476
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ALTERNATIVE NOTATIONS FOR *PRINCIPIA MATHEMATICA* DESCRIPTION THEORY: POSSIBLE MODIFICATIONS

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1 The following are formulas by clauses (1)-(7), pp. 64-65, of a recent paper:[1]

$$[\iota y H^1 y] I^2 x \iota y H^1 y$$
$$[\iota y J^1 y]\, [\iota x H^1 x] I^2 \iota y J^1 y \iota x H^1 x$$
$$\wedge x [\iota y H^1 y] I^2 x \iota y H^1 y$$
$$[\iota y H^1 y] \wedge x I^2 x \iota y H^1 y$$

But the following are *not* formulas by these clauses:

$$[\iota x H^1 x] I^2 x \iota x H^1 x$$
$$[\iota x J^1 x]\, [\iota x H^1 x] I^2 \iota x J^1 x \iota x H^1 x$$
$$\wedge x [\iota x H^1 x] I^2 x \iota x H^1 x$$
$$[\iota x H^1 x] \wedge x I^2 x \iota x H^1 x$$

A connected point is that, by translation rules $\bar{\mathrm{T}}/\iota$ and $\iota/\bar{\mathrm{T}}$, not only is ϕ' a translation of ϕ by $\iota/\bar{\mathrm{T}}$ if and only if ϕ is a translation of ϕ' by $\bar{\mathrm{T}}/\iota$, but each ι-formula has a unique ι-free $\bar{\mathrm{T}}$-translation and vice versa.

Modifications to formation and translation rules are possible, and are given below, that secure as formulas all of the above strings (which may seem a gain) while trading the *unique*-translation feature for a *multiple*-translation feature (which may seem a loss).

2 Replace clause (7) by the following clause (7′):

(a′) An expression $\iota\alpha\phi$, α a variable and ϕ a formula or pseudo-formula, is an ι-description.
(b′) An expression ϕ is a *pseudo-term* (*pseudo-formula*) just in case a term (formula) ϕ' is like ϕ except for having, in place of all occurrences in ϕ of one or more ι-descriptions, occurrences of variables not in ϕ. A term (formula) related to a pseudo-term (pseudo-formula) ϕ in this manner is an *associated term* (*formula*) of ϕ.
(c′) *An occurrence of a variable α is bound* in a term or formula π just in case it stands within an occurrence in π of an expression χ such that (i) either χ is $\wedge\alpha\phi$, $\vee\alpha\phi$, $\urcorner\alpha\phi$, or $\bar{\mathrm{T}}\alpha\phi\psi$, or χ is $[\iota\alpha\phi]\psi$ and the occurrence of

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α in question stands within χ in an occurrence of $\imath\alpha\phi$, (ii) ϕ is a formula or pseudo-formula, (iii) ψ is a formula or pseudo-formula, and (iv) either χ is a term or formula or χ' is an associated term or formula of χ and the occurrence of α in question does not stand in χ in an $\imath$-description that is supplanted by a variable in χ'. An occurrence of a variable α is *free* in a term or formula π just in case it stands in π and is not bound in π.

(d′) If ϕ and ψ are formulas, α is a variable, and ψ' is ψ or comes from ψ by putting $\imath\alpha\phi$ in place of only, but not necessarily any or all, free occurrences of a variable β in ψ, then

$$[\imath\alpha\phi]\psi'$$

is a formula.

Note: Generating $\imath$-formulas by putting $\imath$-descriptions in places marked by free occurrences of variables provides, I think, the most 'natural' generative grammar for the language, and it may be what Russell had more or less in mind. *Cf. Introduction to Mathematical Philosophy* (G. Allen & Unwin, London, 1919), p. 179, and especially *Principia Mathematica* to ***56** (Cambridge University Press, Cambridge, 1962). "Although '$(\imath x)(\phi x)$' has no meaning by itself, it may be substituted for y in any propositional function fy, and we get a significant proposition, though not a value of fy." (p. 68, *PM*.) Of course generating $\imath$-formulas in this manner calls for the simultaneous definition of 'bound occurrence of a variable'. It would be possible with less 'naturalness' and 'historical accuracy' to generate the expressions of clauses (1)-(7′), or just those of clauses (1)-(7), without simultaneously defining 'bound occurrence of a variable'. In any case, however, the definition of 'bound' must be complex. For its complexity derives not from the decision to develop it simultaneously with definitions of 'term' and 'formula', but from the fact that a 'complete $\imath$-symbol' generally involves more than one occurrence of an $\imath$-description and a variable bound in one of these should be bound in all.

