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A MODEL-THEORETIC SEMANTICS FOR MODAL LOGIC

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This paper will deal with the semantics of modal logic in a model-theoretic way. This semantics is clearer than the standard accounts and furthermore will lend itself to further development. From it will flow all the completeness results of modal logic along with a few new results. The paper will presuppose some familiarity with modal logic and model theory.

The key to this semantics lies in the notion of a many-sorted (two-sorted for simplicity) predicate logic. A many-sorted language is simply a language with symbols for sorts and symbols for relations and constants. A structure $\mathfrak{M}$ of similarity type τ ($\mathfrak{M} \in \mathsf{Str}(\tau)$) will associate to each sort symbol of τ a universe of objects. The universe of $\mathfrak{M}$ will be the union of the universes of each sort. $\mathfrak{M}$ will associate to each n-ary relation symbol of τ an n-ary relation over the universe and to each constant symbol of τ an individual in the universe. Satisfaction of many-sorted logic is defined just as in one-sorted logic.

A vector space is a typical example of a structure of such a language. Here there are two different sorts of entities: scalars and vectors. There are also relations on the vectors, relations on the scalars, and relations on both.

Theorem 1 *Let* L *be a predicate logic of type* τ. *Then there is a type* τ^{Σ} *having at least one new sort* I *such that we have three operations satisfying four conditions. The three operations are:*

(i) *from the* L-*sentences of type* τ *to the* L-*sentences of type* τ^{Σ}, $\phi \rightarrow \phi^{\Sigma}$;
(ii) *from a set of structures of type* τ *to a structure of type* τ^{Σ}, $\langle \mathfrak{M}_i \rangle_{i \in \mathsf{I}} \rightarrow \mathfrak{M}^{\Sigma}$;

and

(iii) *from a* τ^{Σ} *structure to a* τ-*structure*, $\mathfrak{M}^{\Sigma} \rightarrow \mathfrak{M}_i$.

The conditions these operations satisfy are as follows:

(i) $\mathfrak{M}^{\Sigma} \in \mathsf{Str}(\tau^{\Sigma})$ *whenever* $\mathfrak{M}_i \in \mathsf{Str}(\tau)$ *for each* $i \in \mathsf{I}$ *and* $\mathfrak{M}^{\Sigma} = (\mathsf{I}, \ldots)$;

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(ii) *if* $\mathfrak{M}^{\Sigma} \in \mathrm{Str}(\tau^{\Sigma})$ *and* $\mathfrak{M}^{\Sigma} = (\mathsf{I}, \ldots)$, *then* $\mathfrak{M}_i \in \mathrm{Str}(\tau)$ *for each* $i \in \mathsf{I}$;
(iii) *if* ϕ *is a formula of type* τ, *then* ϕ^{Σ} *is a formula of type* τ^{Σ} *and* ϕ^{Σ} *has the free variables of* ϕ *together with a new free variable* w *of sort* I;
(iv) *for any* $\mathfrak{M}^{\Sigma} = (\mathsf{I}, \ldots) \in \mathrm{Str}(\tau^{\Sigma})$, $i \in \mathsf{I}$, ϕ *a formula of type* τ *and assignment* s *to the free variables of* ϕ *in* $\mathfrak{M}_i$, *we have* $\mathfrak{M}^{\Sigma} \models \phi^{\Sigma}(i, s)$ *iff* $\mathfrak{M}_i \models \phi(s)$. *Cf.* [1].

Proof: For simplicity we will assume τ is single-sorted with only one binary relation symbol R. Given $\mathfrak{M}_i = (\mathsf{M}_i; R_i)$ for $i \in \mathsf{I}$, take $\mathfrak{M}^{\Sigma} = \left(\mathsf{I}, \bigcup_{i \in I} \mathsf{M}_i; \bigcup_{i \in I} \{i\} \times \mathsf{M}_i, \bigcup_{i \in I} \{i\} \times R_i\right)$. τ has one sort symbol M_i and one binary relation symbol R_i whereas τ^{Σ} has two sort symbols I and M and two relation symbols, a binary M^{Σ} and a ternary R^{Σ}. Inversely given $\mathfrak{M}^{\Sigma} = (\mathsf{I}, \mathsf{M}; M^{\Sigma}, R^{\Sigma})$ of type τ^{Σ} define $\mathfrak{M}_i = (\mathsf{M}_i; R_i)$ by $x \in \mathfrak{M}_i$ iff $\langle i, x\rangle \in M^{\Sigma}$ and $\langle x, y\rangle \in R_i$ iff $\langle i, x, y\rangle \in R^{\Sigma}$. (Note that M^{Σ} is a "belonging" relation and will in the sequel be referred to as B) $\phi^{\Sigma}(w, \ldots)$ is obtained from $\phi(\ldots)$ by replacing each atomic $R(x, y)$ by $R^{\Sigma}(w, x, y)$ and replacing each quantifier $\forall x(\ldots)$ by $\forall x(M^{\Sigma}(w, x) \rightarrow \ldots)$. It should be clear that $\mathfrak{M}^{\Sigma}$ indexes all the models $\mathfrak{M}_i$, $i \in \mathsf{I}$.

