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Frege Against the Booleans

HANS SLUGA

1 "after this piece of writing" Frege finished the preface to his *Begriffsschrift* on December 18, 1878. By that date work on the book itself was presumably also completed. "It was arithmetic, as I remarked at the beginning, that was the point of departure for the train of thought that led to my *Begriffsschrift*," he concluded the preface. "To that science I also intend to apply it first of all" ([6], p. 8). And so he announced confidently: "The further pursuit of the path indicated, the elucidation of the concepts of number, magnitude, etc., will form the subject-matter of further investigations with which I want to step forward immediately after this piece of writing" ([6], p. 8).

But authors do not always foresee the dynamic set in motion by their own writing and so it was also in the case of Frege. It took him another five years before he was ready with the promised elucidation of the concept of number. I have tried to explain elsewhere that the delay was, at least in part, due to his "discovery" that numbers were logical objects and the difficulty he had with incorporating that insight into the logic of the *Begriffsschrift* ([22], pp. 96–128).

There was an additional reason for the delay and that was the predominantly negative reception of the *Begriffsschrift*. The condensed, formal presentation of the material in the book made the understanding of its message difficult, particularly for philosophical readers who were unused to such a high level of formalization and abstraction. Various reviewers—among them Ernst Schröder, Germany's most prominent Boolean logician—also raised the question of the relation between Fregean logic and Boolean algebra.

It is understandable then that between 1879 and 1882 Frege undertook repeated efforts to explain and defend his new logic. He did so in a series of articles of which the longest and most carefully worked out piece is an essay he called "Boole's Calculating Logic and the *Begriffsschrift*". Unfortunately, no scholarly journal was willing to take the piece and so, in the end, it remained unpublished till Frege's *Nachgelassene Schriften* appeared in 1969.[1]

For us today that essay and the others which Frege wrote at the time are of particular interest because they elucidate how he conceived of his work in

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those early years and on what philosophical presuppositions he built. In the face of the critical reception of the *Begriffsschrift* Frege found himself forced to clarify, in particular, how his own logic differed from Boolean algebra. By raising that issue his writings from this period can help us to gain a much clearer understanding of the development of modern logic.

Jean van Heijenoort has argued in his classical paper "Logic as Calculus and Logic as Language" [26] that in the early history of symbolic logic we must sharply distinguish between the Fregean (or logicist) and the Boolean (or algebraist) tradition and that only by doing so can we understand the conditions under which metamathematics emerged in the first decades of this century. Looking at Frege's own assessment of the relations between his logic and Boolean algebra can help us to extend and adjust van Heijenoort's insights.

2 Attack and counterattack When Frege published his *Begriffsschrift* in 1879 Boolean algebra had just begun to make its appearance in Germany. The first two expositions of it in German had come out only two years earlier, the more important of the two being Schröder's concise (37 page) introduction to it in his *Operationskreis des Logikkalkuls* (cf. [23], pp. 344f).

Schröder was seven years older than Frege and by 1879 was solidly established in his position as Professor of Mathematics at the Technical University at Karlsruhe. He had, by that date, seventeen mathematical publications to his credit, including a textbook on arithmetic and algebra (of which only the first of four projected volumes appeared). In that book he had drawn on the work of the brothers Hermann and Robert Grassmann and it was some ideas of the latter, akin in certain ways to Boolean algebra, that he later credited for his interest in logic ([16], p. 256). That interest, just like Frege's, developed only in the middle of the 1870s. And it bore its first fruit in the *Operationskreis*, a work that helped to gain Boolean algebra its belated recognition in Germany. As a result of Schröder's publication two prominent philosophical logicians, Lotze and Wundt, by 1880 considered it necessary to include discussions of it in their much-read logic books.

There were, thus, immediate reasons for rivalry between Frege and Schröder. And they were more than personal. Frege had not come to his interest in logic through the work of Boole and his followers. He had constructed his symbolism on altogether different principles from those employed by the Booleans. He had, in particular, not adopted an algebraic notation to express logical relationships (cf. [23]).

When he became acquainted with Boolean algebra some time before the completion of the *Begriffsschrift* (and quite plausibly as a result of encountering Schröder's *Operationskreis*) he must have been immediately aware of the difference of purpose that separated his work from that of the Booleans. In the preface to the *Begriffsschrift* he attacked what he considered to be the false assimilation of logical functions to algebraic operations. He wrote: "The farthest thing from my mind have been those efforts which try to establish an artificial similarity [between logic and algebra] by conceiving of a concept as a sum of its characteristic marks" ([6], p. 6).

That critical aside was duly noted by Schröder in his long review of the book in the reknowned *Zeitschrift für Mathematik und Physik* [20; 8], the most detailed discussion the *Begriffsschrift* received at the time of its appearance. The review was, in fact, quite unnecessarily hostile and, as a result, may have done substantial damage to Frege's career. Above all, Schröder complained that Frege had not built further on the work undertaken by Boole and himself. "I consider it a shortcoming," he wrote, "that the book is presented in too isolated a manner and not only seeks no serious connection with achievements that have been made in essentially similar directions (namely those of Boole), but even disregards them entirely" ([8], p. 220).

