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# A COMPLETE CLASSIFICATION OF THREE-PLACE FUNCTORS IN TWO-VALUED LOGIC

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1 *Introduction* Sobociński [5] has shown that there exists a functor of four arguments which may define any of the functors of one or two arguments by substitution of the variables $p$, $q$ or constants 0, 1 into its arguments, this functor only being used once in any definition. Such a functor is said to *generate* all the binary functors, and is said to correspond to a universal decision element. Sobociński also proved that no three-place functor can correspond to a universal decision element, since it is unable to generate sufficient binary functors. In [3], the present author considered certain three-place functors which generate particular subsets of the binary functors and proved some results as to which subsets could be generated. In the present paper, we consider all 256 three-place functors and classify them according to the subsets of the binary functors they generate, again subject to the restriction that the functor is only used once in such a definition. The 256 functors are divided into 40 basically distinct classes, it being easily shown that all elements of the same class generate essentially the same set of binary functors.

The detailed derivation of the classes is achieved by investigating 10 basic classes, all the others being deducible from these. For each of the classes, a complete listing of all the binary functors generated is given. Included in this list are all the three-place functors investigated by previous authors. For instance, Church's conditioned disjunction [1] is our class 27 and the sole sufficient operator of Wesselkamper [6] is class 43. The QUDEs given in [3] are classes 45 and 210, which, of course, generate the largest subsets of binary functors. As will be seen, the class numbers yield the truth-tables of the members of the class. The list also indicates which of the classes contain pseudo-Sheffer functions. ($Upq$ is a pseudo-Sheffer function if the calculus based on $U$ and the constants 0, 1 is complete. More details will be found in [4], though our definition is slightly different from that of Rose in that we include the Sheffer functions as a subset of the pseudo-Sheffer functions.)

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For a binary functor $Uxy$, we define its value sequence to be $\langle klmn \rangle$ where $k, l, m, n$ are specified by the truth-table shown ($k, l, m, n \in \{0, 1\}$; 0, 1 being used for both the logical constants and the truth-values they assume).

| $x$ | $y$ | $Uxy$ |
|---|---|---|
| 0 | 0 | $k$ |
| 0 | 1 | $l$ |
| 1 | 0 | $m$ |
| 1 | 1 | $n$ |

Similarly, the three-place functor $\Delta xyz$ specified by the table has a value sequence $\langle abcdefgh \rangle$. For such a functor, we shall identify it by a description number, which is just the decimal equivalent of the binary number $(abcdefgh)$, where $a$ is the most significant bit. Thus 107 is the description number of the functor with value sequence $\langle 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1 \rangle$. $\Delta(107)$ will also be used to denote this functor.

| $x$ | $y$ | $z$ | $\Delta xyz$ |
|---|---|---|---|
| 0 | 0 | 0 | $a$ |
| 0 | 0 | 1 | $b$ |
| 0 | 1 | 0 | $c$ |
| 0 | 1 | 1 | $d$ |
| 1 | 0 | 0 | $e$ |
| 1 | 0 | 1 | $f$ |
| 1 | 1 | 0 | $g$ |
| 1 | 1 | 1 | $h$ |

From $\Delta xyz$, it is possible to define a maximum of nine binary functors by substitution of the variables $p$, $q$ or the constants 0, 1 into its arguments, subject to the following conditions:

a) the resulting functor must contain both $p$ and $q$;
b) the first substitution of $p$ into $\Delta xyz$ is in a place preceding the first substitution of $q$.

The nine possible substitutions and the resulting binary value sequence are listed below:

| substitution | value sequence |
|---|---|
| $x/p,\ y/p,\ z/q$ | $\langle abgh \rangle$ |
| $x/p,\ y/q,\ z/p$ | $\langle acfh \rangle$ |
| $x/p,\ y/q,\ z/q$ | $\langle adeh \rangle$ |
| $x/0,\ y/p,\ z/q$ | $\langle abcd \rangle$ |
| $x/p,\ y/0,\ z/q$ | $\langle abef \rangle$ |
| $x/p,\ y/q,\ z/0$ | $\langle aceg \rangle$ |
| $x/p,\ y/q,\ z/1$ | $\langle bdfh \rangle$ |
| $x/p,\ y/1,\ z/q$ | $\langle cdgh \rangle$ |
| $x/1,\ y/p,\ z/q$ | $\langle efgh \rangle$ |

