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ON THE INDEPENDENCE OF THE FUNDAMENTAL OPERATIONS OF THE ALGEBRA OF SPECIES

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The aim of this article is to prove the results announced in [2]. That is, that the fundamental operations of the algebra of species are independent, in the sense that none of the four operations is definable in terms of the others. The fundamental operations of the algebra of species are: the species implication "$\Rightarrow$", the species union "$\cup$", the species intersection "$\cap$", and the species complement "$-$". For details see [6]. A species algebraic operation, say "$\Rightarrow$", is definable in terms of the other operations, if given any term T of the algebra of species which contains "$\Rightarrow$" and does not contain any of the other three operations, there exists a term T^* which does not contain "$\Rightarrow$", such that the formula $(T \Rightarrow T^* \cap T^* \Rightarrow T) = 1$ is valid in every algebra of species. We shall call this the defining formula of "$\Rightarrow$". We shall in each case prove the independence of each operation by giving a species algebra in which the defining formula of the operation is not valid.

Definition 1 Let $\mathfrak{A} = \langle S, A, \Rightarrow, \cup, \cap, -\rangle$ be an algebra of species, and T a term of the algebra of species. The following formulae define (recursively) a function $F_{T,\mathfrak{A}}$ which correlates an element $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) \in S$ with every infinite sequence of elements $X_1, \ldots, X_n, \ldots \in S$:

(i) $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = X_p$ if $T = T_p (p = 1, 2, 3 \ldots)$;
(ii) $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = F_{T_1,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) \Rightarrow F_{T_2,\mathfrak{A}}(X_1, \ldots, X_n, \ldots)$, if $T = T_1 \Rightarrow T_2$ (where T_1 and T_2 are terms);
(iii) and (iv) Analogously for the operations "$\cup$" and "$\cap$";
(v) $F_{T,\mathfrak{A}}\overline{(X_1, \ldots, X_n, \ldots)} = F_{T_1,\mathfrak{A}}(X_1, \ldots, X_n)$, if $\overline{T} = T_1$.

We say that a term T is verified by the species algebra, in symbol $T \in \mathrm{E}(\mathfrak{A})$, if $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = A$ for all $X_1 \ldots X_n \in S$.

Definition 2 A formula $\mathfrak{H} = (T = 1)$ of the algebra of species is said to be valid in the algebra $\mathfrak{A}$, if T is satisfied by $\mathfrak{A}$, and $\mathfrak{H}$ is said to be valid in the algebra of species if it is valid in every algebra of species.

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Let

$$\mathfrak{A}_1 = \langle S_1, A_1, \Rightarrow, \cup, \cap, - \rangle^1$$

be an algebra of species of a given subspecies of a given species A_1, with $S_1 = \{A_1, A_2, A_3\}$ as shown in the table below. We may obtain a model of $\mathfrak{A}_1$, if we take A_1 to be the species of all real numbers and A_2 to be the species of all real numbers which are known to be rational or irrational. We shall prove the independence of "$-$" by showing that its defining formula is not valid in $\mathfrak{A}_1$. In the algebra $\mathfrak{A}_1$, we have

$$A_1 \Rightarrow A_1 = A_1 \cap A_1 = A_1 \cup A_1 = A_1 \text{ and } \overline{A}_1 = A_3.$$

Suppose that the species complement "$-$" were definable in terms of the other three operations. Then there would exist a term T_1 which does not contain "$-$" and such that $(\overline{A}_1 \Rightarrow T_1 \cap T_1 \Rightarrow \overline{A}) = 1$ would be a valid formula of the algebra of species. Suppose we replaced every species variable in the above formula by A_1. Then T_1 would be reduced to "A_1". Then we would have $(\overline{A}_1 \Rightarrow A_1 \cap A_1 \Rightarrow \overline{A}_1) = A_1$ or $(A_3 \Rightarrow A_1 \cap A_1 \Rightarrow A_3) = A_1$ in the algebra $\mathfrak{A}_1$. But, in $\mathfrak{A}_1$, $(A_3 \Rightarrow A_1 \cap A_1 \Rightarrow A_3) = A_3$. Hence there exists no such term T_1 and the defining formula of "$-$" is not valid in $\mathfrak{A}_1$, and, therefore, "$-$" is not definable in terms of the other three operations.

