Notre Dame Journal of Formal Logic
Volume 29, Number 2, Spring 1988

Generalized Hardy Fields in Several Variables

LEONARDO PASINI

1 Preliminaries

Definition 1.1 A category $\mathcal{C}$ is said to be a smoothness category if the following conditions are satisfied:

(1) The objects of $\mathcal{C}$ are open subsets of finite dimensional real vector spaces; the morphisms of $\mathcal{C}$ are certain differentiable functions and the composition law of morphisms is the usual composition of functions.

(2) If θ is an object of $\mathcal{C}$ and V is a finite dimensional real vector space, then $\mathcal{C}(\theta, V)$ is a linear subspace of the real vector space $C^1(\theta, V)$ of all C^1 functions from θ to V and contains all constant functions from θ to V.

(3) If $V_1, \ldots, V_m$ and W are finite dimensional real vector spaces, then $\mathcal{C}(V_1 \oplus \ldots \oplus V_m, W)$ contains all multilinear functions.

(4) Let θ_1 and θ_2 be open subsets, respectively, of the finite dimensional real vector spaces V_1 and V_2. A function f: $\theta_1 \to \theta_2$ is in $\mathcal{C}(\theta_1, \theta_2)$ if for any $x \in \theta_1$ there is an open subset $\theta_x \subseteq \theta_1$, containing x such that $f_{|\theta_x} \in \mathcal{C}(\theta_x, V_2)$.

(5) If $f_1 \in \mathcal{C}(\theta, V_1)$ and $f_2 \in \mathcal{C}(\theta, V_2)$, then $x \mapsto (f_1(x), f_2(x))$ is in $\mathcal{C}(\theta, V_1 \times V_2)$

(6) If $f \in \mathcal{C}(\theta_1, \theta_2)$ is a bijection from θ_1 to θ_2, then $f^{-1} \in \mathcal{C}(\theta_2, \theta_1)$ if f^{-1} is in C^1 (or equivalently if Df_x is nonsingular for any $x \in \theta_1$).

From the definition we deduce immediately that $\mathcal{C}(\theta, \mathbb{R})$ is a ring with the pointwise defined operations. Moreover, for any smoothness category $\mathcal{C}$, it is possible to prove the implicit function theorem ([5]):

Theorem 1.2 *Let θ be a neighborhood of $(\bar{x}_0, y_0)$ in $\mathbb{R}^{n+1}$ and let $f \in \mathcal{C}(\theta, \mathbb{R})$ with $f(\bar{x}_0, y_0) = 0$ and $(\delta f/\delta y)(\bar{x}_0, y_0) \neq 0$. Then, there are neighborhoods U of $\bar{x}_0$ in $\mathbb{R}^n$ and $g \in \mathcal{C}(U, \mathbb{R})$ with $g(\bar{x}_0) = y_0$ and $f(\bar{x}, g(\bar{x})) = 0$ for each $\bar{x} \in U$.*

Received February 10, 1986; revised June 12, 1986

Examples of smoothness categories are: the categories $\mathcal{C}^k(k = 1,\dots,\infty)$ of C^k functions; the category $\mathcal{C}^\omega$ of analytic functions; the Holder categories $\mathcal{C}^{k+\alpha}(k \in \mathbb{N}^+,\ 0 < \alpha < 1)$; the Lipschitz categories $\mathcal{C}^{k-}$; the category $\mathcal{C}^\Omega$ of Nash functions. The category $\mathcal{C}^\Omega$ turns out to be the intersection of all smoothness categories and, moreover, to be differentially stable.

2 $\mathcal{C}$-Hardy fields in several variables Let O be a point of $(R^n)^+ = \mathbb{R}^n \cup \{\alpha\}$, $n \in \mathbb{N}^+$ and $\alpha \notin \mathbb{R}^n$, the one point compactification of the euclidean space $\mathbb{R}^n$. Let F be a filter of sets of $(\mathbb{R}^n)^+$ with a basis B of open connected subsets of $\mathbb{R}^n$, which converges to O. We denote by $G(F, O)$ the ring of F-germs of real-valued functions defined over $\mathbb{R}^n$.

Definition 2.1 An element ψ of $G(F, O)$ is said to be of class $\mathcal{C}$ if there exists a function f such that:

(1) $f \in \psi$,

(2) $f \in \mathcal{C}(X, \mathbb{R})$ for a certain $X \in F$.