3 *Modified translation rules suited to clause* (7′)

$\imath/\bar{T}$: the term (formula) ϕ' comes from the term (formula) ϕ by translation rule $\imath/\bar{T}$ if, in place of an occurrence of $\bar{T}\alpha\psi\chi$ in ϕ there stands in ϕ' an occurrence of $[\imath\beta\psi']\chi'$, α and β variables, ψ, χ, ψ', and χ' formulas or pseudo-formulas such that there is an expression $\bar{T}\alpha\psi_1\chi_1$ that is, or is an associated formula of, $\bar{T}\alpha\psi\chi$ and an expression $[\imath\beta\psi_1']\chi_1'$ similarly related to $[\imath\beta\psi']\chi'$ wherein (i) ψ_1' comes from ψ_1 by proper substitution of β for α, and ψ_1 comes from ψ_1' by proper substitution of α for β, (ii) χ_1' comes from χ_1 by replacement of each free occurrence of α by an occurrence of $\imath\beta\psi_1'$ and (iii) no occurrence of $\imath\beta\psi_1'$ introduced in (ii) stands in χ_1' to the right of an occurrence in χ_1' of $[\imath\beta\psi_1']$.

$\bar{T}/\imath$: the term (formula) ϕ comes from the term (formula) ϕ' by translation rule $\bar{T}/\imath$ if, in place of an occurrence of $[\imath\beta\psi']\chi'$ in ϕ' there stands in ϕ an occurrence of $\bar{T}\alpha\psi\chi$, α and β variables, ψ, χ, ψ', and χ' formulas or pseudo-formulas such that there is an expression $[\imath\beta\psi_1']\chi_1'$ that is, or is an associated formula of, $[\imath\beta\psi']\chi'$ and an expression $\bar{T}\alpha\psi_1\chi_1$ similarly related to $\bar{\mathbf{T}}\alpha\psi\chi$ such that (i) α is not free in $[\iota\beta\psi_1']\chi_1'$, (ii) ψ_1 comes from ψ_1' by proper substitution of α and β and ψ_1' comes from ψ_1 by proper substitution of β for α, and (iii) χ_1 comes from χ_1' by replacement of each occurrence in χ_1' of $\iota\beta\psi'$ that does not stand in or to the right of an occurrence in χ_1' of $\lfloor\iota\beta\psi'\rfloor$ by a free occurrence of α.

Note: ι- and $\bar{\mathbf{T}}$-formulas have multiple ι-free and $\bar{\mathbf{T}}$-free translations by these rules. But symmetry of translation is preserved: If ϕ is an ι-free formula and ϕ' a $\bar{\mathbf{T}}$-free formula, then ϕ is a translation of ϕ' by applications of $\bar{\mathbf{T}}/\iota$ if and only if ϕ' is a translation of ϕ by applications of $\iota/\bar{\mathbf{T}}$.

4 In place of clauses (1)-(6), (7′), and (8), one can employ clauses (i)-(vi), pp. 68-69, subject to the following adjustments:

References to translation rules become references to the revised rules stated in **3** above.

References to clauses 7 (a) and 7 (b) become references to clauses 7 (a′) and 7 (b′).

The penultimate sentence in (v) becomes: "In particular, for each occurrence of a variable in ϕ there corresponds exactly one occurrence of a variable, but not necessarily the *same* variable, in ϕ'."

Sub-clause (b) in (vi) becomes: "ϕ is an immediate ancestor of ϕ and the occurrence of α in ϕ that is in question corresponds to a bound occurrence of a variable in ϕ'."[2]

NOTES

1. "*Principia Mathematica* description theory: The classical and an alternative notation," *Notre Dame Journal of Formal Logic*, vol. XV (1974), pp. 63-72: In the absence of explicit indication, page references are to this article. [Erratum: p. 65, second line of clause (c), delete "*either*".]

2. Pseudo-terms (formulas) by (1)-(6), (7′), and (8), or equivalently by (i)-(vi) as here revised, do not have *unique* associated terms (formulas). This calls for revisions to the definitions of alphabetic variance, pp. 66 and 69. I state here only a revised version of the shorter definition, p. 69. This revision *both* serves present purposes *and* corrects a deficiency in the shorter definition as originally stated: ϕ and ϕ_1 are ***immediate alphabetic variants*** just in case there are terms or formulas ϕ' and ϕ_1' that are, or are associated terms or formulas of, ϕ and ϕ_1 respectively such that *either* (a) ϕ' and ϕ_1' are $\bar{\mathbf{T}}$-terms or $\bar{\mathbf{T}}$-formulas and $\langle\phi', \phi_1'\rangle$ is $\langle\Lambda\alpha\psi, \Lambda\alpha'\psi'\rangle$, $\langle V\alpha\psi, V\alpha'\psi'\rangle$, $\langle\iota\alpha\psi, \iota\alpha'\psi'\rangle$, or $\langle\bar{\mathbf{T}}\alpha\psi\chi, \bar{\mathbf{T}}\alpha'\psi'\chi'\rangle$, wherein α and α' are variables, ψ, ψ', χ, and χ' are $\bar{\mathbf{T}}$-formulas, ψ' comes from ψ by proper substitution of α' for α and ψ comes from ψ' by proper substitution of α for α', and χ' and χ are similarly related, *or* (b) ϕ' and ϕ_1' have immediate ancestors ϕ'' and ϕ_1'' that are immediate alphabetic variants.

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