Generally, as in the case of logical matrices, (see [1]), if three operations are class-closing on some proper sub-class of the elements of a species algebra $\mathfrak{A}$, while the fourth operation is not class-closing on the same proper sub-class of the elements of the algebra of species $\mathfrak{A}_1$, then the fourth operation is not definable in terms of the other three. In the algebra $\mathfrak{A}_2$ following, $S_2 = \{A_1, A_2, A_3, A_4, A_5\}$ is a species of sub-species of A. To get a model of $\mathfrak{A}_2$ we may take A_1 to be the species of all real numbers, A_2 the species of all known rational numbers, A_3 the species of all known irrational numbers, and A_4 the species of all real numbers which are known to be rational or irrational.

Algebra $\mathfrak{A}_1$

$\Rightarrow$	A_1	A_2	A_3
A_1	A_1	A_2	A_3
A_2	A_1	A_1	A_3
A_3	A_1	A_1	A_1

$\cap$	A_1	A_2	A_3
A_1	A_1	A_2	A_3
A_2	A_2	A_2	A_3
A_3	A_3	A_3	A_3

$\cup$	A_1	A_2	A_3
A_1	A_1	A_1	A_1
A_2	A_1	A_2	A_2
A_3	A_1	A_2	A_3

A	$\overline{A}$
A_1	A_3
A_2	A_3
A_3	A_1

$\Rightarrow \cap \Rightarrow$	A_1	A_2	A_3
A_1	A_1	A_2	A_3
A_2	A_2	A_1	A_3
A_3	A_3	A_3	A_1

[1]The algebraic tables for algebras $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are given on pages 528 and 529.

In the algebra $\mathfrak{A}_2$, the table shows that if A_i and A_j are elements of the subspecies $\{A_1, A_2, A_3, A_5\}$ then $A_i \Rightarrow A_j$ is an element of the subspecies $\{A_1, A_2, A_3, A_5\}$; and the only case where $A_i \Rightarrow A_j = A_4$ is when $A_i = A_1$ and $A_j = A_4$. Similarly $A_i \cap A_j$ is in $\{A_1, A_2, A_3, A_5\}$ when A_i and A_j are in $\{A_1, A_2, A_3, A_5\}$. We note that A_i is never equal to $A_4 (i = 1, 2, 3, 4)$. Thus the operations "$\Rightarrow$", "$\cap$", and "$-$" are class-closing on the class $\{A_1, A_2, A_3, A_5\}$. But $A_2 \cup A_3 = A_4$. Now suppose that "$\cup$" were definable in terms of the other three operations. Then there would exist a term T_1 in which "$\cup$" does not occur, and such that the formula

$$((A_i \cup A_j) \Rightarrow T_1 \cap T_1 \Rightarrow (A_i \cup A_j)) = 1$$

would be a valid formula of the algebra of species. Let T_2 be the term that is obtained from T_1 by replacing "A_i" by "A_2", "A_j" by "A_3" and all other variables (if any) in T_1 by A_3. Then we should have $((A_1 \cup A_2) \Rightarrow T_2 \cap T_2 \Rightarrow (A_2 \cup A_3)) = A_1$ in the algebra $\mathfrak{A}_2$. But since "$\Rightarrow$", "$\cap$", and "$-$" are class-closing on $\{A_1, A_2, A_3, A_5\}$, it follows that T_2 reduces to a subspecies which is distinct from A_4, whereas $A_2 \cup A_3 = A_4$. The defining equation of "$\cup$", which then reduces to $(A_4 \Rightarrow T_2 \cap T_2 \Rightarrow A_4) = 1$, is not valid in $\mathfrak{A}_2$ since $A_4 \neq T_2$ and, therefore, $(A_4 \Rightarrow T_2 \cap T_2 \Rightarrow A_4 \neq A_1$ in $\mathfrak{A}_2$.

From the above we conclude that "$\cup$" is not definable in terms of the other three operations.

We now consider the algebra $\mathfrak{A}_3$ which is the direct product of the algebra A_1 by itself. In the algebra $\mathfrak{A}_3 = \langle S_3, A, \Rightarrow, \cup, \cap, - \rangle$,

$$S_3 = \{A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31}, A_{32}, A_{33}\}.$$

In this algebra we have, for example, $A_{12} \Rightarrow A_{32} = A_{31}$, since $A_1 \Rightarrow A_3 = A_3$ and $A_2 \Rightarrow A_2 = A_1$ in $\mathfrak{A}_1$. Similarly, $A_{22} \cap A_{31} = A_{32}$ in $\mathfrak{A}_3$, since $A_2 \cap A_3 = A_3$ and $A_2 \cap A_1 = A_2$ in $\mathfrak{A}_1$. Further, $\overline{A}_{13} = A_{31}$ in $\mathfrak{A}_3$, since $\overline{A}_1 = A_3$ and $\overline{A}_3 = A_1$ in $\mathfrak{A}_1$. In the algebra $\mathfrak{A}_3$, the three operations "$-$", "$\cup$", and "$\cap$" are