Moreover, we said that $\psi \in G(F, O)$ is semi-algebraic if it contains a semi-algebraic function ([2], [3]). We denote respectively by $G\mathcal{C}(F, O)$ and $G\mathcal{C}_{\text{s.a.}}(F, O)$ the subrings of $G(F, O)$ formed by the elements of class $\mathcal{C}$ and by the semi-algebraic elements of class $\mathcal{C}$.

Definition 2.2 By a "$\mathcal{C}$-field (respectively semi-algebraic $\mathcal{C}$-field) in O for F" we mean a subfield K of the ring $G\mathcal{C}(F, O)$ (respectively $G\mathcal{C}_{\text{s.a.}}(F, O)$).

Definition 2.3 By a "$\mathcal{C}$-Hardy field in n-variables in O for F" we mean a subfield K of the ring $G\mathcal{C}(F, O)$ such that: if $\psi \in K$, then $\psi_i \in K$, where $\psi_i = [\delta f/\delta x_i]_F$ for $i = 1,\dots,n$ and $f \in \psi$.

In the last case we assume $\mathcal{C}$ to be a differentially stable smoothness category. In particular, the class of semi-algebraic $\mathcal{C}$-Hardy fields coincides with the class of $\mathcal{C}^\Omega$-Hardy fields. In fact, if K is a semi-algebraic $\mathcal{C}$-Hardy field, each of its elements is C^∞ semi-algebraic, and hence Nash ([2], [3]).

From now on we denote by K any field belonging to one of the classes defined above.

Proposition 2.4 *The set $P = \{\psi \in K \mid$ there exist $f \in \psi$ and $X \in B$ such that $f(\bar{x}) > 0$ for all $\bar{x} \in X\}$ is a total ordering on K.*

Proof: P is obviously closed by sum and product in K. Let $\psi \in K$ and $\psi \neq 0$; there will be then $\gamma \in K$ such that $\psi \cdot \gamma = 1$ in K, that is if $f \in \psi$ and $g \in \gamma$, $f(\bar{x})g(\bar{x}) = 1$ holds identically over an $X \in B$. Hence $f(\bar{x}) \neq 0$ for all $\bar{x} \in X$. Moreover, we can choose X such that f is continuous over X.

Since X is a connected subset of $\mathbb{R}^n$, one of the two inequalities $f(\bar{x}) > 0$ or $f(\bar{x}) < 0$ holds identically over X. Hence $\psi \in P$ or $-\psi \in P$.

Let $^*\mathbb{R}$ be an enlargement of $\mathbb{R}$ in the sense of nonstandard analysis. We fix an element $\xi \in (^*\mathbb{R})^n$ in the monad $m(F)$ of F: $m(F) = \bigcap\{^*X \mid X \in F\}$. Such an element exists by the properties of enlargements and the transfer theorem. Moreover, by transfer, if $\psi \in G(F, O)$ and $f_1, f_2 \in \psi$, then $^*f_1, {^*f_2}$ are defined and coincide on any $\bar{x} \in m(F)$.

We define now a function ϕ: $G(F,O) \to {}^*\mathbb{R}$ by: $\phi(\psi) = {}^*f(\bar{\xi})$, for $f \in \psi$.

By the transfer theorem ϕ is a homomorphism from $G(F,O)$ to ${}^*\mathbb{R}$ and $\phi_{|K}$ is an injective order-preserving homomorphism.

3 *Theorem on the real closure*

Theorem 3.1 *There exists a real closed field belonging to the same class of K and containing K.*

We denote by $\overline{\phi(K)}$ the real closure of $\phi(K)$ in ${}^*\mathbb{R}$. The set of fields L extending K, and belonging to the class of K such that $\phi(L) \subseteq \overline{\phi(K)}$, is inductive, by Zorn's Lemma it contains some maximal element M. Theorem 3.1 is then a consequence of the following theorem. (For a similar result about Hardy fields on real closed fields see [6].)

Theorem 3.2 $\phi(M) = \overline{\phi(K)}$.

Proof: Let $c \in \overline{\phi(K)} - \phi(M)$ be algebraic of minimal degree, m, over $\phi(M)$.

We suppose $c > 0$; c is a zero of a polynomial ${}^*P(\bar{\xi},y) = {}^*g_0(\bar{\xi}) + \ldots + {}^*g_{m-1}(\bar{\xi})y^{m-1} + y^m$, with $[g_i(\bar{x})] \in M$ for $i = 0,\ldots,m$. Let $Q(\bar{x},y) = g_1(\bar{x}) + 2g_2(\bar{x})y + \ldots + my^{m-1}$ and hence ${}^*Q(\bar{\xi},y) = {}^*g_1(\bar{\xi}) + 2{}^*g_2(\bar{\xi})y + \ldots + my^{m-1}$.

