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THE MODULAR LATTICOIDS

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In this note[1] an algebraic system, henceforth called a modular latticoid, which as far as I know was not yet mentioned in the literature, will be formalized and its essential properties will be discussed.

1 We can define the system mentioned above as follows:

Any algebraic system

$$\mathfrak{A} = \langle A, \cup, \cap \rangle$$

where $\cup$ *and* $\cap$ *are two binary operations defined on the carrier set A, is a modular latticoid, if it satisfies the following two mutually independent postulates:*

$A1$ $[abc]: a, b, c \in A .\supset. (a \cap b) \cup (a \cap c) = ((c \cap a) \cup b) \cap a$
$A2$ $[ab]: a, b \in A .\supset. a = (b \cup a) \cap a$

Remark I: We know that axioms $A1$ and $A2$ are mutually independent since tables $\mathfrak{M}1$ and $\mathfrak{M}2$, given in [4], p. 314, are such that $\mathfrak{M}1$ verifies $A2$ but falsifies $A1$, and $\mathfrak{M}2$ verifies $A1$ and falsifies $A2$.

Remark II: In [3] J. Ričan has proved that, if an algebraic system is based on the postulates $A1$ and

$C1$ $[abc]: a, b, c \in A .\supset. a = (c \cup (b \cup a)) \cap a$

then such a system is a modular lattice. Throughout this paper a modular lattice system based on Ričan's axiomatization will be called system $\mathfrak{B}$.

2 In [4], pp. 313 and 314, section **2**, it has been proved that the following formulas:

$B1$ $[ab]: a, b \in A .\supset. a \cap b = b \cap a$
$B2$ $[ab]: a, b \in A .\supset. a \cup b = b \cup a$

1. An acquaintance with [4] and [5] is presupposed. Throughout this paper the so-called closure axioms are assumed tacitly.

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$B3$ $[ab] : a, b \epsilon A \,.\supset.\, a = a \cap (a \cup b)$
$B4$ $[ab] : a, b \epsilon A \,.\supset.\, a = a \cup (a \cap b)$

are the consequences of the axioms *A1* and *A2*.[2] Thus, *cf.* [1], p. 23, system $\mathfrak{A}$ is a latticoid.

2.1 Obviously, in the field of latticoids the indepotent laws for $\cap$ and for $\cup$ are the consequences of *B3* and *B4*, exactly in the same way as in the field of lattice theory, *cf.* [1], p. 21. Moreover, it is well known that formulas *B1*-*B4* are mutually independent.

2.2 Now, it will be shown that some formulas which we shall need later are the consequences of the set $\{B1; B2; B3; B4\}$, i.e., that they are provable in the field of any latticoid. Namely:

$D1$ $[ab] : a \leqslant b \,.\equiv.\, a \epsilon A \,.\, b \epsilon A \,.\, a = a \cap b$[3] [Definition]
$B5$ $[ab] : a, b \epsilon A \,.\, a \leqslant b \,.\supset.\, a \cup b = b$
PR $[ab] : \mathrm{Hp}(2) \,.\supset.$
3. $a = a \cap b \,.$ [2; *D1*]
$a \cup b = b \cup a = b \cup (a \cap b) = b \cup (b \cap a) = b$ [1; *B2*; 3; *B1*; *B4*, a/b, b/a]
$B6$ $[ab] : a, b \epsilon A \,.\supset.\, a \cap b \leqslant a$
PR $[ab] : \mathrm{Hp}(1) \,.\supset.$
2. $a \cap b \epsilon A \,.$ [1; *cf.* footnote 1]
3. $a \cap b = (a \cap b) \cap ((a \cap b) \cup a) = (a \cap b) \cap (a \cup (a \cap b)) = (a \cap b) \cap a$
[1; *B3*, $a/a \cap b$, b/a; *B2*, $a/a \cap b$, b/a; *B4*]
$a \cap b \leqslant a$ [1; 2; 3; *D1*, $a/a \cap b$, b/a]
$B7$ $[ab] : a, b \epsilon A \,.\supset.\, a \leqslant a \cup b$
PR $[ab] : \mathrm{Hp}(1) \,.\supset.$
2. $a \cup b \epsilon A \,.$ [1; *cf.* footnote 1]
3. $a = a \cap (a \cup b) \,.$ [1; *B3*]
$a \leqslant a \cup b$ [1; 2; 3; *D1*, $b/a \cup b$]
$A2$ $[ab] : a, b \epsilon A \,.\supset.\, a = (b \cup a) \cap a$ [*B3*; *B1*, $b/b \cup a$; *B2*]

3 We know, *cf.*, e.g., [1], Example 3, that there are latticoids such that in their fields one of the associative laws holds, but obviously not both, i.e., either for $\cap$ or for $\cup$, and such systems are the proper subsystems of lattice theory. This is not the case in the field of system $\mathfrak{A}$, since in [4], p. 315, Remark II, I proved that an addition of one of the associative laws, i.e., either for $\cap$ or for $\cup$, as a new axiom to the postulates *A1* and *A2*, generates a modular lattice. On the other hand, I was unable to deduce the associative laws from the set $\{A1; A2\}$, and in [4], p. 312, I conjectured that system $\mathfrak{A}$ is not a modular lattice.