Algebra $\mathfrak{A}_2$

$\Rightarrow$	A_1	A_2	A_3	A_4	A_5
A_1	A_1	A_2	A_3	A_4	A_5
A_2	A_1	A_1	A_3	A_1	A_3
A_3	A_1	A_2	A_3	A_1	A_5
A_4	A_1	A_2	A_3	A_1	A_5
A_5	A_1	A_1	A_1	A_1	A_1

$\cap$	A_1	A_2	A_3	A_4	A_5
A_1	A_1	A_2	A_3	A_4	A_5
A_2	A_2	A_2	A_5	A_2	A_5
A_3	A_3	A_5	A_3	A_3	A_5
A_4	A_4	A_2	A_3	A_4	A_5
A_5	A_5	A_5	A_5	A_5	A_5

$\cup$	A_1	A_2	A_3	A_4	A_5
A_1	A_1	A_1	A_1	A_1	A_1
A_2	A_1	A_2	A_4	A_4	A_2
A_3	A_1	A_4	A_3	A_4	A_3
A_4	A_1	A_4	A_4	A_4	A_4
A_5	A_1	A_2	A_3	A_4	A_5

A	$\overline{A}$
A_1	A_5
A_2	A_3
A_3	A_2
A_4	A_5
A_5	A_1

$\Rightarrow\cap\Rightarrow$	A_1	A_2	A_3	A_4	A_5
A_1	A_1	A_2	A_3	A_4	A_5
A_2	A_2	A_1	A_5	A_2	A_3
A_3	A_3	A_5	A_1	A_3	A_2
A_4	A_4	A_2	A_3	A_1	A_5
A_5	A_5	A_3	A_2	A_5	A_1

Algebra $\mathfrak{A}_3$

$\Rightarrow$	A_{11}	A_{12}	A_{13}	A_{21}	A_{22}	A_{23}	A_{31}	A_{32}	A_{33}
A_{11}	A_{11}	A_{12}	A_{13}	A_{21}	A_{22}	A_{23}	A_{31}	A_{32}	A_{33}
A_{12}	A_{11}	A_{11}	A_{13}	A_{21}	A_{21}	A_{23}	A_{31}	A_{31}	A_{33}
A_{13}	A_{11}	A_{11}	A_{11}	A_{21}	A_{21}	A_{21}	A_{31}	A_{31}	A_{31}
A_{21}	A_{11}	A_{12}	A_{13}	A_{11}	A_{12}	A_{13}	A_{31}	A_{32}	A_{33}
A_{22}	A_{11}	A_{11}	A_{13}	A_{11}	A_{11}	A_{13}	A_{31}	A_{31}	A_{33}
A_{23}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{31}	A_{31}	A_{31}
A_{31}	A_{11}	A_{12}	A_{11}	A_{11}	A_{12}	A_{13}	A_{11}	A_{12}	A_{13}
A_{33}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}

$\cap$	A_{11}	A_{12}	A_{13}	A_{21}	A_{22}	A_{23}	A_{31}	A_{32}	A_{33}
A_{11}	A_{11}	A_{12}	A_{13}	A_{21}	A_{22}	A_{23}	A_{31}	A_{32}	A_{33}
A_{12}	A_{12}	A_{12}	A_{13}	A_{22}	A_{22}	A_{23}	A_{32}	A_{32}	A_{33}
A_{13}	A_{13}	A_{13}	A_{13}	A_{23}	A_{23}	A_{23}	A_{33}	A_{33}	A_{33}
A_{21}	A_{21}	A_{22}	A_{23}	A_{21}	A_{22}	A_{23}	A_{31}	A_{32}	A_{33}
A_{22}	A_{22}	A_{22}	A_{23}	A_{22}	A_{22}	A_{23}	A_{32}	A_{32}	A_{33}
A_{23}	A_{23}	A_{23}	A_{23}	A_{23}	A_{23}	A_{23}	A_{33}	A_{33}	A_{33}
A_{31}	A_{31}	A_{32}	A_{33}	A_{31}	A_{32}	A_{33}	A_{31}	A_{32}	A_{33}
A_{32}	A_{32}	A_{32}	A_{33}	A_{32}	A_{32}	A_{33}	A_{32}	A_{32}	A_{33}
A_{33}	A_{33}	A_{33}	A_{33}	A_{33}	A_{33}	A_{33}	A_{33}	A_{33}	A_{33}