For any $\bar{x}$ where the coefficients are defined $Q(\bar{x},y)$ is the derivative of $P(\bar{x},y)$ with respect to y. Since ${}^*Q(\bar{\xi},c) \neq 0$ we may suppose ${}^*Q(\bar{\xi},c) > 0$. Let $d_1,\ldots,d_k$, with $d_i \neq d_j$ for $i \neq j$, be the distinct roots of ${}^*Q(\bar{\xi},y)$ *in* $\overline{\phi(K)}$. Since deg ${}^*Q(\bar{\xi},y) = m - 1$, $d_i \in \phi(M)$, for $i = 1,\ldots,k$.

Therefore $d_i = {}^*h_i(\bar{\xi})$, with $[h_i(\bar{x})] \in M$. Since c is algebraic over $\phi(M)$, there are $[u(x)]$, $[v(x)] \in M$ such that: ${}^*u(\bar{\xi}) < c < {}^*v(\bar{\xi})$, ${}^*u(\bar{\xi}) > 0$ and ${}^*h_i(\bar{\xi}) \notin [{}^*u(\bar{\xi}),{}^*v(\bar{\xi})]$ for $i = 1,\ldots,k$.

Proposition 3.3
(1) ${}^*Q(\bar{\xi},y) > 0$ *for every* $y \in {}^*\mathbb{R}$ *in* $[{}^*u(\bar{\xi}), {}^*v(\bar{\xi})]$.
(2) ${}^*P(\bar{\xi},{}^*u(\bar{\xi})) < 0$ *and* ${}^*P(\bar{\xi},{}^*v(\bar{\xi})) > 0$.

Proof: (1) Let $y_0 \in {}^*\mathbb{R}$ with ${}^*Q(\bar{\xi},y_0) \leq 0$ and $y_0 \in [{}^*u(\bar{\xi}),{}^*v(\bar{\xi})]$. By the intermediate value property, since $Q(\bar{\xi},c) > 0$ there is $z \in {}^*\mathbb{R}$, ${}^*u(\bar{\xi}) \leq z \leq {}^*v(\bar{\xi})$ and ${}^*Q(\bar{\xi},z) = 0$. Since $\overline{\phi(K)}$ is real closed, $z \in \overline{\phi(K)}$, contrary to the choice of ${}^*u(\bar{\xi})$ and ${}^*v(\bar{\xi})$. (2) follows from (1); apply the mean value theorem to ${}^*P(\bar{\xi},y)$, bearing in mind that ${}^*P(\bar{\xi},c) = 0$.

Proposition 3.4 *There exists* $X \in B$ *such that for all* $\bar{x} \in X$:
(1) $Q(\bar{x},y) > 0$ *for every real* $y \in [u(\bar{x}),v(\bar{x})]$.
(2) $P(\bar{x},u(\bar{x})) < 0$ *and* $P(\bar{x},v(\bar{x})) > 0$.

Proof: (1) Since the roots of ${}^*Q(\bar{\xi},y)$ in $\overline{\phi(K)}$ are in $\phi(M)$, we have ${}^*Q(\bar{\xi},y) = m\prod_{j=1}^{k} (y - {}^*h_j(\bar{\xi}))^{i_j}({}^*a_0(\bar{\xi}) + {}^*a_1(\bar{\xi})y + \ldots + y^r)$ with ${}^*a_0(\bar{\xi}) + {}^*a_1(\bar{\xi})y + \ldots + y^r > 0$ for all $y \in {}^*\mathbb{R}$, since it is the product of monic irreducible polynomials in $\overline{\phi(K)}[y]$. Then, the coefficients ${}^*g_i(\bar{\xi})$ of ${}^*Q(\bar{\xi},y)$ are entire rational expressions of the ${}^*h_j(\bar{\xi})$'s and the ${}^*a_t(\bar{\xi})$'s.

Hence, there exists $X \in B$ such that for all $\bar{x} \in X$:

(I) $Q(\bar{x},y) = m \prod_{j=1}^{k} (y - h_j(\bar{x}))^{i_j}(a_0(\bar{x}) + a_1(\bar{x})y + \ldots + y^r)$ and

(II) $a_0(\bar{x}) + a_1(\bar{x})y + \ldots + y^r > 0$ for any y in $\mathbb{R}$.

The formula $\alpha(\bar{z})$: $\forall y(z_0 + z_1 y + \ldots + y^r > 0)$ holds in ${}^*\mathbb{R}$ for $z_t = {}^*a_t(\xi)$.