4 *This conjecture of mine was proved by* T. A. *Sudkamp. Namely, in* [5] *he* presented a finite algebraic table which verifies *A1* and *A2*, but falsifies *C1*

2. In [4], pp. 313 and 314, section **2.2**, the enumerations of *B1* - *B4* are *A22*, *A21*, *A19* and *A15* respectively.
3. It should be noted that in the fields of modular latticoids the relation $\leqslant$ is not transitive.

and both laws of associativity. Hence, due to this result of Sudkamp, we know that system $\mathfrak{A}$ is not a modular lattice. Therefore, *cf.* Remark II, in Ryčan's system $\mathfrak{B}$ its postulate *C1* cannot be substituted by *A2*. Hence, system $\mathfrak{A}$ is a proper subsystem of $\mathfrak{B}$.

5 It is well known that in the field of lattice theory the proper axiom of modular lattice

F1 $[abc]: a, b, c \in A . a \leqslant c .\supset. a \cup (b \cap c) = (a \cup b) \cap c$

is inferentially equivalent to each of the several other modular formulas, e.g., to *A1* and to each of the formulas

G1 $[abc]: a, b, c \in A .\supset. (a \cup b) \cap (a \cup c) = a \cup (b \cap (c \cup a))$

H1 $[abc]: a, b, c \in A .\supset. ((a \cap b) \cup c) \cap b = ((c \cap b) \cup a) \cap b$

K1 $[abc]: a, b, c \in A .\supset. a \cap (b \cup c) = a \cap ((b \cap (a \cup c)) \cup c)$

K2 $[a, b, c, d]: a, b, c, d \in A .\supset. ((a \cap b) \cap c) \cup (a \cap d) = ((d \cap a) \cup (c \cap b)) \cap a$[4]

In the field of latticoid $\{B1; B2; B3; B4\}$ these inferential equivalences do not hold for all formulas given above. Namely, all modular formulas, each of which is inferentially equivalent to *F1* in the field of lattice theory, can be divided into three classes, viz.: class (A) contains the formulas mentioned above such that in the field of $\{B1; B2; B3; B4\}$ each of them is inferentially equivalent to *F1*; class (B) contains these formulas such that, in the field of the postulate-system given above, *F1* implies each of them, but in the same field no member of this class implies *F1*; class (C) contains those formulas which are not the consequences of the system $\{B1; B2; B3; B4; F1\}$. Below, we shall show that each of these classes is not empty.

5.1 Let us assume that $\{B1; B2; B3; B4\}$ holds in all deductions presented in this subsection. We shall prove that in the assumed field each of the formulas *F1*, *A1*, and *G1* is inferentially equivalent to the other ones.

5.1.1 Assume *F1*. We have also *B6*, *cf.* section **2.2**. Then:

A1 $[abc]: a, b, c \in A .\supset. (a \cap b) \cup (a \cap c) = ((c \cap a) \cup b) \cap a$

PR $[abc]: \mathrm{Hp}(1) .\supset.$

2. $a \cap c \leqslant a .$ $[1; B6, b/c]$

$$(a \cap b) \cup (a \cap c) = (a \cap c) \cup (a \cap b) = (a \cap c) \cup (b \cap a) = ((a \cap c) \cup b) \cap a = ((c \cap a) \cup b) \cap a$$

$[1; B2, a/a \cap b, b/a \cap c; B1; F1, a/a \cap c; 2; B1, b/c]$

5.1.2 Assume *A1*. We have also *D1*, *cf.* section **2.2**. Then:

F1 $[abc]: a, b, c \in A . a \leqslant c .\supset. a \cup (b \cap c) = (a \cup b) \cap c$

PR $[abc]: \mathrm{Hp}(2) .\supset.$

3. $a = a \cap c .$ $[2; D1, b/c]$

4. Concerning the formulas *A1*, *G1*, *H1*, *K1* and *K2*, *cf.* [3], [1], pp. 31 and 39, and [2].

$$a \cup (b \cap c) = (a \cap c) \cup (b \cap c) = (c \cap a) \cup (c \cap b) = (c \cap b) \cup (c \cap a)$$
$$= ((a \cap c) \cup b) \cap c = (a \cup b) \cap c$$
[1; 3; *B1*, b/c; *B1*, a/b, b/c; *B2*, $a/c \cap a$, $b/c \cap b$; *A1*, a/c, c/a; 3]

5.1.3 Again assume *F1*. We have also *B7*, *cf.* section **2.2**. Then:

G1 $[abc] : a, b, c \epsilon A . \supset . (a \cup b) \cap (a \cup c) = a \cup (b \cap (c \cup a))$
PR $[abc]$: Hp 1 $. \supset .$
2. $a \leqslant a \cup c$. [1; *B7*, b/c]
$(a \cup b) \cap (a \cup c) = a \cup (b \cap (a \cup c)) = a \cup (b \cap (c \cup a))$
[1; 2; *F1*, $c/a \cup c$; *B2*, b/c]