2 *Notation for the binary functors* We shall use Łukasiewicz's notation for the 16 binary functors, and it is given in the table below:

| | notation | value sequence | functor |
|---|---|---|---|
| 1 | $O$ | $\langle 0000 \rangle$ | |
| 2 | $K$ | $\langle 0001 \rangle$ | conjunction |
| 3 | $L$ | $\langle 0010 \rangle$ | nonimplication |
| 4 | $I$ | $\langle 0011 \rangle$ | |
| 5 | $M$ | $\langle 0100 \rangle$ | nonimplication |
| 6 | $H$ | $\langle 0101 \rangle$ | |

| | notation | value sequence | functor |
|---|---|---|---|
| 7 | $J$ | $\langle 0110\rangle$ | exclusive or (nonequivalence) |
| 8 | $A$ | $\langle 0111\rangle$ | disjunction |
| 9 | $X$ | $\langle 1000\rangle$ | joint denial |
| 10 | $E$ | $\langle 1001\rangle$ | equivalence |
| 11 | $G$ | $\langle 1010\rangle$ | |
| 12 | $B$ | $\langle 1011\rangle$ | implication |
| 13 | $F$ | $\langle 1100\rangle$ | |
| 14 | $C$ | $\langle 1101\rangle$ | implication |
| 15 | $D$ | $\langle 1110\rangle$ | incompatibility |
| 16 | $V$ | $\langle 1111\rangle$ | |

3 *Derivation of the conjugacy classes* From a particular three-place functor $\Delta xyz$ with value sequence $\langle abcdefgh\rangle$, it is possible to obtain five further functors by permutations of the input variables $x$, $y$, $z$. These have value sequences $\langle abefcdgh\rangle$, $\langle aecgbfdh\rangle$, $\langle acbdegfh\rangle$, $\langle aebfcgdh\rangle$, $\langle acegbdfh\rangle$ resulting from the cycles $(xy)$, $(xz)$, $(yz)$, $(xyz)$, $(xzy)$ respectively. If all six of these value sequences are distinct, then $\Delta xyz$ is *fully conjugated* (this term is introduced in [2]). If all the value sequences are identical, then the functor is *unconjugated* while the only other possibility is that there are three distinct functors amongst the six value sequences. This is shown by the result below and, in this case, the functor is called *half-conjugated.* A class is described as fully conjugated if all its elements are fully conjugated, and similarly for the other two terms.

The following result is very straightforward and we do not include all the details of the proof.

Theorem 3.1 *All three-place binary functors are either fully conjugated, half-conjugated, or unconjugated.*

It is necessary to show that any functor which is not fully conjugated must either be half-conjugated or unconjugated. Initially, if the six value sequences are not all distinct, then at least one of the following conditions must hold:

$$\text{(i)}\quad e = c,\ d = f,$$
$$\text{(ii)}\quad e = b,\ d = g,$$
$$\text{(iii)}\quad b = c,\ g = f.$$

The three cases are similar and we only consider (i). In this case, the six value sequences reduce to $\langle abcdcgh\rangle$, $\langle accgbddh\rangle$, and $\langle acbdcgdh\rangle$ so that the functor is half-conjugated, if these are distinct, while any further identification which makes two of them identical also results in the third being identical, yielding an unconjugated functor.

The functors defined by the six value sequences form a conjugacy class which we say is generated by any of these value sequences. It is clear from the structure of the six value sequences that a number of other classes may be deduced from any given one. $\{\langle abcdefgh\rangle\}$ denotes the conjugacy class generated by $\langle abcdefgh \rangle$, and $\bar{a}, \bar{b}, \ldots, \bar{h}$ denote the opposite constants from $a, b, \ldots, h$ respectively.

Clearly, if $\alpha = \{\langle abcdefgh \rangle\}$ is a conjugacy class, then so are:

$$\begin{aligned}
\alpha_1 &= \{\langle abcdefg\bar{h} \rangle\}, \\
\alpha_2 &= \{\langle a\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}\bar{g}h \rangle\}, \\
\alpha_3 &= \{\langle a\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}\bar{g}\bar{h} \rangle\}, \\
\alpha_4 &= \{\langle \bar{a}bcdefgh \rangle\}, \\
\alpha_5 &= \{\langle \bar{a}bcdefg\bar{h} \rangle\}, \\
\alpha_6 &= \{\langle \bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}\bar{g}h \rangle\}, \\
\alpha_7 &= \{\langle \bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}\bar{g}\bar{h} \rangle\}.
\end{aligned}$$

It is obvious that all eight classes are disjoint and it follows that all the conjugacy classes may be deduced by limiting consideration to those of the form $\{\langle 00cdefg0 \rangle\}$.

