Notre Dame Journal of Formal Logic
Volume XVII, Number 2, April 1976
NDJFAM

SEMANTICS FOR S4.04, S4.4, AND S4.3.2

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1 In what follows, semantics for modal systems S4.04, S4.4, and S4.3.2 are offered in the style of Kripke, employing the terminology and techniques of Hughes and Cresswell in [2]. Semantics for S4.4 are offered by Zeman in [6] which he calls the "end of the world matrix." In constructing a model for S4.4 Zeman distinguishes between two kinds of worlds:

> . . . the distinction depending upon the properties of the accessibility relation for the respective worlds. One kind of world will have one and only one representative in any S4.4 model (this is the 'real' world); this world has access to all worlds in the model including itself; the other kind of world may have any number of representatives in an S4.4 model; this kind of world has access to all worlds in the model *except* for the real world. (*Cf.* [8], p. 454)

In section **3** of this paper an alternative interpretation for S4.4 is offered which does not distinguish between two kinds of worlds, but rather is characterized by the additional requirements it imposes upon the accessibility relation in an S4 model structure. In [8] Zeman also offers an interpretation for modal system S4.3.2 which is similar to the end of the world matrix in that it distinguishes between two kinds of worlds, but differs essentially in that it admits of no limit to the number of worlds of either kind that may belong to the model. In section **4** an alternative interpretation for S4.3.2 is also offered; again one which does not distinguish between two kinds of worlds, but merely imposes an additional requirement on the accessibility relation in an S4 model structure. In proving completeness theorems for the respective modal systems, we employ the Henkin-style completeness techniques, lemmata, and terminology given in [2], pp. 150-159.

2 In [7] Zeman constructs modal system S4.04 by adding, to some base for S4 containing a primitive rule of necessitation, the following formula:

L1 *CLMLpCpLp*

Now it is quite obvious that *L1* is deductively equivalent to

Received November 24, 1973

L3 $CNpCMpMLMp$

in a field at least as weak as T; thus modal system S4.04 may alternatively be axiomatized by replacing *L1* by *L3*. We point this out because our completeness proof for S4.04 will make use of *L3*.

In order to construct a semantic model for S4.04, we need only impose the requirement of what we shall call "remote symmetricality" on the accessibility relation in an S4 model $\langle W, R, \mathbf{V}\rangle$, where W is a set of objects (worlds), R is a reflexive and transitive relation defined over the members of W, and $\mathbf{V}$ is a value assignment satisfying the conditions specified in [2], p. 73. We say that R is *remotely symmetrical* iff for every $w_i, w_j, w_k \epsilon W$, if w_iRw_j and w_jRw_k, then either w_kRw_j or $w_i = w_j$. Since modal system S4.04 is a proper extension of S4, we can demonstrate the soundness of our interpretation by simply showing that *L1* is S4.04-valid. We do this in the following fashion.

Assume for the sake of reductio that $\mathbf{V}(CLMLpCpLp, w_i) = 0$. Clearly it follows that

$$(1)\quad \mathbf{V}(LMLp, w_i) = 1$$
$$(2)\quad \mathbf{V}(CpLp, w_i) = 0$$

From (2) we have

$$(3)\quad \mathbf{V}(p, w_i) = 1$$
$$(4)\quad \mathbf{V}(Lp, w_i) = 0$$

Now from (4) we have

$$(5)\quad \mathbf{V}(p, w_j) = 0$$

and from (1) it follows that

$$(6)\quad \mathbf{V}(MLp, w_j) = 1$$

Thus from (6) we have

$$(7)\quad \mathbf{V}(Lp, w_k) = 1$$

Now since R is remotely symmetrical, we have either w_kRw_j or $w_i = w_j$. If w_kRw_j, then it follows from (7) that

$$(8)\quad \mathbf{V}(p, w_j) = 1$$

which is inconsistent with (5). If $w_i = w_j$, then clearly (3) and (5) are inconsistent. Either way we have an inconsistency and so $\mathbf{V}(CLMLpCpLp, w_i) = 1$.