By the quantifier elimination for the theory of real closed fields, there exist a finite number of finite systems $S_j(\bar{z})$ of the form $\bigwedge_{u,v} (p_u(\bar{z}) = 0 \wedge q_v(\bar{z}) > 0)$ with $p_u(\bar{z})$ and $q_v(\bar{z})$ formal polynomials with integer coefficients such that, if L is any real closed field, $a_0, \ldots, a_{r-1} \in L$, $\alpha(a_0, \ldots, a_{r-1})$ is true in L iff one of the systems $S_j(\bar{z})$ holds at $z_t = a_t$.

The proof of (II) follows, then, by noting that a finite system $\bigwedge_{u,v} (p_u({}^*a_0(\xi), \ldots, {}^*a_{r-1}(\xi)) = 0 \wedge q_v({}^*a_0(\xi), \ldots, {}^*a_{r-1}(\xi)) > 0)$ holds in ${}^*\mathbb{R}$ iff there is $X \in B$ such that $\bigwedge_{u,v} (p_u(a_0(\bar{x}), \ldots, a_{r-1}(\bar{x})) = 0 \wedge q_v(a_0(\bar{x}), \ldots, a_{r-1}(\bar{x})) > 0)$ holds in $\mathbb{R}$ for all $\bar{x} \in X$. Moreover, we can choose $X \in B$ such that for all $\bar{x} \in X$:

(III) $h_j(\bar{x}) \notin [u(\bar{x}), v(\bar{x})]$ with $j = 1, \ldots, k$;

(IV) $Q(\bar{x}, u(\bar{x})) > 0$.

If $Q(\bar{x}_0, y_0) \leq 0$ for $\bar{x}_0 \in X$ and $y_0 \in [u(\bar{x}_0), v(\bar{x}_0)]$, then $Q(\bar{x}_0, y_1) = 0$ for some $y_1 \in [u(x_0), v(x_0)]$; that is, $y_1 = h_j(x_0)$ by (I) and (II), contradicting (III).

(2) This follows from Proposition 3.3(2).

Proposition 3.5 *There exists only one function $y(\bar{x})$ defined over X such that for all $\bar{x} \in X$: $u(\bar{x}) < y(\bar{x}) < v(\bar{x})$ and $P(\bar{x}, y(\bar{x})) = 0$.*

Proof: By the intermediate value property and Proposition 3.4(1), there is a unique $y(\bar{x}) \in [u(\bar{x}), v(\bar{x})]$ such that $P(\bar{x}, y(\bar{x})) = 0$.

Then, for any $\bar{x} \in X$: $P(\bar{x}, y(\bar{x})) = 0$ and $(\delta P/\delta y)(\bar{x}, y(\bar{x})) \neq 0$. Hence, by the implicit function theorem for the category $\mathcal{C}$, which characterizes the field K, for any $\bar{x}_0 \in X$ there are neighborhoods $U_{\bar{x}_0} \subseteq X$ and $f \in \mathcal{C}(U_{\bar{x}_0}, \mathbb{R})$ such that $f(\bar{x}_0) = y(\bar{x}_0)$ and $P(\bar{x}, f(\bar{x})) = 0$ for all $\bar{x} \in U_{\bar{x}_0}$. Since $u(\bar{x})$, $v(\bar{x})$, $f(\bar{x})$ are continuous on $U_{\bar{x}_0}$, $\theta_{\bar{x}_0} = \{\bar{x} \in U_{\bar{x}_0} \mid u(\bar{x}) < f(\bar{x}) < v(\bar{x})\}$ is an open subset of $\mathbb{R}^n$ containing $\bar{x}_0$ and $f(\bar{x}) = y(\bar{x})$ for all $\bar{x} \in \theta_{\bar{x}_0}$.

Thus $y_{|\theta_{\bar{x}_0}}(\bar{x}) \in \mathcal{C}(\theta_{\bar{x}_0}, \mathbb{R})$ and, by Definition 1.1(4), we have $y(\bar{x}) \in \mathcal{C}(X, \mathbb{R})$.

