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Λ-ELIMINATION IN ILLATIVE COMBINATORY LOGIC

M. W. BUNDER

The system of propositional (and predicate) calculus developed in [2] on the basis of the illative combinatory logic in [1] lacked the Λ-elimination rules:

$$\Lambda XY \vdash X \tag{1}$$

and

$$\Lambda XY \vdash Y \tag{2}$$

which are required in some of the applications of this work.[1] In this paper we show that the axioms in [1] allow for stronger versions of

$$\Lambda XY, \mathbf{H}X, \mathbf{H}Y \vdash X \tag{3}$$

and

$$\Lambda XY, \mathbf{H}X, \mathbf{H}Y \vdash Y \tag{4}$$

(which were used in [2]) and that the addition of further axioms can lead to stronger versions still. To obtain (1) and (2), however, we need a strengthening of the axioms which alters the interpretation of the system in terms of the 3 valued truth tables given in [1].

In [1], (3) and (4) were derived using the definition of Λ:

$$\Lambda = [x, y]\, \mathbf{H}z \supset_z . (x \supset . y \supset x) \supset z$$

and the theorem

$$\mathbf{H}X, \mathbf{H}Y \vdash X \supset . Y \supset X \tag{5}$$

The first aim of [2], however, was to prove the general result:

1. In [3] the extra axiom:

$$\vdash \Lambda xy \supset_{x,y} x$$

was added.

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If $\vdash X$ *is a theorem of pure intuitionistic logic and* $x_1, x_2, \ldots, x_n$ *are the free propositional variables in* X*, then* $\mathbf{H}x_1, \mathbf{H}x_2, \ldots, \mathbf{H}x_n \vdash X$ *is a theorem of the present system,*

for which (5) was sufficient.

Axiom 3 of [1], however, also leads to the following more general version of (5):

$$\mathbf{H}X \vdash X \supset . \ Y \supset X$$

which in turn allows us to prove

$$\Lambda XY, \mathbf{H}X \vdash X \tag{6}$$

and

$$\Lambda XY, \mathbf{H}Y \vdash Y \tag{7}$$

In addition we can allow the axioms

$$\vdash \Gamma(\Pi y) \supset_y . \ \mathbf{H}(\Xi xy) \supset_x \mathbf{L}x^2 \tag{8}$$

and

$$\vdash \mathbf{H}(\Xi xy) \supset_{y,x} . \ xu \supset_u \mathbf{H}(yu) \tag{9}$$

which lead to

$$\Gamma Y, \mathbf{H}(X \supset Y) \vdash \mathbf{H}X \tag{10}$$

and

$$\mathbf{H}(X \supset Y) \vdash X \supset \mathbf{H}Y \tag{11}$$

and fit the three valued truth table of [1]. We then obtain

$$\mathbf{H}(\Gamma X) \vdash \mathbf{H}X$$

which helps us to prove

$$\Lambda XY, \mathbf{H}X \vdash Y$$

and

$$\Lambda XY, \mathbf{H}Y \vdash X,$$

but (1) and (2) seem to be unattainable.

This system, as it was described in [1] has the advantage of being able to deal with certain nonpropositions or with terms that may or may not be propositions. However, this flexibility does not work its way through to the general result above and has not been used in applications, so if we restrict the axioms of [1] so that they lead only to theorems in the form given in this general result and if we add

2. In [1] $\mathbf{L}$ was defined to be $\mathbf{FAH}$ and in [4] $\mathbf{L}$ was taken as primitive and $\mathbf{H}$ defined to be $\mathbf{BLK}$.

$$\vdash \mathbf{H}(x \supset y) \supset_{x,y} \mathbf{H}x \tag{12}$$
$$\vdash \mathbf{H}(x \supset y) \supset_{x,y} \mathbf{H}y \tag{13}$$

or the corresponding forms in terms of Ξ, we do not lose much. In particular these new axioms do not conflict with the axioms (8) and (9) suggested above.

Clearly using only $\vdash \mathbf{H}(\Xi\mathbf{HI})$ we have

$$\Lambda XY \vdash (X \supset. Y \supset \Xi\mathbf{HI}) \supset \Xi\mathbf{HI}$$

so that (12) and (13) give us

$$\Lambda XY \vdash \mathbf{H}X$$

and

$$\Lambda XY \vdash \mathbf{H}Y$$

and using (3) and (4) we get (1) and (2).

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The University of Wollongong
Wollongong, New South Wales, Australia