has finitely many unknowns and each unknown ranges over a nonempty finite domain. For every equation of system (4) *let there be an equation of system* (4), *possibly in more unknowns, which has a solution. Then system* (4) *has a partial solution.*

Proof. Let us call a nonempty family $(x_j = d_j)_{j \in P}$ with $d_j \in D_j$ a *partial assignment* of system (4) if and only if for every nonempty finite subset P'

of P and every finite subset V' of V there exists an equation (with L' a finite subset of V)

(8) $G_i(\ldots, x_j, \ldots) = c_i$ *with* $j \in (P' \cup V' \cup L')$

of system (4) such that

(9) $(x_j = d_j)_{j \in P'}$ *and* $(x_j = e_j)_{j \in V'}$ *and* $(x_j = h_j)_{j \in L'}$

is a solution of equation (8) with $e_j \in D_j$ and $h_j \in D_j$.

We first show that the set D of all the partial assignments of system (4) is nonempty. Let us consider an unknown x_u where again, without loss of generality, we may take $D_j = \{p, q, r\}$. Assume on the contrary that the set D of all the partial assignments of system (4) is empty. Thus, in view of (8) and (9) and corresponding to

(10) $(x_j = p)_{j \in \{u\}}$ *and* $(x_j = q)_{j \in \{u\}}$ *and* $(x_j = r)_{j \in \{u\}}$

and finite subsets V_p' and V_q' and V_r' of V there does not exist an equation (with L'' a finite subset of V)

(11) $G_i(\ldots, x_j, \ldots) = c_i$ *with* $j \in (\{u\} \cup V_p' \cup V_q' \cup V_r' \cup L'')$

such that none of the equalities listed in (10) is a partial solution of equation (11). But this contradicts the hypothesis of the theorem stating that system (4) is expanding and that for every equation of system (4) there exists an equation of system (4) possibly in more unknowns which has a solution. Thus, our assumption is false and D is nonempty. Again, if we partial order D in an obvious manner, we see that (since each equation has finitely many unknowns) every simply ordered subset of D has a least upper bound and therefore by Zorn's lemma, system (4) has a maximal set $(x_j = d_j)_{j \in M}$ of partial assignments. However, by a reasoning analogous to the above which showed that D is nonempty, it is easily seen that $M = V$. But then $(x_j = d_j)_{j \in M}$ is a partial solution of system (4), as desired.

Corollary 2. *Consider a (finite or infinite) expanding system of equations*

(4) $G_i(\ldots, x_j, \ldots) = c_i$ *with* $i \in H$

where each equation has finitely many unknowns and each unknown ranges over a nonempty finite domain. Then system (4) *has a partial solution if and only if for every equation of system* (4) *there is an equation of system* (4) *possibly in more unknowns which has a solution.*

Proof. Let system (4) have a partial solution given by (5). Let $G_i(\ldots, x_j, \ldots) = c_i$ with $j \in F$ be an equation of system (4). But then there exists an equation of system (4) possibly in more unknowns, say, such as the one given by (6) which has a solution as indicated by (7). The converse follows from Theorem 2.

Recalling that by an equation we mean an equation in finitely many unknowns each of which ranges over a nonempty finite domain, we prove the following corollary.

Corollary 3. *If every equation of an expanding system of equations has a solution then the system has a partial solution.*

Proof. Clearly, if every equation of the system has a solution then for every equation of the system there is an equation of the system (namely the equation under consideration) possibly in more unknowns which has a solution. But then the conclusion of Corollary 3 follows from Theorem 2.

A much weaker version of Corollary 2 is given by the following corollary.

Corollary 4. *Let every two distinct equations of an expanding system of equations have at least one noncommon unknown. Moreover let every equation of the system have an unique solution. Then the system has a partial solution.*

We prove below that the following theorem of Rado [1] is a direct consequence of Corollary 4.

