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A Unified Approach to Relative Interpolation

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1 Introduction In general, languages $L_{\kappa\lambda}$ do not have the interpolation property that, for $L_{\omega\omega}$, was proven by Craig ([3]). At this moment interpolation is known to hold for $L_{\omega\omega}, L_{\omega_1\omega}$ ([10]; see [12] for a combinatoric proof) and for countable admissible fragments of $L_{\infty\omega}$ (see [1]). Other infinitary languages just do not seem to have the property, as was partly shown by Malitz ([13]), who gave counterexamples for languages $L_{\kappa\omega}$ with $\kappa > \omega_1$, and $L_{\kappa\lambda}$ with $\kappa \geqq \lambda > \omega$.

As a result of this situation attention has been paid to more restrictive forms of interpolation. For strong limits κ with $\mathrm{cf}(\kappa) = \omega$, for instance, Karp ([11]) proved an extension of Craig's theorem to $L_{\kappa\kappa}$ using the notions of consistency property with respect to ω-chains of structures and ω-satisfiability (introduced in [10]). That is, for $L_{\kappa\kappa}$-sentences ϕ and ψ with $\models^{\omega}\phi \rightarrow \psi$ there exists an interpolating $L_{\kappa\kappa}$-sentence θ such that $\models\phi \rightarrow \theta$ and $\models\theta \rightarrow \psi$. Cunningham improved this to $\models^{\omega}\phi \rightarrow \theta$ and $\models^{\omega}\theta \rightarrow \psi$, using the notion of chain consistency property (see [4]). Ferro introduced the seq-consistency property (see [7]) to prove Cunningham's result and extend it to second-order logic.

This paper concentrates on another approach: that of 'relative' interpolation (i.e., there exists an interpolating sentence, but in a stronger language). Dickmann ([5]) uses $\mathrm{Interp}(L_{\kappa\lambda}, L_{\kappa'\lambda'})$ to denote the property that for every pair of $L_{\kappa\lambda}$-sentences there is an interpolating sentence in $L_{\kappa'\lambda'}$.

Malitz ([13]) outlines a combinatoric proof of $\mathrm{Interp}(L_{\kappa\omega}, L_{(2^{<\kappa})^+\kappa})$ for regular κ. For $\mathrm{cf}(\kappa) = \omega$, Friedman ([8]) proves $\mathrm{Interp}(L_{\kappa^+\omega}, L_{(2^{<\kappa})^+\kappa})$. Chang ([2]), using special and ω_1-saturated models, proves $\mathrm{Interp}(L_{\kappa^+\omega}, L_{\kappa^+\kappa})$ for strong limits κ with $\mathrm{cf}(\kappa) = \omega$, which—although proven independently—is a direct consequence of Friedman's result.

This paper contains a straightforward proof, using only basic model-theoretic notions, of a somewhat stronger version of $\mathrm{Interp}(L_{\kappa\omega}, L_{(2^{<\kappa})^+\kappa})$ for regular κ. It will be shown that this proof can be easily modified to obtain the

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theorems of Friedman and Chang, thus providing a unified method to yield all known relative interpolation results.

Moreover, the reason for the bound $(2^{<\kappa})^+$ is explained in a natural way by this method (a model-construction in the vein of [9]).

2 Relative interpolation Every nonlogical symbol in an $L_{\kappa\lambda}$-formula is in the range of only a finite number of negations. This justifies the definition of *positive* (*negative*) occurrence of a nonlogical symbol, i.e. being within the range of an even (odd) number of negations. (A nonlogical symbol can occur positively, negatively, both, or not at all in an $L_{\kappa\lambda}$-formula.)

Atomic formulas and negations thereof are called *basis-formulas*. $B(\Gamma)$ is the set of all basis-formulas in a set Γ of formulas. A *basis-sentence* is a basis-formula containing no free variables.