5.1.4 Assume *G1*. We have also *B5*, *cf.* section **2.2**. Then:

F1 $[abc] : a, b, c \epsilon A . a \leqslant c . \supset . a \cup (b \cap c) = (a \cup b) \cap c$
PR $[abc]$: Hp(2) $. \supset .$
3. $a \cup c = c$. [1; 2; *B5*, b/c]
$a \cup (b \cap c) = a \cup (b \cap (a \cup c)) = a \cup (b \cap (c \cup a)) = (a \cup b) \cap (a \cup c)$
$= (a \cup b) \cap c$ [1; 3; *B2*, b/c; *G1*; 3]

5.1.5 It follows from the deductions given in sections **2.2** and **5.1.1-5.1.4** that class (A) is not empty and, moreover, that

$$\{A1; A2\} \rightleftarrows \{B1; B2; B3; B4; A1\} \rightleftarrows \{B1; B2; B3; B4; F1\} \rightleftarrows \{B1; B2; B3; B4; G1\}$$

The fact that class (A) is not empty suggests that the modular formulas which belong to this class can be called the latticoidal modular formulas.

5.2 Now, we shall prove that class (B) is not empty.

5.2.1 Assume system $\{B1; B2; B3; B4; A1\}$. Then:

H1 $[abc] : a, b, c \epsilon A . \supset . ((a \cap b) \cup c) \cap b = ((c \cap b) \cup a) \cap b$
PR $[abc]$: Hp(1) $. \supset .$
$((a \cap b) \cup c) \cap b = (b \cap c) \cup (b \cap a) = (b \cap a) \cup (b \cap c)$
$= ((c \cap b) \cup a) \cap b$
[1; *A1*, a/b, b/c, c/a; *B2*, $a/b \cap c$, $b/b \cap a$; *A1*, a/b, b/a]

Thus, *H1* is provable in the field of system $\mathfrak{A}$.

5.2.2 On the other hand, the following algebraic table[5]

$\mathfrak{M}$I

$\cup$	α	β	γ	δ	η
α	α	β	γ	δ	η
β	β	β	η	δ	η
γ	γ	η	γ	δ	η
δ	δ	δ	δ	δ	η
η	η	η	η	η	η

$\cap$	α	β	γ	δ	η
α	α	α	α	α	α
β	α	β	α	β	β
γ	α	α	γ	γ	γ
δ	α	β	γ	δ	δ
η	α	β	γ	δ	η

5. It is self-evident that this table is isomorphic with the dual of the diagram given in [1], p. 22, Figure 5.

verifies *B1, B2, B3, B4,* and *H1*, but falsifies, e.g., *A1* for a/δ, b/β, and c/γ: (i) $(\delta \cap \beta) \cup (\delta \cap \gamma) = \beta \cup \gamma = \eta$, (ii) $((\gamma \cap \delta) \cup \beta) \cap \delta = (\gamma \cup \beta) \cap \delta = \eta \cap \delta = \delta$.

5.2.3 Hence, class (B) is not empty. In particular, with respect to formula *H1*, we obtain an even stonger result. Namely, $\mathfrak{M}1$ verifies also

$$N1 \quad [abc] : a, b, c \in A \;.\supset.\; a \cap (b \cap c) = (a \cap b) \cap c$$

but falsifies

$$M1 \quad [abc] : a, b, c \in A \;.\supset.\; a \cup (b \cup c) = (a \cup b) \cup c$$

for a/β, b/γ, and c/δ: (i) $\beta \cup (\gamma \cup \delta) = \beta \cup \delta = \delta$, (ii) $(\beta \cup \gamma) \cup \delta = \eta \cup \delta = \eta$. From this follows that a latticoid with the additional postulates *N1* and *H1* is not a modular lattice, *cf.* [4], p. 315.

5.3 In order to prove that class (C) is not empty we use Sudkamp's table $\mathfrak{M}1$, *cf.* [5]. $\mathfrak{M}1$, as we know, verifies *A1* and *A2*, but it falsifies *K1* for a/α, b/δ, and c/β: (i) $\alpha \cap (\delta \cup \beta) = \alpha \cap \delta = 0$, (ii) $\alpha \cap ((\delta \cap (\alpha \cup \beta)) \cup \beta) = \alpha \cap ((\delta \cap \beta) \cup \beta) = \alpha \cap (\delta \cup \beta) = \alpha \cap \beta = \alpha$, and it falsifies *K2* for a/α, b/β, c/γ, and $d/0$: (i) $((\alpha \cap \beta) \cap \gamma) \cup (\alpha \cap 0) = (\alpha \cap \gamma) \cup 0 = \alpha \cap \gamma = 0$, (ii) $((0 \cap \alpha) \cup (\gamma \cap \beta)) \cap \alpha = (0 \cup \beta) \cap \alpha = \beta \cap \alpha = \alpha$.

Hence, class (C) is not empty.

Remark III: In this note I shall not discuss the metaalgebraic properties of the modular latticoids. It is rather self-evident that some metatheorems concerning the modular lattices hold for the modular latticoids, but several such theorems which are valid for the former systems obviously fail for the latter ones.

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