Before proceeding with the completeness theorem for S4.04, we state and prove the following additional lemma concerning maximal consistent sets:

Lemma 4 If Γ is maximal consistent relative to S (where S contains some adequate axiomatic version of **PC**), then if $A\alpha\beta \epsilon \Gamma$ then either $\alpha \epsilon \Gamma$ or $\beta \epsilon \Gamma$.

Proof:[1] We prove this lemma by showing that if neither $\alpha \in \Gamma$ nor $\beta \in \Gamma$ then $A\alpha\beta \notin \Gamma$. If neither $\alpha \in \Gamma$ nor $\beta \in \Gamma$ then (by Lemma 2) $N\alpha \in \Gamma$ and $N\beta \in \Gamma$. But $CN\alpha CN\beta NA\alpha\beta$ is a thesis of **PC**, thus (by corollary of Lemma 2; *cf.* [2], p. 153) $CN\alpha CN\beta NA\alpha\beta \in \Gamma$ and hence (by Lemma 3) $NA\alpha\beta \in \Gamma$. Therefore, (by Lemma 1) $A\alpha\beta \notin \Gamma$.

In dealing with the completeness theorem for S4.04, we must require that R be remotely symmetrical. We therefore have to say that whenever Γ_j is a subordinate of Γ_i and Γ_k subordinate to Γ_j, then either $w_k R w_j$ or $w_i = w_j$. This means that we have to add to the S4 proof that Theorem 2 holds for L, (*cf.* [2], pp. 157-158), a proof that if Γ_j is subordinate to Γ_i and Γ_k subordinate to Γ_j, then either if $L\beta \in \Gamma_k$ then $\beta \in \Gamma_j$ or $L\beta \in \Gamma_j$ iff $L\beta \in \Gamma_i$ (Γ_i, Γ_j, and Γ_k are all assumed to be maximal consistent with respect to S4.04).

The proof proceeds by showing that if $\beta \notin \Gamma_j$, then either $L\beta \notin \Gamma_k$ or $L\beta \in \Gamma_j$ iff $L\beta \in \Gamma_i$. If $\beta \notin \Gamma_j$ then (by Lemma 2) $N\beta \in \Gamma_j$. Now since $CN\beta CM\beta MLM\beta$ is a thesis of S4.04, it follows (by corollary of Lemma 2) that $CN\beta CM\beta MLM\beta \in \Gamma_j$ and so (by Lemma 3) $CM\beta MLM\beta \in \Gamma_j$. But $\vdash CCM\beta MLM\beta ANM\beta MLM\beta$, thus (by corollary of Lemma 2) $CCM\beta MLM\beta ANM\beta MLM\beta \in \Gamma_j$ and so (by Lemma 3) $ANM\beta MLM\beta \in \Gamma_j$. Consequently (by Lemma 4) either $NM\beta \in \Gamma_j$ or $MLM\beta \in \Gamma_j$. If $NM\beta \in \Gamma_j$ then since $\vdash CNM\beta LNL\beta$ we have (by corollary of Lemma 2) $CNM\beta LNL\beta \in \Gamma_j$ and so (by Lemma 3) $LNL\beta \in \Gamma_j$. Therefore (by construction of Γ_k) we have $NL\beta \in \Gamma_k$ and so (by Lemma 1) $L\beta \notin \Gamma_k$.

Now we wish to show that if $MLM\beta \in \Gamma_j$ then $L\beta \in \Gamma_j$ iff $L\beta \in \Gamma_i$. Assume for the sake of reductio that it is not the case that $L\beta \in \Gamma_j$ iff $L\beta \in \Gamma_i$, then clearly it follows that either $L\beta \notin \Gamma_j$ and $L\beta \in \Gamma_i$ or $L\beta \in \Gamma_j$ and $L\beta \notin \Gamma_i$. If $L\beta \notin \Gamma_j$ and $L\beta \in \Gamma_i$ then (by construction of Γ_j) $\beta \in \Gamma_j$. But $\beta \notin \Gamma_j$ (by hypothesis). If $L\beta \in \Gamma_j$ and $L\beta \notin \Gamma_i$ then since $\vdash CL\beta\beta$ we have (by Lemma 3) $\beta \in \Gamma_j$. But again $\beta \notin \Gamma_j$ (by hypothesis). Either way then we have an inconsistency and so $L\beta \in \Gamma_i$ iff $L\beta \in \Gamma_j$.