A formula is in *negation normal form* (is an *nnf*) if it is composed of basis-formulas by means of $\bigvee$, $\exists$, $\bigwedge$, and $\forall$. An *nns* is an nnf containing no free variables. It is easy to show the following

Lemma 2.1 *For every $L_{\kappa\lambda}$-formula ϕ there exists an $L_{\kappa\lambda}$-nnf ϕ' with the same positive (negative) occurrences of relation-symbols and with the same occurrences of constants as ϕ, such that $\models\phi \leftrightarrow \phi'$.*

For convenience we will make no use of function-symbols, nor of the identity.

Before proving the relative interpolation result for $L_{\kappa\omega}$ we state a model existence lemma with only 'break-down clauses' (like the 'mixed lemma' in [6]) that exactly meets our requirements:

Lemma 2.2 *Let Γ and Δ be sets of nns's of a fragment of $L_{\kappa\omega}$ for a language $L(C)$ with C a nonempty set of constants. Suppose the following conditions hold:*

(Γ1) $\exists x\phi(x) \in \Gamma \Rightarrow \phi(c) \in \Gamma$ *for some* $c \in C$
(Γ2) $\bigvee\Phi \in \Gamma \Rightarrow \phi \in \Gamma$ *for some* $\phi \in \Phi$
(Γ3) $\forall x\phi(x) \in \Gamma \Rightarrow \phi(c) \in \Gamma$ *for all* $c \in C$
(Γ4) $\bigwedge\Phi \in \Gamma \Rightarrow \phi \in \Gamma$ *for all* $\phi \in \Phi$
(Δ1) $\exists x\phi(x) \in \Delta \Rightarrow \phi(c) \in \Delta$ *for all* $c \in C$
(Δ2) $\bigvee\Phi \in \Delta \Rightarrow \phi \in \Delta$ *for all* $\phi \in \Phi$
(Δ3) $\forall x\phi(x) \in \Delta \Rightarrow \phi(c) \in \Delta$ *for some* $c \in C$
(Δ4) $\bigwedge\Phi \in \Delta \Rightarrow \phi \in \Delta$ *for some* $\phi \in \Phi$
(5) $\mathfrak{A}_0 \models \bigwedge B(\Gamma) \wedge \neg\bigvee B(\Delta)$ *for some model* $\mathfrak{A}_0$.

Then $\mathfrak{A} \models \bigwedge\Gamma \wedge \neg\bigvee\Delta$ *for some model* $\mathfrak{A}$.

Proof: Let $\mathfrak{A}_0 = \langle A_0, \ldots \rangle$, $A = \{c^{\mathfrak{A}_0} \mid c \in C\}$ and $\mathfrak{A} = \langle A, \ldots \rangle$. Then we have $\mathfrak{A} \models \bigwedge B(\Gamma) \wedge \neg\bigvee B(\Delta)$. For convenience, we identify constants with their interpretation in $\mathfrak{A}$ (i.e., in $\mathfrak{A}_0$).

By induction on the complexity of the nns ϕ we prove

$$(\phi \in \Gamma \Rightarrow \mathfrak{A} \models \phi)\ \&\ (\phi \in \Delta \Rightarrow \mathfrak{A} \models \neg\phi). \tag{1}$$

(a) If ϕ is a basis-sentence then (1) holds by $\mathfrak{A} \models \bigwedge B(\Gamma) \wedge \neg\bigvee B(\Delta)$. Suppose ϕ is not a basis-sentence, and the induction hypothesis is

$$(1) \text{ holds for } \psi \text{ with } c(\psi) < c(\phi), \tag{2}$$

where $c(\phi)$ is the complexity of ϕ. We distinguish the following cases:

(b) $\phi = \exists x\psi(x)$: $\exists x\psi(x) \in \Gamma \Rightarrow \psi(c) \in \Gamma$ for some $c \in C \Rightarrow \mathfrak{A} \models \psi(c)$ for some $c \in C$ (by (2)) $\Rightarrow \mathfrak{A} \models \exists x\psi(x)$; $\exists x\psi(x) \in \Delta \Rightarrow \psi(c) \in \Delta$ for all $c \in C \Rightarrow \mathfrak{A} \models \neg\psi(c)$ for all $c \in C$ (by (2)) $\Rightarrow \mathfrak{A} \models \neg\exists x\psi(x)$ (for $A \subset C$).