$\cup$	A_{11}	A_{12}	A_{13}	A_{21}	A_{22}	A_{23}	A_{31}	A_{32}	A_{33}
A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}	A_{11}
A_{12}	A_{11}	A_{12}	A_{12}	A_{11}	A_{12}	A_{12}	A_{11}	A_{12}	A_{12}
A_{13}	A_{11}	A_{12}	A_{13}	A_{11}	A_{12}	A_{13}	A_{11}	A_{12}	A_{13}
A_{21}	A_{11}	A_{11}	A_{11}	A_{21}	A_{21}	A_{21}	A_{21}	A_{21}	A_{21}
A_{22}	A_{11}	A_{12}	A_{12}	A_{21}	A_{22}	A_{22}	A_{21}	A_{22}	A_{22}
A_{23}	A_{11}	A_{12}	A_{13}	A_{21}	A_{22}	A_{23}	A_{21}	A_{22}	A_{23}
A_{31}	A_{11}	A_{11}	A_{11}	A_{21}	A_{21}	A_{21}	A_{31}	A_{31}	A_{31}
A_{32}	A_{11}	A_{12}	A_{12}	A_{21}	A_{22}	A_{22}	A_{31}	A_{32}	A_{32}
A_{33}	A_{11}	A_{12}	A_{13}	A_{21}	A_{22}	A_{23}	A_{31}	A_{32}	A_{33}

A	$\bar{A}$
A_{11}	A_{33}
A_{12}	A_{33}
A_{13}	A_{31}
A_{21}	A_{33}
A_{22}	A_{33}
A_{23}	A_{31}
A_{31}	A_{13}
A_{32}	A_{13}
A_{33}	A_{11}

$\Rightarrow\cap\Rightarrow$	A_{11}	A_{12}	A_{13}	A_{21}	A_{22}	A_{23}	A_{31}	A_{32}	A_{33}
A_{11}	A_{11}	A_{12}	A_{13}	A_{21}	A_{22}	A_{23}	A_{31}	A_{32}	A_{33}
A_{12}	A_{12}	A_{11}	A_{13}	A_{22}	A_{21}	A_{23}	A_{32}	A_{31}	A_{33}
A_{13}	A_{13}	A_{13}	A_{11}	A_{23}	A_{23}	A_{21}	A_{33}	A_{33}	A_{31}
A_{21}	A_{21}	A_{22}	A_{23}	A_{11}	A_{12}	A_{13}	A_{31}	A_{32}	A_{33}
A_{22}	A_{22}	A_{21}	A_{23}	A_{12}	A_{11}	A_{13}	A_{32}	A_{31}	A_{33}
A_{23}	A_{23}	A_{23}	A_{21}	A_{13}	A_{13}	A_{11}	A_{33}	A_{33}	A_{31}
A_{31}	A_{31}	A_{32}	A_{33}	A_{31}	A_{32}	A_{33}	A_{11}	A_{12}	A_{13}
A_{32}	A_{32}	A_{31}	A_{33}	A_{32}	A_{31}	A_{33}	A_{12}	A_{11}	A_{13}
A_{33}	A_{33}	A_{33}	A_{31}	A_{33}	A_{33}	A_{31}	A_{13}	A_{13}	A_{11}

class-closing on the class $\{A_{11}, A_{12}, A_{22}, A_{33}\}$, while $A_{12} \Rightarrow A_{22} = A_{21}$, and, by similar arguments as shown above, we conclude that "$\Rightarrow$" is not definable in terms of the other three operations.

Finally "$-$", "$\cup$", and "$\Rightarrow$" are class-closing on the class $\{A_{11}, A_{12}, A_{13}, A_{33}\}$, while $A_{12} \cap A_{31} = A_{32}$. Hence, we conclude, by similar arguments as shown above, that "$\cap$" is not definable in terms of the other three operations. This concludes the proof that none of the four fundamental operations of the algebra of species is definable in terms of the others.

We note also that since the algebra of species is isomorphic with the intuitionistic propositional calculus (see [5]), the independence of the propositional functions $\rightarrow$, $\vee$, $\wedge$, $\sim$ imply the independence of the species-algebraic operations $\Rightarrow$, $\cup$, $\cap$, $-$. Further, the completeness of the algebra of species (see [3] and [4]), together with the above results, means for example that a formula which contains "$-$" cannot be equivalent to one which does not contain "$-$".

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