The sentences of predicate modal logic are built up inductively as in predicate logic but with two modal operators P and N added. That is if ϕ is a sentence or formula of type τ so is $\mathsf{N}\phi$ and $\mathsf{P}\phi$. (We will consider dyadic modal operators in another paper.) There are a plethora of modal logics depending on the kind and strength of the modality being considered, but the intended interpretation in all is roughly the same. "$\mathsf{N}\phi$" is true iff ϕ is necessarily true iff ϕ is true in all "possible worlds". "$\mathsf{P}\phi$" is true iff ϕ is possibly true iff ϕ is true in some "possible world".

The τ-structures ("possible worlds") of a modal logic of type τ are simply the τ-structures of a nonmodal logic. But to define truth of modal τ-sentences requires us to consider $\tau^{\Sigma+}$ structures and $\tau^{\Sigma+}$ sentences. Here $\tau^{\Sigma+}$ is type τ^{Σ} with an extra binary relation symbol H and an extra constant 0. The binary relation associated with H will be on the universe of sort I and will tell us how the $\mathfrak{M}_i$ are related (in terms of "co-possibility"). The individual associated to 0 will be the index of the "real world". The connection between truth of τ modal sentences in τ-modal structures and the truth of $\tau^{\Sigma+}$ sentences in $\tau^{\Sigma+}$ structures is of course provided by the proposition. Thus we define

$$\mathfrak{M}_0 \models \mathsf{N}\phi(x) \text{ iff } \mathfrak{M}^{\Sigma} \models \forall z(\mathsf{H}(z, 0) \rightarrow \phi^{\Sigma}(z, x))$$

and

$$\mathfrak{M}_0 \models \mathsf{P}\,\phi(x) \text{ iff } \mathfrak{M}^{\Sigma} \models \exists z(\mathsf{H}(z, 0) \wedge \phi^{\Sigma}(z, x)).$$

The $\tau^{\Sigma+}$ structures $\mathfrak{M}^{\Sigma+}$ are called model systems. Note that we cannot determine $\mathfrak{M}_i \models \phi$ without considering the model system it is "imbedded" in. Thus the model systems are the structures of prime importance in determining truth, validity, etc. of τ-modal sentences and not the τ-structures themselves.

Which modal logic we are considering determines, of course, what axioms we accept. The axioms can be written in either of two different ways depending upon whether the theorems themselves or the conditions on H and B are to be emphasized. Let us look at some examples. One axiom would surely be $\mathsf{N}\phi(x) \rightarrow \mathsf{P}\phi(x)$. This would be rendered as $\forall w(\mathsf{H}(w, 0) \rightarrow \phi^{\Sigma}(w, x)) \rightarrow \exists w(\mathsf{H}(w, 0) \wedge \phi^{\Sigma}(w, x))$. Another axiom that may be considered is the Barcan formula which would be rendered as follows: $\exists w(\mathsf{H}(w, 0) \wedge \exists x(\mathsf{B}(w, x) \wedge \phi(w, x))) \rightarrow \exists x(\mathsf{B}(0, x) \wedge \exists w(\mathsf{H}(w, 0) \wedge \phi(w, x)))$. Conditions on H and B are more important in applications than the theorems themselves. Some such conditions might be $\forall x \forall y \forall z(\mathsf{H}(x, y) \wedge \mathsf{H}(y, z) \rightarrow \mathsf{H}(x, z))$ or $\forall x \forall y \forall z(\mathsf{B}(y, x) \wedge \mathsf{H}(z, y) \rightarrow \mathsf{B}(z, x))$.

