Notre Dame Journal of Formal Logic
Volume XVII, Number 3, July 1976
NDJFAM

THE MODALITIES OF $KT4_nMG$

J. B. BEARD

1 In [1], p. 260, Hughes and Cresswell state the following result due to Sugihara [5], namely, that any $S4_n$ system (henceforth $KT4_n$ in the terminology of Segerberg [3]), obtained by adding to KT an axiom of the form $L^n p \supset L^{n+1}p$, has an infinite number of non-equivalent modalities if $n > 1$. In this paper* it is shown that the addition to each $KT4_n$ of the axioms:

M: $LMp \supset MLp$

and

G: $MLp \supset LMp$

and hence of the modality reduction principle:

MG: $LM = ML$

results in a distinct system $KT4_nMG$ with only finitely many non-equivalent modalities.

2 $KT4_0$ is the trivial system (collapsed into **PC**) which has two non-equivalent modalities and $KT4_1MG$ is the system called K2 in Sobociński [4]. Both systems are, of course, extensions of $KT4_1$ (i.e., S4) and are covered in the study of Pledger [2]. When $n > 1$, $KT4_nMG$ is independent of $KT4_1$. This, together with the distinctness of all the $KT4_nMG$ systems, can be proved as follows. It is easy to check that for each $n \in Nat$ the following is a frame for $KT4_nMG$:

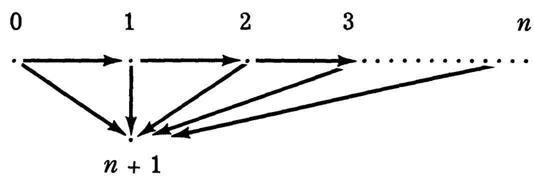

Figure 1

*I am indebted to Professors G. E. Hughes and M. J. Cresswell, Dr. R. I. Goldblatt and Mr. K. E. Pledger for some valuable discussions on the topic of this paper.

Received October 28, 1975

$4_{n-1}(n > 1)$, i.e., $L^{n-1}p \supset L^n p$, however, is falsified at 0 in the model on such a frame in which $V(p) = \{0, 1, \ldots, n-1, n+1\}$. Hence 4_{n-1} is not a theorem of $KT4_nMG$; and in particular 4_1: $Lp \supset L^2p$, the $KT4_1$ axiom, is not a theorem of $KT4_nMG$ when $n > 1$. Moreover, as is well-known, neither **M** nor **G** is a theorem of $KT4_1$; hence $KT4_1$ does not contain any $KT4_nMG$ system. Figure 2 illustrates the containment relations holding between the systems considered in this paper.

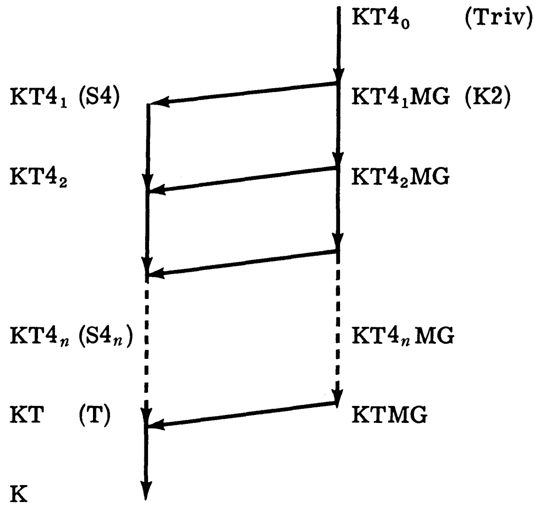

Figure 2

3 Sugihara shows that $KT4_2$ has infinitely many non-equivalent modalities in the sequence:

$$Lp;\ MLp;\ LMLp;\ MLMLp;\ LMLMLp;\ etc.,$$

and its dual obtained by replacing each L by M and vice versa. In each $KT4_nMG$, however, the following is a theorem schema: $L^aM^bp \equiv LMp$ $(a, b \geqslant 1)$.

Proof:

T:	(1) $(L^2M(p \supset Lp) \supset ML^2(p \supset Lp)) \supset (Lp \supset ML^3p)$
K:	(2) $L^2M(p \supset Lp) \supset L^2M(p \supset Lp)$
(2) MG:	(3) $L^2M(p \supset Lp) \supset ML^2(p \supset Lp)$
(1) (3) MP:	(4) $Lp \supset ML^3p$
(4) MG:	(5) $Lp \supset L^3Mp$
(4) Dual:	(6) $LM^3p \supset Mp$
(5) US:	(7) $LM^2p \supset L^3M^3p$
(6) US, MG:	(8) $L^3M^3p \supset L^2Mp$
T:	(9) $L^2Mp \supset LMp$
T:	(10) $LMp \supset LM^2p$
(7) (8) (9) (10) Syll:	(11) $L^2Mp \equiv LMp$
(11) Dual, MG:	(12) $LMp \equiv LM^2p$

(11) MI:	(13) $L^aMp \equiv LMp$ $(a, b \geqslant 1)$
(12) MI:	(14) $LMp \equiv LM^bp$ $(a, b \geqslant 1)$
(13) (14) Syll:	(15) $L^aM^b \equiv LMp$ $(a, b \geqslant 1)$

Q.E.D.

This means that any affirmative modality containing at least one L and at least one M is equivalent to LM (and so by **MG** to ML). Since $L^{n+m} = L^n$ and $M^{n+m} = M^n$, this entails that there are only finitely many modalities in each $KT4_nMG$. Furthermore, both $p \supset LMp$ and $Mp \supset LMp$ are falsified at 0 in the model on the frame of section **2** in which $V(p) = \{0\}$, and $LMp \supset p$ and $LMp \supset Lp$ are falsified at 0 in the model in which $V(p) = \{n + 1\}$. Hence I, the improper or empty modality, and LM are independent, and L, LM, and M are all distinct, in every $KT4_nMG$. Consequently the modality patterns can be read off from the following diagram:

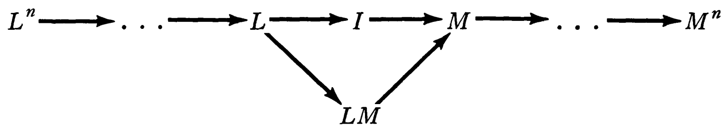

Figure 3

It is easy to see that the total number of non-equivalent modalities (including negative onces) is $4(n + 1)$ in each case.

REFERENCES

[1] Hughes, G. E., and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen, London (1968).

[2] Pledger, K. E., "Modalities of systems containing S3," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 18 (1972), pp. 267-283.

[3] Segerberg, K., *An Essay in Classical Modal Logic*, 3 Vols., Filosofiska Studier, Uppsala (1971).

[4] Sobociński, B., "Remarks about axiomatizations of certain modal systems," *Notre Dame Journal of Formal Logic*, vol. V (1964), pp. 71-80.

[5] Sugihara, T., "The number of modalities in T supplemented by the axiom CL^2pL^3p," *The Journal of Symbolic Logic*, vol. 27 (1962), pp. 407-408.

Victoria University of Wellington
Wellington, New Zealand