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A SHORT EQUATIONAL AXIOMATIZATION OF ORTHOMODULAR LATTICES

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By definition, *cf.* e.g., [2], p. 53, an orthomodular lattice is an ortholattice satisfying the following formula:[1]

K1 $[ab]: a, b \in A . a \leqslant b .\supset. a \cup (a^{\perp} \cap b) = b$

In this note it will be proved that:

(**A**) *Any algebraic system*

$$\mathfrak{A} = \langle A, \cup, \cap, {}^{\perp} \rangle$$

where $\cup$ *and* $\cap$ *are two binary operations and* $^{\perp}$ *is a unary operation defined on the carrier set* A, *is an orthomodular lattice, if it satisfies the following three mutually independent postulates:*

C1 $[abcd]: a, b, c, d \in A .\supset. a \cup ((a \cup ((b \cup c) \cup d)) \cap a^{\perp}) =$
$((d^{\perp} \cap c^{\perp})^{\perp} \cup b) \cup a$

C2 $[ab]: a, b \in A .\supset. a = a \cup (b \cap b^{\perp})$

C3 $[ab]: a, b \in A .\supset. a = a \cap (a \cup b)$[2]

Proof of (**A**):

1 Clearly, postulates *C2* and *C3* are the theses of any ortholattice. It remains to prove that, in the field of an arbitrary ortholattice, *C1* is inferentially equivalent to formula *K1*.

1.1 First, we shall prove that in the field of any lattice *K1* is inferentially equivalent to formula *R1* given below.

1.1.1 Assume **L**. Then we have at our disposal:

1. Throughout this paper A indicates an arbitrary but fixed carrier set, **L** a lattice, and **OL** an ortholattice. The so-called closure axioms are assumed tacitly.

2. Of course, in this postulate-system the operations $\cup$, $\cap$ and $^{\perp}$ are not mutually independent.

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$D1$ $[ab]:a \leqslant b \;.\!\equiv.\; a \cup b = b \,.\, a \in A \,.\, b \in A$ [**L**]
$L1$ $[ab]:a,\, b \in A \,.\, a \leqslant b \;.\!\supset.\; a \cup b = b$ [$D1$]
$L2$ $[ab]:a,\, b \in A \;.\!\supset.\; a \leqslant a \cup b$ [**L**; $D1$]
$L3$ $[ab]:a,\, b \in A \;.\!\supset.\; a \cap b = b \cap a$ [**L**]

1.1.2 Now, assume **L** and $K1$. Then:

$R1$ $[ab]:a,\, b \in A \;.\!\supset.\; a \cup (a^\perp \cap (a \cup b)) = a \cup b$
PR $[ab]:$ Hp (1) $.\!\supset.$
2. $a \leqslant a \cup b\,.$ [1; $L2$]
$a \cup (a^\perp \cap (a \cup b)) = a \cup b$ [1; 2; $K1$, $b/a \cup b$]

1.1.3 Let us assume **L** and $R1$. Then:

$K1$ $[ab]:a,\, b \in A \,.\, a \leqslant b \;.\!\supset.\; a \cup (a^\perp \cap b) = b$
PR $[ab]:$ Hp (2) $.\!\supset.$
3. $a \cup b = b\,.$ [1; 2; $L1$]
$a \cup (a^\perp \cap b) = a \cup (a^\perp \cap (a \cup b)) = a \cup b = b$ [1; 3; $R1$; 3]

1.1.4 Thus, from the deductions presented above, it follows that in the field of any lattice we have

$$\{K1\} \rightleftarrows \{R1\}$$

1.2 Now, let us assume an arbitrary ortholattice. Then, obviously, in its field, the equivalence $\{K1\} \rightleftarrows \{R1\}$ and the formula $L3$ hold. Moreover, we have at our disposal:

$M1$ $[abcd]:a,\, b,\, c,\, d \in A \;.\!\supset.\; a \cup ((b \cup c) \cup d) = ((d^\perp \cap c^\perp)^\perp \cup b) \cup a$
[**OL**, *cf.* [1], p. 251]
$M2$ $[b]:b \in A \;.\!\supset.\; (b \cup (b \cap b^\perp)) \cup (b \cap b^\perp) = b$ [**OL**]
$M3$ $[ab]:a,\, b \in A \;.\!\supset.\; (((b \cap b^\perp)^\perp \cap (b \cap b^\perp)^\perp)^\perp \cup b) \cup a = a \cup b$ [**OL**]