(c) $\phi = \bigvee\Phi$: $\bigvee\Phi \in \Gamma \Rightarrow \phi \in \Gamma$ for some $\phi \in \Phi \Rightarrow \mathfrak{A} \models \phi$ for some $\phi \in \Phi$ (by (2)) $\Rightarrow \mathfrak{A} \models \bigvee\Phi$; $\phi \in \Delta \Rightarrow \phi \in \Delta$ for all $\phi \in \Phi \Rightarrow \mathfrak{A} \models \neg\phi$ for all $\phi \in \Phi \Rightarrow \mathfrak{A} \models \neg\bigvee\Phi$ (by (2)).

(d) $\phi = \forall x\psi(x)$: similar to (b).

(e) $\phi = \bigwedge\Phi$: similar to (c).

Hence $\mathfrak{A} \models \bigwedge\Gamma \wedge \neg\bigvee\Delta$.

Let $\Gamma \models \phi$ (respectively $\phi \models \Gamma$) abbreviate $\bigwedge\Gamma \models \phi$ (respectively $\phi \models \bigvee \Gamma$) for formulas ϕ and sets of formulas Γ.

Theorem 2.3 *Let κ be regular. For $L_{\kappa\omega}$-sentences ϕ and ψ with $\models\phi \rightarrow \psi$, there exists an $L_{(2^{<\kappa})^+\kappa}$-sentence θ such that*

(i) $\models\phi \rightarrow \theta$ *and* $\models\theta \rightarrow \psi$

(ii) *every relation-symbol occurring positively (negatively) in θ occurs positively (negatively) in both ϕ and ψ*

(iii) *every constant occurring in θ occurs in both ϕ and ψ.*

Proof: From Lemma 2.1 we can assume that ϕ and ψ are nns's. Supposing there is no $L_{(2^{<\kappa})^+\kappa}$-nns θ satisfying (i)–(iii) above permits us to construct a model of $\phi \wedge \neg\psi$.

For this purpose we form two countable chains of sets of nns's

$$\{\phi\} = \Gamma_0 \subset \Gamma_1 \subset \ldots$$

and

$$\{\psi\} = \Delta_0 \subset \Delta_1 \subset \ldots$$

and a countable chain of sets of constants

$$C_L = C_0 \subset C_1 \subset \ldots$$

where C_L is the set of all constants in L (the basic set of nonlogical symbols from which we form the languages $L_{\kappa\lambda}$). We can assume that L contains only symbols occurring in either ϕ or ψ, so that $|C_L| < \kappa$.

It is our intention that $\Gamma = \bigcup_{n\in\omega} \Gamma_n$, $\Delta = \bigcup_{n\in\omega} \Delta_n$ and $C = \bigcup_{n\in\omega} C_n$ satisfy the conditions of Lemma 2.2, assuring us of the existence of a model of $\bigwedge\Gamma \wedge \neg\bigvee\Delta$, and hence of $\phi \wedge \neg\psi$.

First a definition: An $L_{(2^{<\kappa})^+\kappa}(C_p)$-nns θ *separates* Γ_n and Δ_m *relative to* C_p if $\Gamma_n \models \theta \models \Delta_m$ and every relation-symbol occurring positively (negatively) in θ occurs positively (negatively) in both Γ_n and Δ_m $(n, m, p \in \omega)$; if such a θ does not exist, Γ_n and Δ_m are *inseparable relative to* C_p; Γ_n and Δ_n are *inseparable* if they are inseparable relative to C_n.