3 Modal system S4.2 is axiomatized by adding, to some base for S4 containing a primitive rule of necessitation, the following formula:

G1 $CMLpLMp$

Now in [3], pp. 27-29, Prior discusses a temporal interpretation for S4.2 which imposes the requirement of convergence on the accessibility relation in addition to those requirements needed for an S4 model. In [1] Hazen provides a semantic interpretation for S4.2 by adding to the requirements for an S4 model the stipulation that for any two worlds in an S4.2 model structure there is a third world accessible from both of them. He then demonstrates that S4.2 is both sound and complete on this interpretation.

Now clearly we might sum up Hazen's interpretation as follows: for any w_i, w_j, $w_k \in W$, if $w_i R w_j$ and $w_i R w_k$, then there exists a $w_l \in W$ such that

1. The lemmas 1, 2 and 3 used in this proof are given in [2], pp. 152-153.

$w_j R w_l$ and $w_k R w_l$. But this is just what Prior means by convergence. Thus, a semantic model for S4.2 is afforded by imposing the requirement of convergence on the accessibility relation in an S4 model structure.

In [5] Sobociński introduces modal system S4.4 by adding

R1 *CMLpCpLp*

to the basis of S4. In [4], p. 354, Sobociński remarks that the addition of *L1* as an axiom to any extension of S4 containing the proper axiom of S4.2, viz. *G1*, yields S4.4. This consideration suggests that a semantic model for S4.4 can be constructed by simply imposing the requirement of remote symmetricality to the accessibility relation of an S4.2 model structure where the accessibility relation is reflexive, transitive, and convergent.

The soundness theorem for S4.4 is easily demonstrated by merely proving that *R1* is S4.4-valid. We prove it thus: Assume for the sake of reductio that $\mathsf{V}(CMLpCpLp, w_i) = 0$, it then follows that

$$(1)\quad \mathsf{V}(MLp, w_i) = 1$$
$$(2)\quad \mathsf{V}(CpLp, w_i) = 0$$

Hence it follows from (2) that

$$(3)\quad \mathsf{V}(p, w_i) = 1$$
$$(4)\quad \mathsf{V}(Lp, w_i) = 0$$

Now from (1) it follows that

$$(5)\quad \mathsf{V}(Lp, w_j) = 1$$

and from (4) that

$$(6)\quad \mathsf{V}(p, w_k) = 0$$

Since R is convergent, it follows that there exists a $w_l \epsilon W$ such that $w_j R w_l$ and $w_k R w_l$. But R is also remotely symmetrical, thus if $w_i R w_k$ and $w_k R w_l$, then either $w_l R w_k$ or $w_i = w_k$. If $w_l R w_k$, then since both $w_j R w_l$ and R is transitive, we have from (5) that

$$(7)\quad \mathsf{V}(p, w_k) = 1$$

But this is inconsistent with (6). If $w_i = w_k$, then (3) and (6) are inconsistent with each other. Hence either way we have an inconsistency and so $\mathsf{V}(CMLpCpLp, w_i) = 1$.

Quite obviously the completeness theorem for S4.4 proceeds in similar fashion as that of S4.04.

4 Zeman introduces modal system S4.3.2 in [9] by appending

F1 *ALCLpqCMLqp*

to an axiomatic basis of S4. A semantic model for S4.3.2 is easily constructed by merely imposing the requirement of what I shall call "non-branching" to the accessibility relation in an S4 model structure. To say that R is non-branching is to say that for any w_i, w_j, $w_k \in W$, if $w_i R w_j$ and $w_i R w_k$, then either $w_j R w_i$ or $w_k R w_j$.