1.2.1 Assume **OL** and $R1$. Then:

$C1$ $[abcd]:a,\, b,\, c,\, d \in A \;.\!\supset.\; a \cup ((a \cup ((b \cup c) \cup d)) \cap a^\perp) =$
$((d^\perp \cap c^\perp)^\perp \cup b) \cup a$
PR $[abcd]:$ Hp (1) $.\!\supset.$
$a \cup ((a \cup ((b \cup c) \cup d)) \cap a^\perp) = a \cup (a^\perp \cap (a \cup ((b \cup c) \cup d)))$
[1; $L3$, $a/a \cup ((b \cup c) \cup d)$, $b/a^\perp$]
$= a \cup ((b \cup c) \cup d)$ [$R1$, $b/(b \cup c) \cup d$]
$= ((d^\perp \cap c^\perp)^\perp \cup b) \cup a$ [$M1$]

1.2.2 Assume **OL** and $C1$. Then:

$R1$ $[ab]:a,\, b \in A \;.\!\supset.\; a \cup (a^\perp \cap (a \cup b)) = a \cup b$
PR $[ab]:$ Hp (1) $.\!\supset.$
$a \cup (a^\perp \cap (a \cup b)) = a \cup ((a \cup b) \cap a^\perp)$ [1; $L3$, $a/a^\perp$, $b/a \cup b$]
$= a \cup ((a \cup ((b \cup (b \cap b^\perp)) \cup (b \cap b^\perp))) \cap a^\perp)$ [$M2$]
$= (((b \cap b^\perp)^\perp \cap (b \cap b^\perp)^\perp)^\perp \cup b) \cup a$
[$C1$, $c/b \cap b^\perp$, $d/b \cap b^\perp$]
$= a \cup b$ [$M3$]

1.3 From sections **1.1** and **1.2** it follows at once that in the field of any ortholattice

$$\{K1\} \rightleftarrows \{R1\} \rightleftarrows \{C1\}$$

Therefore, it is proved that formulas *C1*, *C2*, and *C3* are the theses of any orthomodular lattice.

2 Now, let us assume *C1*, *C2*, and *C3*. Then:

C4 $[ab]:a,\ b \in A\ .\supset.\ a = ((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp} \cup a$

PR $[ab]: \mathrm{Hp}\ (1)\ .\supset.$

$a = a \cup (a \cap a^{\perp}) = a \cup ((a \cup (b \cap b^{\perp})) \cap a^{\perp})$ [1; *C2*, *b*/*a*; *C2*]

$= a \cup ((a \cup ((b \cap b^{\perp}) \cup (b \cap b^{\perp}))) \cap a^{\perp})$ [$C2,\ a/b \cap b^{\perp}$]

$= a \cup ((a \cup (((b \cap b^{\perp}) \cup (b \cap b^{\perp})) \cup (b \cap b^{\perp}))) \cap a^{\perp})$ [$C2,\ a/b \cap b^{\perp}$]

$= (((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp} \cup (b \cap b^{\perp})) \cup a$

[$C1,\ b/b \cap b^{\perp},\ c/b \cap b^{\perp},\ d/b \cap b^{\perp}$]

$= ((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp} \cup a$ [$C2,\ a/((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp}$]

C5 $[b]:b \in A\ .\supset.\ b \cap b^{\perp} = ((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp}$

PR $[b]: \mathrm{Hp}\ (1)\ .\supset.$

$b \cap b^{\perp} = ((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp} \cup (b \cap b^{\perp})$ [1; $C4,\ a/b \cap b^{\perp}$]

$= ((b \cap b^{\perp}) \cap (b \cap b^{\perp})^{\perp})^{\perp}$ [$C2,\ a/((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp}$]

C6 $[ab]:a,\ b \in A\ .\supset.\ a = (b \cap b^{\perp}) \cup a$ [*C4*; *C5*]

C7 $[abcd]:a,\ b,\ c,\ d \in A\ .\supset.\ ((a \cup c) \cup d) \cap (b \cap b^{\perp})^{\perp} = (d^{\perp} \cap c^{\perp})^{\perp} \cup a$

PR $[abcd]: \mathrm{Hp}\ (1)\ .\supset.$

$((a \cup c) \cup d) \cap (b \cap b^{\perp})^{\perp} = ((b \cap b^{\perp}) \cup ((a \cup c) \cup d))) \cap (b \cap b^{\perp})^{\perp}$

[1; $C6,\ a/(a \cup c) \cup d$]

$= (b \cap b^{\perp}) \cup (((b \cap b^{\perp}) \cup ((a \cup c) \cup d)) \cap (b \cap b^{\perp})^{\perp})$

[$C6,\ a/((b \cup b^{\perp}) \cup ((a \cup c) \cup d)) \cap (b \cap b^{\perp})^{\perp}$]