The construction of the chains is such that for all $n \in \omega$ the following are satisfied:

(1_n) $|\Gamma_n|$, $|\Delta_n|$, $|C_n| < \kappa$ and $\Gamma_n, \Delta_n \subset L_{\kappa\omega}(C_n)$
(2_n) Γ_n and Δ_n are inseparable
$(\Gamma 1_n)$ $\exists x\eta(x) \in \Gamma_n \Rightarrow \eta(c) \in \Gamma_{n+1}$ for some $c \in C_{n+1}$ if $n = 0$ (mod 8)
$(\Gamma 2_n)$ $\bigvee\Phi \in \Gamma_n \Rightarrow \eta \in \Gamma_{n+1}$ for some $\eta \in \Phi$ if $n = 1$ (mod 8)
$(\Gamma 3_n)$ $\forall x\eta(x) \in \Gamma_n \Rightarrow \eta(c) \in \Gamma_{n+1}$ for all $c \in C_{n+1}$ if $n = 2$ (mod 8)
$(\Gamma 4_n)$ $\bigwedge\Phi \in \Gamma_n \Rightarrow \eta \in \Gamma_{n+1}$ for all $\eta \in \Phi$ if $n = 3$ (mod 8)
$(\Delta 1_n)$ $\exists x\eta(x) \in \Delta_n \Rightarrow \eta(c) \in \Delta_{n+1}$ for all $c \in C_{n+1}$ if $n = 4$ (mod 8)
$(\Delta 2_n)$ $\bigvee\Phi \in \Delta_n \Rightarrow \eta \in \Delta_{n+1}$ for all $\eta \in \Phi$ if $n = 5$ (mod 8)
$(\Delta 3_n)$ $\forall x\eta(x) \in \Delta_n \Rightarrow \eta(c) \in \Delta_{n+1}$ for some $c \in C_{n+1}$ if $n = 6$ (mod 8)
$(\Delta 4_n)$ $\bigwedge\Phi \in \Delta_n \Rightarrow \eta \in \Delta_{n+1}$ for some $\eta \in \Phi$ if $n = 7$ (mod 8).

$\Gamma_0 = \{\phi\}$, $\Delta_0 = \{\psi\}$, and $C_0 = C_{\mathrm{L}}$ satisfy (1_0) and (2_0): suppose θ separates $\{\phi\}$ and $\{\psi\}$ relative to C_{L}, then $\models\phi \rightarrow \theta$ and $\models\theta \rightarrow \psi$.

Say $\theta = \theta(D_1, D_2)$ where D_1 (respectively D_2) is the set of all (less than κ) constants in θ not occurring in ϕ (respectively ψ). Then $\forall E \exists Y\, \theta(X, Y)$ is an interpolating $\mathrm{L}_{(2^{<\kappa})^+\kappa}$-sentence for ϕ and ψ, contradicting our assumption.

Suppose Γ_n, Δ_n, and C_n are formed and satisfy (1_n) and (2_n).

$(\Gamma 1_n)$: If $n = 0$ (mod 8), choose a set $C' = \{c_\eta \mid \exists x\eta(x) \in \Gamma_n\}$ of constants, all different, such that C' and C_n are disjoint. Take $\Gamma_{n+1} = \Gamma_n \cup \{\eta(c_\eta) \mid \exists x\, \eta(x) \in \Gamma_n\}$, $\Delta_{n+1} = \Delta_n$, and $C_{n+1} = C_n \cup C'$; then (1_{n+1}) is satisfied, as well as (2_{n+1}): Suppose θ separates Γ_{n+1} and Δ_{n+1} relative to C_{n+1}, so that $\Gamma_n \cup \{\eta(c_\eta) \mid \exists x\eta(x) \in \Gamma_n\} \models \theta \models \Delta_n$. Say $\theta = \theta(D)$, where D is the set of all constants from C' occurring in θ. Then, from the choice of C', $\Gamma_n \models \exists X\theta(X) \models \Delta_n$. But then $\exists X\theta(X)$ is an $\mathrm{L}_{(2^{<\kappa})^+\kappa}$-nns separating Γ_n and Δ_n relative to C_n: indeed, every relation-symbol occurring positively (negatively) in $\exists X\theta(X)$ occurs positively (negatively) in both Γ_n and Δ_n on account of the corresponding property of θ, Γ_{n+1}, and Δ_{n+1}, contradicting (2_n).