If $\alpha$ denotes a conjugacy class, we shall use $\beta$ to denote the set containing all the description numbers of the functors in $\alpha$. Each conjugacy class whose members all have description numbers less than 128 will be identified by the smallest of these numbers, and each class whose members all have description numbers greater than 127 will be identified by the largest of these numbers. For a class $\beta$, the class identifier is denoted by $\beta^*$. Clearly, no class can contain elements with description numbers $p, q$ such that $p < 128$ and $q \geqslant 128$, since $a$ is the same for all members of a class. The purpose of making the distinction at 128 in the definition of the class identifier is to ensure that the dual of a class $\beta^*$ is $255 - \beta^*$.

If $\alpha = \{\langle 00cdefg0 \rangle\}$, then the other seven classes may be defined by:

$$\begin{aligned}
\beta_1 &= \{p \mid p = q + 1,\ q \in \beta\}, \\
\beta_2 &= \{p \mid p = 126 - q,\ q \in \beta\}, \\
\beta_3 &= \{p \mid p = 127 - q,\ q \in \beta\}, \\
\beta_4 &= \{p \mid p = 128 + q,\ q \in \beta\}, \\
\beta_5 &= \{p \mid p = 129 + q,\ q \in \beta\}, \\
\beta_6 &= \{p \mid p = 254 - q,\ q \in \beta\}, \\
\beta_7 &= \{p \mid p = 255 - q,\ q \in \beta\}.
\end{aligned}$$

If we use the above to deduce eight conjugacy classes from any given one, there are only ten distinct classes to consider. These are listed below:

a) fully conjugated functors

$$\begin{aligned}
10 &= \{10, 12, 34, 48, 68, 80\}, \\
26 &= \{26, 28, 38, 52, 70, 82\};
\end{aligned}$$

b) half-conjugated functors

$$\begin{aligned}
02 &= \{02, 04, 16\}, \\
06 &= \{06, 18, 20\}, \\
08 &= \{08, 32, 64\}, \\
14 &= \{14, 50, 84\}, \\
24 &= \{24, 36, 66\}, \\
30 &= \{30, 54, 86\};
\end{aligned}$$

c) unconjugated functors

$$0 = \{0\},$$
$$22 = \{22\}.$$

These ten classes contain 32 functors and may be used to deduce another 70 classes containing the remaining 224 functors. For example, from 14, we may deduce the following eight classes:

$$14 = \{14, 50, 84\},$$
$$15 = \{15, 51, 85\},$$
$$42 = \{42, 76, 112\},$$
$$43 = \{43, 77, 113\},$$
$$212 = \{142, 178, 212\},$$
$$213 = \{143, 179, 213\},$$
$$240 = \{170, 204, 240\},$$
$$241 = \{171, 205, 241\}.$$

In the table below, see p. 434, we explicitly list the class that includes $\Delta(i)$ for $0 \leqslant i \leqslant 127$ for those cases in which $i$ is not the class identifier. The dual functors and their classes are easily deduced from this table, for example, $\Delta(205)$ is the dual of $\Delta(50)$ which is in class 14, so $\Delta(205)$ is in class 241. The duals of a class $\alpha$ and a functor $U$ will be denoted $\alpha^{\mathrm{D}}$ and $U^{\mathrm{D}}$ respectively.

**4** *Binary functors generated* Initially, it is easy to see that all members of a conjugacy class generate essentially the same set of binary functors, except possibly for the order of the variables for the noncommutative functors. Consequently, for each of the four pairs of noncommutative binary functors $L, M$; $I, H$; $G, F$; $B, C$; it is only necessary to generate one of each pair (since $Lxy = Myx$ etc.) and we shall write $L, I, G, B$ for $L$ or $M$, $I$ or $H$, $G$ or $F$, $B$ or $C$ respectively in the following work.

Although there are 80 distinct conjugacy classes, it is only necessary to consider 40 of them since it is clear that if class $\alpha$ generates binary functor $U$ (meaning every element of $\alpha$ generates $U$), then $\alpha^{\mathrm{D}}$ will generate $U^{\mathrm{D}}$. In the classification below, see p. 435, we list all of these 40 classes and the binary functors they generate. Results for the dual classes are easily deduced; for example, $\Delta(205)$ is the dual of $\Delta(50)$ which is in class 14. Consequently, $\Delta(205)$ generates $K$, $G$, $B$, and $V$ being the duals of $D$, $I$, $L$, and $O$.