Thus a modal logic $\mathbf{L}$ can be specified by the (usually finite) collection of its logical axioms and a theory $\mathbf{T}$ in $\mathbf{L}$ can be specified by certain extra non-logical axioms. We say that $\mathfrak{M}^{\Sigma+}$ is a model system of $\mathbf{T}$ iff $\mathfrak{M}^{\Sigma+} \models \mathbf{T}^{\Sigma+}$ where $\mathbf{T}^{\Sigma+} = \{\phi^{\Sigma+} | \phi \in \mathbf{T}\}$. Hence in analogy with predicate logic we say that ϕ is a valid consequence of $\mathbf{T}$ iff $\phi^{\Sigma+}$ is true in all model systems $\mathfrak{M}^{\Sigma+}$ such that $\mathfrak{M}^{\Sigma+} \models \mathbf{T}^{\Sigma+}$.

Theorem 2 *Any modal logic* $\mathbf{L}$ *of type* τ *whose axioms are* $\tau^{\Sigma+}$*-expressible in predicate logic is complete.* (Thus S4, S5, and all standard monadic modal logics are complete.)

Proof: From the definition of validity, Theorem 1, and the completeness theorem for predicate logic of type $\tau^{\Sigma+}$, the valid sentences of $\mathbf{L}$ are recursively enumerable.

Given the above model-theoretic treatment of the semantics of modal logic we can now use results from the model theory of predicate logic to get results on modal logics. Of course the modal logic analogues of most of these results are not very interesting, but there are some which are. Compactness follows from completeness.

Theorem 3 *Given any modal logic* $\mathbf{L}$ *and set of sentences* $\mathbf{T}$, $\mathbf{T}$ *has a* $\tau^{\Sigma+}$ *model system iff every finite subset of* $\mathbf{T}$ *does.*

Very often in modal logic the existence or nonexistence of a certain type of individual is important. The relevant model-theoretic notion is of a consistent set of formulas in one free variable.

Theorem 4 (Omitting "worlds" and individuals) *Let* $\mathbf{T}$ *be a theory in a modal logic* $\mathbf{L}$ *and let* Γ *be a consistent infinite set of formulas of* $\tau^{\Sigma+}$ *with one free variable—either of sort* $\mathbf{I}$ *or sort* $\mathbf{M}$. *(Γ is a description of a possible "world" or possible individual.) Assume that for every* $\tau^{\Sigma+}$ *formula* $\phi(x)$ *we have that if* $\exists x \phi(x)$ *is consistent with* $\mathbf{T}^{\Sigma+}$*, then there exists a* $\gamma \in \Gamma \ni \exists x(\phi(x) \wedge \neg\gamma(x))$ *that is consistent with* $\mathbf{T}^{\Sigma+}$. *($\mathbf{T}$ locally omits Γ.) Then* $\mathbf{T}$ *has a (countable) model system in which no "world" or individual satisfies all the formulas of* Γ.

Proof: This is just a restatement of the omitting types theorem of predicate logic.

By the extended omitting types theorem we can similarly insure that if T locally omits each set of formulas, Γ_n, $n < w$, then T has a model system omitting each of the infinitely many "worlds" or individuals.

Moreover, these results can be restated in terms of model systems $\mathfrak{M}^{\Sigma+}$. Thus the above theorem can be put as follows: given any model system $\mathfrak{M}^{\Sigma+}$, there is a model system $\mathfrak{N}^{\Sigma+}$, elementarily equivalent to $\mathfrak{M}^{\Sigma+}$, which omits any set Γ (or sets Γ_n, $n < w$) that the theory of $\mathfrak{M}^{\Sigma+}$ locally omits. The theory of $\mathfrak{M}^{\Sigma+}$ is the set of sentences true in $\mathfrak{M}^{\Sigma+}$.

On the other hand, we can insure by compactness that T has a model system which realizes a set of formulas Γ if every finite subset of Γ is consistent with T.

More generally we can show the following:

Theorem 5 *Given any model system $\mathfrak{M}^{\Sigma+}$, there is a model system $\mathfrak{N}^{\Sigma+}$, elementarily equivalent to $\mathfrak{M}^{\Sigma+}$, which is saturated (realizes all "worlds" or individuals consistent with $\mathfrak{M}^{\Sigma+}$).*

Proof: This comes from the theorem on saturated models of predicate logic.

This result can, of course, be restated in terms of theories T. Further analogues of model-theoretic results (interpolation, generic model systems) as well as model-theoretic accounts of dyadic modal operators will be the subject of a future paper.

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