$(\Gamma 2_n)$: If $n = 1$ (mod 8), choose $C_{n+1} = C_n$. Assertion: there exists a choice-function f for $\{\Phi \mid \bigvee\Phi \in \Gamma_n\}$ such that $\Gamma_{n,f} = \Gamma_n \cup \{f\Phi \mid \bigvee\Phi \in \Gamma_n\}$ and Δ_n are inseparable relative to C_n. Suppose the assertion does not hold; i.e., for all such f there exists a θ_f separating $\Gamma_{n,f}$ and Δ_n relative to C_n. Thus, for all such f,

$$\Gamma_n \cup \{f\Phi \mid \bigvee\Phi \in \Gamma_n\} \models \theta_f \models \Delta_n;$$

i.e.,

$$\Gamma_n \cup \left\{\bigwedge_{\bigvee\Phi \in \Gamma_n} f\Phi\right\} \models \theta_f \models \Delta_n.$$

Consequently,

$$\Gamma_n \cup \left\{\bigvee_f \bigwedge_{\bigvee\Phi \in \Gamma_n} f\Phi\right\} \models \bigvee_f \theta_f \models \Delta_n.$$

Because of

$$\bigwedge_{\bigvee\Phi \in \Gamma_n} \bigvee\Phi \models \bigvee_f \bigwedge_{\bigvee\Phi \in \Gamma_n} f\Phi \qquad \text{and} \qquad \Gamma_n \models \bigwedge_{\bigvee\Phi \in \Gamma_n} \bigvee\Phi,$$

we already have

$$\Gamma_n \models \bigvee_f \theta_f \models \Delta_n.$$

The cardinality of the disjunction $\bigvee_f$, i.e. that of the set of possible choice-functions, is

$$\left| \prod_{\bigvee\Phi\in\Gamma_n} \Phi \right|;$$

and

$$(*) \qquad \left| \prod_{\bigvee\Phi\in\Gamma_n} \Phi \right| \leqq \prod_{\Gamma_n} \kappa = \kappa^{|\Gamma_n|} = \sum_{\lambda<\kappa} \lambda^{|\Gamma_n|} \leqq \sum_{\lambda<\kappa} 2^\lambda = 2^{<\kappa} < (2^{<\kappa})^+$$

by (1_n) and the regularity of κ.

Therefore, $\bigvee_f\theta_f$ is an $L_{(2^{<\kappa})^+\kappa}(C_n)$-nns. If a relation-symbol R occurs positively (negatively) in $\bigvee_f\theta_f$, say in θ_f, then R occurs positively (negatively) in both $\Gamma_n \cup \{f\Phi \mid \bigvee\Phi \in \Gamma_n\}$ and Δ_n, and consequently in both Γ_n and Δ_n (for $f\Phi \in \Phi$). Therefore, $\bigvee_f\theta_f$ separates Γ_n and Δ_n relative to C_n, contradicting (2_n).

Take $\Gamma_{n+1} = \Gamma_{n,f}$ and $\Delta_{n+1} = \Delta_n$, then (1_{n+1}) and (2_{n+1}) are satisfied.

$(\Gamma 3_n)$: If $n = 2 \pmod 8$, choose $\Gamma_{n+1} = \Gamma_n \cup \{\eta(c) \mid \forall x\eta(x) \in \Gamma_n \ \& \ c \in C_n\}$, $\Delta_{n+1} = \Delta_n$, and $C_{n+1} = C_n$, then (1_{n+1}) and (2_{n+1}) are satisfied: Suppose θ separates Γ_{n+1} and Δ_{n+1} relative to C_{n+1}, so that

$$\Gamma_n \cup \{\eta(c) \mid \forall x\eta(x) \in \Gamma_n \ \& \ c \in C_n\} \models \theta \models \Delta_n;$$

then we already have

$$\Gamma_n \models \theta \models \Delta_n;$$

more: θ separates Γ_n and Δ_n relative to C_n, contradicting (2_n).