However, a more useful classification than this can be made. Initially, we define a *true k-place functor* ($k > 1$) to be one which does not reduce to an $m$-place functor for some $m < k$. In the table, see p. 435,

a) no distinction is made between fully conjugated, half-conjugated, and unconjugated functors;
b) $I$, being just an input variable regenerated, is not a true binary functor;
c) similarly, $G$, an inverted input variable, is not a true binary functor. Further, if we are interested in generating $N$ (negation), then there are $\Delta$'s

| functor | class | functor | class | functor | class | functor | class | functor | class |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 02 | 38 | 26 | 68 | 10 | 86 | 30 | 108 | 106 |
| 05 | 03 | 39 | 27 | 69 | 11 | 87 | 31 | 109 | 107 |
| 12 | 10 | 48 | 10 | 70 | 26 | 88 | 44 | 112 | 42 |
| 13 | 11 | 49 | 11 | 71 | 27 | 89 | 45 | 113 | 43 |
| 16 | 02 | 50 | 14 | 72 | 40 | 90 | 60 | 114 | 46 |
| 17 | 03 | 51 | 15 | 73 | 41 | 91 | 61 | 115 | 47 |
| 18 | 06 | 52 | 26 | 74 | 44 | 92 | 46 | 116 | 46 |
| 19 | 07 | 53 | 27 | 75 | 45 | 93 | 47 | 117 | 47 |
| 20 | 06 | 54 | 30 | 76 | 42 | 94 | 62 | 118 | 62 |
| 21 | 07 | 55 | 31 | 77 | 43 | 95 | 63 | 119 | 63 |
| 28 | 26 | 56 | 44 | 78 | 46 | 96 | 40 | 120 | 106 |
| 29 | 27 | 57 | 45 | 79 | 47 | 97 | 41 | 121 | 107 |
| 32 | 08 | 58 | 46 | 80 | 10 | 98 | 44 | 122 | 110 |
| 33 | 09 | 59 | 47 | 81 | 11 | 99 | 45 | 123 | 111 |
| 34 | 10 | 64 | 08 | 82 | 26 | 100 | 44 | 124 | 110 |
| 35 | 11 | 65 | 09 | 83 | 27 | 101 | 45 | 125 | 111 |
| 36 | 24 | 66 | 24 | 84 | 14 | 102 | 60 | | |
| 37 | 25 | 67 | 25 | 85 | 15 | 103 | 61 | | |

| class | binary functors | class | binary functors |
|---|---|---|---|
| 00 | $O$ | 40 | $O\ L\ J\ X$ |
| 01 | $O\ K$ | 41 | $K\ L\ I\ J\ E$ |
| 02 | $O\ K\ L$ | 42 | $O\ L\ A\ G$ |
| 03 | $O\ K\ I$ | 43 | $K\ L\ I\ A\ B$ |
| 06 | $O\ K\ L\ J$ | 44 | $O\ L\ I\ J\ X\ G$ |
| 07 | $O\ K\ I\ A$ | 45 | $K\ L\ I\ J\ A\ E\ B$ |
| 08 | $O\ L\ X$ | 46 | $L\ I\ J\ A\ G\ D$ |
| 09 | $O\ K\ L\ I\ E$ | 47 | $L\ I\ A\ B\ V$ |
| 10 | $O\ L\ I\ G$ | 60 | $O\ I\ J\ G$ |
| 11 | $O\ K\ L\ I\ B$ | 61 | $K\ I\ J\ A\ B$ |
| 14 | $O\ L\ I\ D$ | 62 | $L\ I\ J\ A\ D$ |
| 15 | $O\ I\ V$ | 63 | $I\ A\ V$ |
| 22 | $K\ L\ J$ | 104 | $L\ J\ X$ |
| 23 | $K\ I\ A$ | 105 | $I\ J\ E$ |
| 24 | $O\ K\ L\ J\ X$ | 106 | $L\ J\ A\ X\ G$ |
| 25 | $K\ L\ J\ A\ E$ | 107 | $I\ J\ A\ E\ B$ |
| 26 | $O\ K\ L\ I\ J\ G$ | 110 | $L\ J\ A\ G\ D$ |
| 27 | $K\ L\ I\ A\ B$ | 111 | $I\ J\ A\ B\ V$ |
| 30 | $K\ L\ I\ J\ D$ | 126 | $J\ A\ D$ |
| 31 | $K\ I\ A\ V$ | 127 | $A\ V$ |

which generate $N$, but not $G$ (since $G$ is obtained as a degenerate binary functor, and we did not include unary functors in our substitutions), such as $\Delta(41)$.

Hence, the following table is divided into three sections, one for fully conjugated, one for half-conjugated, and one for unconjugated functors. Further, for each class, the list of functors generated is restricted to $K$, $L$, $J$, $A$, $X$, $E$, $B$, $D$, and $N$; that is, all the true binary functors and negation. Because of negation, it is necessary to list the dual classes explicitly.