$= ((d^{\perp} \cap c^{\perp})^{\perp} \cup a) \cup (b \cap b^{\perp}) = (d^{\perp} \cup c^{\perp})^{\perp} \cup a$

[$C1,\ a/b \cap b^{\perp},\ b/a;\ C2,\ a/(d^{\perp} \cap c^{\perp})^{\perp} \cup a$]

C8 $[ab]:a,\ b \in A\ .\supset.\ a \cap (b \cap b^{\perp})^{\perp} = a$

PR $[ab]: \mathrm{Hp}\ (1)\ .\supset.$

$a \cap (b \cap b^{\perp})^{\perp} = (a \cup (b \cap b^{\perp})) \cap (b \cap b^{\perp})^{\perp}$ [1; *C2*]

$= ((a \cup (b \cap b^{\perp})) \cup (b \cap b^{\perp})) \cap (b \cap b^{\perp})^{\perp}$ [*C2*]

$= ((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp} \cup a = a$

[$C7,\ c/b \cap b^{\perp},\ d/b \cap b^{\perp};\ C4$]

C9 $[abc]:a,\ b,\ c \in A\ .\supset.\ (a \cup b) \cup c = (c^{\perp} \cap b^{\perp})^{\perp} \cup a$

PR $[abc]: \mathrm{Hp}\ (1)\ .\supset.$

$(a \cup b) \cup c = ((a \cup b) \cup c) \cap (b \cap b^{\perp})^{\perp} = (c^{\perp} \cap b^{\perp})^{\perp} \cup a$

[1; $C8,\ a/(a \cup b) \cup c;\ C7,\ c/b,\ d/c$]

3 Since, on the basis of deductions presented in [3], L. Beran has proved in [1] that any algebraic system which satisfies theses *C9*, *C3*, and *C2* is an ortholattice, it follows immediately from sections **1** and **2** that any algebraic system which satisfies postulates *C1*, *C2*, and *C3* is an orthomodular lattice.

4 The mutual independence of axioms *C1*, *C2*, and *C3* is established by using the following algebraic tables:[3]

$\mathfrak{M}1$

$\cup$	α	β
α	α	α
β	β	β

$\cap$	α	β
α	α	β
β	α	β

x	$x^{\perp}$
α	β
β	α

$\mathfrak{M}2$

$\cup$	α	β
α	β	β
β	β	β

$\cap$	α	β
α	α	α
β	α	β

x	$x^{\perp}$
α	β
β	α

$\mathfrak{M}3$

$\cup$	α	β
α	α	β
β	β	α

$\cap$	α	β
α	β	α
β	α	β

x	$x^{\perp}$
α	β
β	α

Namely:

(a) $\mathfrak{M}1$ verifies *C2* and *C3*, but falsifies *C1* for a/α, b/β, c/β, and d/β: (i) $\alpha \cup ((\alpha \cup ((\beta \cup \beta) \cup \beta)) \cap \alpha^{\perp}) = \alpha \cup ((\alpha \cup (\beta \cup \beta)) \cap \beta) = \alpha \cup ((\alpha \cup \beta) \cap \beta) = \alpha \cup (\alpha \cap \beta) = \alpha \cup \beta = \alpha$, (ii) $((\beta^{\perp} \cap \beta^{\perp})^{\perp} \cup \beta) \cup \alpha = ((\alpha \cap \alpha)^{\perp} \cup \beta) \cup \alpha = (\alpha^{\perp} \cup \beta) \cup \alpha = (\beta \cup \beta) \cup \alpha = \beta \cup \alpha = \beta$.
(b) $\mathfrak{M}2$ verifies *C1* and *C3*, but falsifies *C2* for a/α and b/α: (i) $\alpha = \alpha$, (ii) $\alpha \cup (\alpha \cap \alpha^{\perp}) = \alpha \cup (\alpha \cap \beta) = \alpha \cup \alpha = \beta$.
(c) $\mathfrak{M}3$ verifies *C1* and *C2*, but falsifies *C3* for a/α and b/α: (i) $\alpha = \alpha$, (ii) $\alpha \cap (\alpha \cup \alpha) = \alpha \cap \alpha = \beta$.

5 It follows immediately from sections **1**, **3**, and **4** that the proof of (**A**) is complete.

Remark: We have to note that, although, clearly, axiom *C1* is constructed in a rather mechanical way by combining formulas *R1* and *C9*, *C1* is an organic formula in the sense defined in [4], p. 60, point (c).

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[4] Sobociński, B., "On well constructed axiom systems," *Yearbook of the Polish Society of Arts and Sciences Abroad*, vol. VI (1956), pp. 54-65.

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3. Concerning $\mathfrak{M}1$ and $\mathfrak{M}3$, *cf.* [3], p. 143.