$(\Gamma 4_n)$: If $n = 3 \pmod 8$, choose $\Gamma_{n+1} = \Gamma_n \cup \bigcup \{\Phi \mid \bigwedge\Phi \in \Gamma_n\}$, $\Delta_{n+1} = \Delta_n$ and $C_{n+1} = C_n$, then (1_{n+1}) is satisfied on account of

$$|\Gamma_{n+1}| \leqq |\Gamma_n| + \sum_{\bigwedge\Phi\in\Gamma_n} |\Phi| < \kappa$$

(from the regularity of κ); and so is (2_{n+1}): Suppose θ separates Γ_{n+1} and Δ_{n+1} relative to C_{n+1}; so that

$$\Gamma_n \cup \bigcup \{\Phi \mid \bigwedge\Phi \in \Gamma_n\} \models \theta \models \Delta_n,$$

then we already have

$$\Gamma_n \models \theta \models \Delta_n;$$

more: θ separates Γ_n and Δ_n relative to C_n, contradicting (2_n).

Similarly—but dually, according to the conditions of Lemma 2.2—we enrich Δ_n if $n = 4 \pmod 8$, $5 \pmod 8$, $6 \pmod 8$, $7 \pmod 8$. The construction is such that, for all $n \in \omega$, (1_n), (2_n), $(\Gamma 1_n)$-$(\Gamma 4_n)$, and $(\Delta 1_n)$-$(\Delta 4_n)$ hold. We check up on the conditions of Lemma 2.2 for Γ, Δ, and C:

(Γ1) $\exists x\eta(x) \in \Gamma$, say $\exists x\eta(x) \in \Gamma_n$ for some $n = 0$ (mod 8). Then $\eta(c) \in \Gamma_{n+1}$ for some $c \in C_{n+1}$, hence $\eta(c) \in \Gamma$ for some $c \in C$.
(Γ2) $\bigvee\Phi \in \Gamma$, say $\bigvee\Phi \in \Gamma_n$ for some $n = 1$ (mod 8). Then $\eta \in \Gamma_{n+1}$ for some $\eta \in \Phi$, hence $\eta \in \Gamma$ for some $\eta \in \Phi$.
(Γ3) $\forall x\eta(x) \in \Gamma$, say $\forall x\eta(x) \in \Gamma_n$ for some $n = 2$ (mod 8). Then $\forall x\eta(x) \in \Gamma_m$ for all $m \geqq n$ with $m = 2$ (mod 8). Let $c \in C$ be arbitrary, say $c \in C_p$. Choose an $m = 2$ (mod 8) such that $m + 1 \geqq p$; then $\eta(c) \in \Gamma_{m+1} \subset \Gamma$.
(Γ4) $\bigwedge\Phi \in \Gamma$, say $\bigwedge\Phi \in \Gamma_n$ for some $n = 3$ (mod 8). Then $\eta \in \Gamma_{n+1} \subset \Gamma$ for all $\eta \in \Phi$.
(Δ1) $\exists x\eta(x) \in \Delta$, say $\exists x\eta(x) \in \Delta_n$ for some $n = 4$ (mod 8). Then $\exists x\eta(x) \in \Delta_m$ for all $m \geqq n$ with $m = 4$ (mod 8). Let $c \in C$ be arbitrary, say $c \in C_p$. Choose an $m = 4$ (mod 8) such that $m + 1 \geqq p$; then $\eta(c) \in \Gamma_{m+1} \subset \Gamma$.
(Δ2)–(Δ4) similarly.
(5) Suppose $\bigwedge B(\Gamma) \wedge \neg\bigvee B(\Delta)$ has no model, i.e. $\models \bigwedge B(\Gamma) \to \bigvee B(\Delta)$. Then the Lyndon interpolation theorem for finitary predicate logic (proposition logic is even sufficient!) provides an interpolating sentence θ for $\bigwedge B(\Gamma)$ and $\bigvee B(\Delta)$. So $B(\Gamma) \models \theta \models B(\Delta)$. From the compactness theorem for finitary predicate logic there are already finite $\Gamma' \subset \Gamma$ and $\Delta' \subset \Delta$ such that $B(\Gamma') \models \theta \models B(\Delta')$. Then we can choose an $n \in \omega$ such that $B(\Gamma') \subset \Gamma_n$ and $B(\Delta') \subset \Delta_n$, and therefore $\Gamma_n \models \theta \models \Delta_n$. But then θ separates Γ_n and Δ_n relative to C_n, contradicting (2_n). Consequently, there exists a model $\mathfrak{A}_0 \models \bigwedge B(\Gamma) \wedge \neg\bigvee B(\Delta)$.