Two further items are also included. Those functors which are pseudo-Sheffer are indicated by 'P' following the class identifer, and finally, those functors which are not true three-place functors are marked by an *.

a) Fully conjugated

| class | functors | dual class | functors |
|---|---|---|---|
| 10*P | $L\ N$ | 245*P | $B\ N$ |
| 11P | $K\ L\ B\ N$ | 244P | $L\ B\ D\ N$ |
| 26P | $K\ L\ J\ N$ | 229P | $E\ B\ D\ N$ |
| 27P | $K\ L\ A\ B\ N$ | 228P | $L\ X\ B\ D\ N$ |
| 44P | $L\ J\ X\ N$ | 211P | $A\ E\ B\ N$ |
| 45P | $K\ L\ J\ A\ E\ B\ N$ | 210P | $L\ J\ X\ E\ B\ D\ N$ |
| 46P | $L\ J\ A\ D\ N$ | 209P | $K\ X\ E\ B\ N$ |
| 47P | $L\ A\ B\ N$ | 208P | $L\ X\ B\ N$ |

b) Half-conjugated

| class | functors | dual class | functors |
|---|---|---|---|
| 02 P | $K\ L\ N$ | 253 P | $B\ D\ N$ |
| 03* | $K$ | 252* P | $D\ N$ |
| 06 P | $K\ L\ J\ N$ | 249 P | $E\ B\ D\ N$ |
| 07 | $K\ A$ | 248 P | $X\ D\ N$ |
| 08 P | $L\ X\ N$ | 247 P | $A\ B\ N$ |
| 09 P | $K\ L\ E\ N$ | 246 P | $J\ B\ D\ N$ |
| 14 P | $L\ D\ N$ | 241 P | $K\ B\ N$ |
| 15* | | 240* | $N$ |
| 24 P | $K\ L\ J\ X\ N$ | 231 P | $A\ E\ B\ D\ N$ |
| 25 P | $K\ L\ A\ E\ N$ | 230 P | $J\ X\ B\ D\ N$ |
| 30 P | $K\ L\ J\ D\ N$ | 225 P | $K\ E\ B\ D\ N$ |
| 31 | $K\ A$ | 224 P | $X\ D\ N$ |
| 40 P | $L\ J\ X\ N$ | 215 P | $A\ E\ B\ N$ |
| 41 P | $K\ L\ J\ E\ N$ | 214 P | $J\ E\ B\ D\ N$ |
| 42 P | $L\ A\ N$ | 213 P | $X\ B\ N$ |
| 43 P | $K\ L\ A\ B\ N$ | 212 P | $L\ X\ B\ D\ N$ |
| 60* | $J\ N$ | 195* | $E\ N$ |
| 61 P | $K\ J\ A\ B\ N$ | 194 P | $L\ X\ E\ D\ N$ |
| 62 P | $L\ J\ A\ D\ N$ | 193 P | $K\ X\ E\ B\ N$ |
| 63* | $A$ | 192* P | $X\ N$ |
| 106 P | $L\ J\ A\ X\ N$ | 149 P | $A\ X\ E\ B\ N$ |
| 107 P | $J\ A\ E\ B\ N$ | 148 P | $L\ J\ X\ E\ N$ |
| 110 P | $L\ J\ A\ D\ N$ | 145 P | $K\ X\ E\ B\ N$ |
| 111 P | $J\ A\ B\ N$ | 144 P | $L\ X\ E\ N$ |

c) Unconjugated

| class | functors | dual class | functors |
|---|---|---|---|
| 0* | | 255* | |
| 01 | $K$ | 254 P | $D\ N$ |
| 22 P | $K\ L\ J\ N$ | 233 P | $E\ B\ D\ N$ |
| 23 P | $K\ A$ | 232 P | $X\ D\ N$ |
| 104 P | $L\ X\ E\ N$ | 151 P | $J\ A\ B\ N$ |
| 105 | $J\ E\ N$ | 150 | $J\ E\ N$ |
| 126 P | $J\ A\ D\ N$ | 129 P | $K\ X\ E\ N$ |
| 127 | $A$ | 128 P | $X\ N$ |

A number of immediate results may now be stated concerning three-place binary functors. Of the 256 such functors, 226 are pseudo-Sheffer, and this sort of ratio appears also to be true for $n$-valued logic ($n > 2$).

Of the 256 functors, 238 will generate $N$ immediately, and the remainder are unable to do so, even if more occurrences of $\Delta$ are allowed.

It is also interesting to note that all fully conjugated functors are pseudo-Sheffer and it is thought that this result is true in general for $n$-valued logic.

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