Theorem 3 (Rado). *Let $(D_j)_{j\in V}$ be a nonempty family of nonempty finite sets D_j. Let for every nonempty finite subset F of V a choice function $(d_{j(F)})_{j\in F}$ of $(D_j)_{j\in F}$ be given where $d_{j(F)}\in D_j$. Then there exists a choice function $(d_j)_{j\in V}$ of $(D_j)_{j\in V}$ where $d_j\in D_j$ such that for every nonempty finite subset P of V there exists a finite subset F of V with*

(12) $P \subset F$ *and* $(d_j)_{j\in P} \equiv (d_{j(F)})_{j\in F}$

Proof. With every nonempty finite subset F of V we associate an unique equation $G_F(\ldots, x_j, \ldots) = c_F$ with $j \in F$ whose unique solution is given by $(x_j = d_{j(F)})_{j\in F}$. As mentioned earlier, we do not specify the range of the function $G_F(\ldots, x_j, \ldots)$ since it is not needed for our purpose. Clearly, the system of equations (where F is a nonempty finite subset of V)

(13) $G_F(\ldots, x_j, \ldots) = c_F$ *with* $F \subset V$

is an expanding system every two distinct equations of which have at least one noncommon unknown. Moreover, every equation of the system has an unique solution. But then by Corollary 4, system (13) has a partial solution $(d_j)_{j\in V}$, which in view of (5), (6) and (7) satisfies (12), as desired.

Corollary 5. *Theorem 3 is equivalent to Corollary 4.*

Proof. The proof of Theorem 3 shows that Corollary 4 implies Theorem 3. Thus, it suffices to show that Theorem 3 implies Corollary 4. Let V be the set of all the indices of the unknowns appearing in the expanding system of equations mentioned in Corollary 4. Consider the set of all finite subsets of V. Corresponding to a nonempty finite subset F of V if there is no equation of the system in the unknowns x_j with $j \in F$, we adjoin to the system an equation in the unknowns x_j with $j\in F$ and we assign an unique solution to it. But then clearly, with respect to the resulting system of equations, the hypothesis of Theorem 3 is satisfied and the choice function $(d_j)_{j\in V}$ mentioned in Theorem 3 is the required partial solution mentioned in Corollary 4.

We observe that Theorem 2 is proved based on the axiom of choice (specifically, in its Zorn's lemma form). We observe also that from Corollary 5 it follows that Theorem 2 implies Theorem 3.

Finally, without the use of the axiom of choice (however, with the use of the axiom of choice for finite sets [2] which states that there exists a choice-function for every family of nonempty finite sets) we prove the following theorem.

Theorem 4. *Rado's theorem is equivalent to Theorem* 2.

Proof. As mentioned above, Theorem 2 implies Rado's theorem (i.e., Theorem 3). Next, based on the axiom of choice for finite sets, we prove that Theorem 3 implies Theorem 2. Consider expanding system (4) mentioned in Theorem 2 and assume the hypothesis of Theorem 2. Let V be the set of all the indices of the unknowns appearing in system (4). Let F be a nonempty finite subset of V. If there exists an equation of system (4) in the unknowns x_j with $j \in F$ then clearly the set of all the solutions of all such equations is a finite set. From this finite set (by virtue of the axiom of choice for finite sets) we choose a solution $(x_j = d_{j(F)})_{j \in F}$. If there is no equation of system (4) in the unknowns x_j with $j \in F$, then, again (by virtue of the axiom of choice for finite sets) we consider $(x_j = d_{j(F)})_{j \in F}$ with $d_{j(F)} \in D_j$. Thus, for every nonempty finite subset F of V there corresponds a unique family $(x_j = d_{j(F)})_{j \in F}$ with $d_{j(F)} \in D_j$. But then by Theorem 2 there exists a family $(x_j = d_j)_{j \in F}$ with $d_j \in D_j$ such that for every nonempty finite subset P of V there exists a finite subset F of V such that $P \subset F$ and $(d_j)_{j \in P} \equiv (d_{j(F)})_{j \in F}$. Clearly, this implies that $(d_j)_{j \in V}$ is a partial solution of system (4), as desired.

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