So Lemma 2.2 provides a model $\mathfrak{A} \models \Gamma \wedge \neg\bigvee\Delta$. In particular, $\mathfrak{A} \models \phi \wedge \neg\psi$, contradicting $\models \phi \to \psi$.

Remark: The explanation (*) for the bound $(2^{<\kappa})^+$ is connected in a natural way to the use of choice-functions for the construction of Γ_{n+1} in $(\Gamma 2_n)$.

Corollary 2.4 *Theorem 2.3 remains valid for formulas ϕ, ψ, and θ, and with (iii) extended to free variables.*

Proof: Let $A = \{a \mid a$ occurs as free variable in ϕ or ψ; and consider $L(C)$ where $C = \{c_a \mid a \in A\}$ is a set of constants, all different. If ϕ' (respectively ψ') is the sentence that originates from ϕ (respectively ψ) by replacing all free variables a by constants c_a, then $\phi' \to \psi'$ and Theorem 2.3 provides an interpolating $L_{(2^{<\kappa})^+\kappa}$-sentence θ' for ϕ' and ψ'. Hence $\phi' \to \theta'$ and $\theta' \to \psi'$. Then also $\phi \to \theta$ and $\theta \to \psi$ if θ originates from θ' by replacing all constants c_a by variables a. Then θ is an interpolating $L_{(2^{<\kappa})^+\kappa}$-formula for ϕ and ψ.

Lemma 2.5 *For κ singular and $\phi \in L_{\kappa^+\lambda}$, there exists a $\phi' \in L_{\kappa\lambda}$ such that $\models \phi \leftrightarrow \phi'$.*

Proof: By induction on the complexity of ϕ. The only interesting case is $\phi = \bigvee_{\mu<\kappa} \phi_\mu$. Let $\mathrm{cf}(\kappa) = \nu < \kappa$, say $\lim_{\gamma\to\nu} \kappa_\gamma = \kappa$ $(\kappa_\gamma < \kappa)$. Then take

$$\phi' = \bigvee_{\gamma<\nu}\left(\bigvee_{\mu<\kappa_\gamma} \phi'_\mu\right),$$

then $\phi' \in L_{\kappa\lambda}$ and $\models \phi \leftrightarrow \phi'$.

This result tells us that languages $L_{\kappa^+\lambda}$ and $L_{\kappa\lambda}$ have the same expressive power if κ is singular. (Hence the demand that κ in $L_{\kappa\lambda}$ is regular does not restrict us if the axiom of choice is at our disposal, for in that case κ^+ is regular.)