Since ${}^*P(\xi, y)$ is irreducible, the smallest subring of $G(F, O)$ containing M and $[y(\bar{x})]$ is a field whose elements are of the form $q([y(\bar{x})])$ with $q(y) \in M[y]$ and $\deg q(y) < m$. The elements of $M([y(\bar{x})])$ are then of class $\mathcal{C}$. The same is true for the semi-algebraic case because of the definition of the function $y(\bar{x})$. If K is a $\mathcal{C}$-Hardy field, $M[y(\bar{x})]$ is also differentially stable. Since $y(\bar{x}) \in \mathcal{C}^\infty(\bar{x}, \mathbb{R})$, for all $\bar{x} \in X$: $(\delta y/\delta x_i)(\bar{x}) = -((\delta P/\delta x_i)(\bar{x}, y(\bar{x}))/(\delta P/\delta Y)(\bar{x}, y(\bar{x}))$; hence $[y(\bar{x})]_i \in M([y(\bar{x})])$, for $i = 1, \ldots, n$.

Since $\phi_{|M([y(\bar{x})])}$ is an order-preserving embedding, by Propositions 3.3(1) and 3.5, it follows that $\phi([y(\bar{x})]) = c$. Since $[y(\bar{x})] \notin M$, this contradicts the maximality of M and proves Theorem 3.2.

4 ***Characterization of the real closure*** We denote by $GC(F,O)$ the subring of $G(F,O)$ of germs of continuous functions, and by ${}^{GC(F,O)}K$ the relative algebraic closure of K in $GC(F,O)$.

Theorem 4.1 *Let M be any real closure of K belonging to the class of K. Then: $M = {}^{GC(F,O)}K$.*

Proof: Obviously $M \subseteq {}^{GC(F,O)}K$. Let $[y(\bar{x})] \in GC(F,O)$ and let $P(y) \in K[y]$ be monic, with $P([y(\bar{x})]) = 0$. Then, there is $X \in B$ such that $y(\bar{x})$ is continuous over X and $P(\bar{x}, y(\bar{x})) = 0$ for all $\bar{x} \in X$. Since M is real closed and $P(y) \in M[y]$, $X \in B$ can be chosen so that, in addition, the following hold for all $\bar{x} \in X$:

$$P(\bar{x}, y) = \prod_{j=1}^{k} (y - h_j(\bar{x}))^{s_j} \prod_{i=1}^{r} [(y + a_i(\bar{x}))^2 + b_i^2(\bar{x})]$$

with pairwise distinct $h_j(x)$'s

$b_i(\bar{x}) \neq 0$ for $i = 1, \ldots, r$.

Then for all $\bar{x} \in X$ we have: $\prod_{j=1}^{k} (y(\bar{x}) - h_j(\bar{x})) = 0$, and then $X \subseteq \bigcup_{j=1}^{k} Z(y(\bar{x}) - h_j(\bar{x}))$. Since M is ordered, by Proposition 2.4, we can choose $X \in B$ so that in addition $Z(y(\bar{x}) - h_j(\bar{x})) \cap Z(y(\bar{x}) - h_i(\bar{x})) \cap X = \emptyset$ for $j \neq i$. The set $Z(y(\bar{x}) - h_j(\bar{x})) \cap X$ is closed in X, which is connected.

Hence $X \subseteq Z(y(\bar{x}) - h_j(\bar{x}))$ for some $j \in \{1, \ldots, k\}$, that is $[y(\bar{x})] \in M$.

REFERENCES

[1] Bourbaki, N., *Eléments de mathémathique*, Vol. XII, Book IV, Hermann, Paris, 1951.

[2] Brumfield, G. W., *Partially Ordered Rings and Semi-Algebraic Geometry*, London Mathematical Society Lecture Notes Series 37, Cambridge University Press, Cambridge, 1979.

[3] Dickmann, M. A., *Applications of Model Theory to Real Algebraic Geometry*, Lecture Notes in Mathematics 1130, Springer-Verlag, Berlin, 1985.

[4] Hardy, G. H., *Orders of Infinity*, Tracts in Mathematics and Mathematical Physics No. 12, 2nd Ed., Cambridge University Press, Cambridge, 1924.

[5] Palais, R., "Equivarient, real algebraic, differential topology," Part I: "Smoothness categories and Nash manifolds," Notes from Brandeis University, 1972.

[6] Pasini, L., "Sulla chiusura reale dei campi generalizzati di Hardy," *Bollettino Unione Matematica Italiana, Algebra e Geometria*, ser. VI, vol. I-D (1982), pp. 147–159.

[7] Robinson, A., "On the real closure of Hardy field," pp. 427–433 in *Theory of Sets and Topology: Hausdorff Memorial Volume*, ed., G. Asser, et al., VEB Deutscher Verlag der Wissenschaft, Berlin, 1972.

Dipartimento di Matematica
Università di Camerino
62032 Camerino, Italy