We prove the soundness theorem for S4.3.2 by showing that $F1$ is S4.3.2-valid. Assume for the sake of reductio that $\mathsf{V}(ALCLpqCMLqp, w_i) = 0$. Then clearly it follows that

$$(1)\ \mathsf{V}(LCLpq, w_i) = 0$$
$$(2)\ \mathsf{V}(CMLqp, w_i) = 0$$

From (1) it follows that

$$(3)\ \mathsf{V}(CLpq, w_j) = 0$$

and so

$$(4)\ \mathsf{V}(Lp, w_j) = 1$$
$$(5)\ \mathsf{V}(q, w_j) = 0$$

From (2) it follows that

$$(6)\ \mathsf{V}(MLq, w_i) = 1$$
$$(7)\ \mathsf{V}(p, w_i) = 0$$

From (6) it follcws that

$$(8)\ \mathsf{V}(Lq, w_k) = 1$$

Now since R is non-branching, we have either $w_j R w_i$ or $w_k R w_j$. If $w_j R w_i$, it follows from (4) that

$$(9)\ \mathsf{V}(p, w_i) = 1$$

But this is inconsistent with (7). If $w_k R w_j$, then it follows from (8) that

$$(10)\ \mathsf{V}(q, w_j) = 1$$

which is inconsistent with (5). Hence it follows that $\mathsf{V}(ALCLpqCMLpq, w_i) = 1$.

We now turn to the completeness theorem for S4.3.2. To deal with this system we must require that R not only be reflexive and transitive, but non-branching as well. We therefore have to say that whenever both Γ_j and Γ_k are subordinates of Γ_i, then either $w_j R w_i$ or $w_k R w_j$. This means that we have to add to the S4 proof that Theorem 2 holds for L a proof that if both Γ_j and Γ_k are subordinates to Γ_i, then either if $L\beta \in \Gamma_j$, then $\beta \in \Gamma_i$ or if $L\gamma \in \Gamma_k$ then $\gamma \in \Gamma_j$. Alternatively, we say that if both Γ_j and Γ_k are subordinates of Γ_i, then if both $L\beta \in \Gamma_j$ and $L\gamma \in \Gamma_k$, then either $\beta \in \Gamma_i$ or $\gamma \in \Gamma_j$.

We prove this by showing that if neither $\beta \in \Gamma_i$ nor $\gamma \in \Gamma_j$, then either $L\beta \notin \Gamma_j$ or $L\gamma \notin \Gamma_k$. Assume that neither $\beta \in \Gamma_i$ nor $\gamma \in \Gamma_j$. It clearly follows from this that $\beta \notin \Gamma_i$ and $\gamma \notin \Gamma_j$ and so (by Lemma 2) we have $N\beta \in \Gamma_i$ and $N\gamma \in \Gamma_j$. Now since $ALCL\beta\gamma CML\gamma\beta$ is a thesis of S4.3.2, it follows (by corollary of Lemma 2) that $ALCL\beta\gamma CML\gamma\beta \in \Gamma_i$ and so (by Lemma 4) either $LCL\beta\gamma \in \Gamma_i$ or $CML\gamma\beta \in \Gamma_i$. If $LCL\beta\gamma \in \Gamma_i$ then (by construction of Γ_j) we have $CL\beta\gamma \epsilon \Gamma_j$. But $\vdash CCL\beta\gamma CN\gamma NL\beta$, thus (by corollary of Lemma 2) $CCL\beta\gamma CN\gamma NL\beta \epsilon \Gamma_j$ and so (by Lemma 3) $CN\gamma NL\beta \epsilon \Gamma_j$. Now $N\gamma \epsilon \Gamma_j$ (by hypothesis), hence (again by Lemma 3) $NL\beta \epsilon \Gamma_j$ and so (by Lemma 1) $L\beta \notin \Gamma_j$. If $CML\gamma\beta \epsilon \Gamma_i$, then since $\vdash CCML\gamma\beta CN\beta NML\gamma$, we have (by corollary of Lemma 2) $CCML\gamma\beta CN\beta NML\gamma \epsilon \Gamma_i$ and so (by Lemma 3) $CN\beta NML\gamma \epsilon \Gamma_i$. Now $N\beta \epsilon \Gamma_i$ (by hypothesis) hence (by Lemma 3) $NML\gamma \epsilon \Gamma_i$. But $\vdash CNML\gamma LNL\gamma$, thus (by corollary of Lemma 2) $CNML\gamma LNL\gamma \epsilon \Gamma_i$ and so (by Lemma 3) we have $LNL\gamma \epsilon \Gamma_i$. Now (by construction of Γ_k) we have $NL\gamma \epsilon \Gamma_k$ and so (by Lemma 1) $L\gamma \notin \Gamma_k$.

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