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ON A MODAL SYSTEM OF R. A. BULL'S

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Bull [1] mentions, in passing, having discovered the weakest extension of S4 that both contains S4.3 and is obtainable by extending S4 with an axiom involving a single sentential variable. I shall call the axiom in question

F3 *CMLpALCpLpLCLCpLpLp*

By an S4**F**-*model* I mean an S4-model $\langle W, R, V\rangle$ (see, e.g., [2]) wherein

$$\forall x\forall y\forall z((xRy \,.\, xRz) \rightarrow (zRy \vee yRx)) \qquad \text{(F)}$$

Lemma 1 *Each theorem of* S4 + **F3** *is valid in every* S4**F**-*model.*

I content myself with showing **F3** cannot fail in such a model $\langle W, R, V\rangle$. If it does, then for some $x \in W$ (1) $V(MLp, x) = 1$, (2) $V(LCpLp, x) = 0$ and (3) $V(LCLCpLpLp, x) = 0$. By (1) there exists $z \in W$ such that xRz and (4) $V(Lp, z) = 1$. By (3), on the other hand, there exists $y \in W$ such that xRy, (5) $V(LCpLp, y) = 1$ and (6) $V(Lp, y) = 0$. It follows from (4) that $V(LLp, z) = 1$ and so, by (6), $z \not R y$. According to (F), then, yRx. But from (5) I have $V(LLCpLp, y) = 1$ and so now $V(LCpLp, x) = 1$, contradicting (2).

Lemma 2 *Each formula valid in all* S4**F**-*models is provable in* S4 + **F3**.

I prove only what is not already familiar from the literature: (**F**) holds in the canonical model $\langle W, R, V\rangle$ of S4 + **F3**. Otherwise, there exist $x, y, z \in W$ with xRy, xRz, $z \not R y$, and $y \not R x$ so that for some formulas q and r, $Lr \in z$, $r \notin y$, $Lq \in y$, and $q \notin x$. Since $Lr \in z$, $LCqLr \in z$ and so $MLCqLr \in x$. By **F3**, then, $LCCqLrLCqLr \in x$ or $LCLCCqLrLCqLrLCqLr \in x$.

$CqLr \in x$ since $q \notin x$; $CqLr \notin y$, however, so $LCqLr \notin x$ and the first alternative is impossible: $LCCqLrLCqLr \notin x$. It must be, then, that $LCLCCqLrLCqLrLCqLr \in x$, and $CLCCqLrLCqLrLCqLr \in y$. As before, $LCqLr \notin y$, so $LCCqLrLCqLr \notin y$. There must then exist $y' \in W$ such that yRy', $CqLr \in y'$, and $LCqLr \notin y'$. However, $Lq \in y$ so that $q \in y'$. Hence $Lr \in y'$ and so $LCqLr \in y'$, which is also impossible.

Thus Bull's system has been independently introduced and studied in more recent literature:

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Theorem S4 + **F3** *is the system* S4.3.2 (= S4 + *ALCLpqCMLqp*) *of Zeman's* [3].

Proof: Immediate from the lemmas and the known result ([2], p. 161, where S4.3.2 is called "S4F") that S4F-models characterize S4.3.2.

REFERENCES

[1] Bull, R. A. "On three related extensions of S4," *Notre Dame Journal of Formal Logic*, vol. VIII (1967), pp. 330-334.

[2] Segerberg, K., *An Essay in Classical Modal Logic*, Filosofiska Studier, Uppsala (1971).

[3] Zeman, J. Jay, "The propositional calculus **MC** and its modal analog," *Notre Dame Journal of Formal Logic*, vol. IX (1968), pp. 294-298.

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