Next, we show how the proof of Theorem 2.3 can be modified to obtain results by Friedman ([8]) and Chang ([2]), respectively (A) and (B) in the next theorem:

Theorem 2.6 *For $L_{\kappa^+\omega}$-sentences ϕ and ψ with $\models\phi \rightarrow \psi$, there exists a sentence θ satisfying (i)–(iii) in Theorem 2.3 and*
(A) $\mathrm{cf}(\kappa) = \omega \Rightarrow \theta \in L_{(2^{<\kappa})^+\kappa}$
(B) $\mathrm{cf}(\kappa) = \omega$ & *κ is a strong limit (i.e., $\lambda < \kappa \Rightarrow 2^\lambda < \kappa$)* $\Rightarrow \theta \in L_{\kappa^+\kappa}$.

Proof: Let $\mathrm{cf}(\kappa) = \omega$. If $\kappa = \omega$, then $(2^{<\kappa})^+ = \omega_1$ and the assertion is the interpolation theorem for $L_{\omega_1\omega}$. If $\kappa > \omega$, then κ is singular. Let ϕ' and ψ' be $L_{\kappa\omega}$-sentences originated from ϕ and ψ as indicated in the proof of Lemma 2.5, so that $\models\phi \leftrightarrow \phi'$ and $\models\psi \leftrightarrow \psi'$. Then for (A) it suffices to know that there exists an interpolating $L_{\kappa^+\kappa}$-sentence for ϕ' and ψ', for the operation $'$ does not change the positive (negative) occurrence of relation-symbols or the occurrence of constants. Theorem 2.3 unfortunately does not provide such an interpolating sentence, for κ is singular. However, we can modify the proof of Theorem 2.3 as follows:

Write $\kappa = \bigcup_{n\in\omega} \kappa_n$ ($\kappa_n < \kappa$) and replace ($\Gamma 2_n$) and ($\Gamma 4_n$) by the weaker

$(\Gamma 2_n)'$ $\quad \bigvee\Phi \in \Gamma_n$ & $|\Phi| < \kappa_n \Rightarrow \eta \in \Gamma_{n+1}$ for some $\eta \in \Phi$ if $n = 1 \pmod 8$

and

$(\Gamma 4_n)'$ $\quad \bigwedge\Phi \in \Gamma$ & $|\Phi| < \kappa_n \Rightarrow \eta \in \Gamma_{n+1}$ for all $\eta \in \Phi$ if $n = 3 \pmod 8$.

Then ($\Gamma 2$) and ($\Gamma 4$) in 2.2 are warranted: in the end, all $\bigvee\Phi \in \Gamma$ and $\bigwedge\Phi \in \Gamma$ come in for their turn because of $\kappa = \bigcup_{n\in\omega} \kappa_n$.

The construction of $(\Gamma 2_n)'$ is like that of ($\Gamma 2_n$); however, we can restrict the set of choice-functions to

$$\prod_{\substack{\bigvee\Phi\in\Gamma_n\\ |\Phi|<\kappa_n}} \Phi;$$

and

$$\left|\prod_{\substack{\bigvee\Phi\in\Gamma_n\\ |\Phi|<\kappa_n}} \Phi\right| \leqq \prod_{\Gamma_n} \kappa_n = \kappa_n^{|\Gamma_n|} \leqq 2^{<\kappa} < (2^{<\kappa})^+,$$

which is the desired inequality.

In the case of $(\Gamma 4_n)'$ we observe that

$$|\Gamma_{n+1}| \leqq |\Gamma_n| + \sum_{\substack{\bigwedge\Phi\in\Gamma_n\\ |\Phi|<\kappa_n}} |\Phi| \leqq \kappa$$

is still guaranteed.

Similar arguments hold for $(\Delta 2_n)'$ and $(\Delta 4_n)'$.
(B) is implied by (A): $2^{<\kappa} = \kappa$ for strong limits κ.

Notice that (A) and (B) in Theorem 2.6 are both generalizations of the interpolation theorem for $L_{\omega_1\omega}$—with which they coincide for $\kappa = \omega$ (although, in that case, the proof of Theorem 2.6 does not work)—unlike Theorem 2.3, which for $\kappa = \omega$ provides an interpolating sentence in $L_{(2^\omega)^+\omega_1}$.

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