He was convinced that Frege's system "does not differ essentially from Boole's formula language" ([8], p. 221), adding: "With regard to its major content the *Begriffsschrift* could actually be considered a *transcription* of the Boolean formula language. With regard to its appearance, though, the former is different beyond recognition—and not to its advantage." Indeed it struck him that Frege's "schemata are ornate with symbols" ([8], p. 230).

Schröder granted that Boolean algebra was defective when it came to expressing existential judgments. But he saw no major achievement in Frege's quantifier notation. "Frege correctly lays down stipulations that permit him to express such judgments precisely," he wrote, but "the analogous modification or extension can easily be achieved in Boolean notation as well" ([8], pp. 229f).[2]

He was contemptuous of Frege's project to use his new logic for the analysis of arithmetical concepts and propositions. The Fregean definition of the notion of following in a series struck him "as very abstruse" ([8], p. 230). Frege's plan to test how far one could get in arithmetic by means of logical deductions alone, he said finally, was already out of date. "If I have properly understood what the author wishes to do, then this point would also be, in large measure, already settled—namely, through the perceptive investigations of Hermann Grassmann" ([8], p. 231).

It was obviously of paramount importance for Frege to defend himself against such charges. Schröder could be considered an established and respected authority in the field while the *Begriffsschrift* had been Frege's first substantial publication. He could hardly stand by idly as his work was being demolished in the name of Boolean algebra.

In the essay "Boole's Calculating Logic and the *Begriffsschrift*" he tried to establish that his logic could match Boolean algebra in various respects.[3] He showed through examples that his notation is capable of expressing complex mathematical propositions ([11], pp. 21–27); he gave a formal derivation of an arithmetical proposition (pp. 27–32) and he examined a logical problem discussed by Boole and Schröder in order to illustrate how it can be dealt with more easily in his own logic (pp. 39–45). In all those respects Frege considered his logic equal or superior to Boolean algebra. But above all, he wanted to establish that the conception and design of his work was original and went far beyond that of the algebraic conception of Boolean logic.

In arguing that point Frege was actually addressing Schröder as much as Boole.[4] In fact, he cast his response from the start in terms of a Leibnizian distinction between calculus and characteristic language that Schröder had adopted in his critique of Frege.

Schröder had aimed his remarks at Frege's expressed belief that his *Begriffsschrift* could be considered a partial realization of the Leibnizian idea "of a universal characteristic, a *calculus philosophicus* or *ratiocinator*" ([6], p. 6). He had said in his review that Frege's work promised to advance "toward Leibniz's ideal of a universal language", but that its title "promises too much" ([8], p. 218). In fact, according to him, Frege's "title does not correspond at all to the content. Instead of leaning toward a universal characteristic, the present work (perhaps unknown to the author himself) definitely leans towards Leibniz's '*calculus ratiocinator*'" ([8], pp. 219f). Schröder thought that in a universal language one should be able to "construct all complex concepts by means of a few simple, completely determinate and clearly classified operations from the fewest possible fundamental concepts with clearly delimited extensions" ([8], p. 219). But Frege's symbolism was, in his eyes, no such language. It was at best a calculus whereas Boole's could be considered to have made an advance towards a logical, or characteristic, language.

In contrast, Frege argued that the situation was just the other way around. Boolean algebra was a mere abstract logic, a mere calculus, whereas his own was capable of expressing an actual content and could be considered a partial realization of a characteristic language.[5]

3 Calculus and characteristic language Frege and Schröder were, thus, expressing their disagreement in terms they had derived from Leibniz—which is not altogether surprising since they had relied on one and the same essay by Trendelenburg for their understanding of the logical symbolism. In an article "On Leibniz's Design of a Universal Characteristic" [25] Trendelenburg had described Leibniz's aim as the construction of a logical calculus or characteristic language. It was this passage that served to provide the terminology for their disagreement.[6]

Before one considers the nature of that disagreement it is worth emphasizing that both relied on the same source for the terms in which they interpreted their dispute and that each of them insisted that his symbolism constituted an approximation to a characteristic language whereas the other's was a mere calculus. This shows that, behind their disagreement, there was a good deal of common ground. Both thought that the logical symbolism *should* be a characteristic language; their disagreement was over the question what such a language would look like.[7]

Leibniz had described the characteristic language as a language in which the logical structure of the expressions reflects the structure of the represented objects, whereas a calculus was for him a mechanism for determining the truth of our assertions. A characteristic language, he had said, is a language in which concepts and things are brought together in the appropriate order ([3], p. 162). Such a language had, in particular, to reflect the fact that all our concepts are built up from simples through the operations of negation, conjunction, and disjunction. Leibniz had described a calculus, in contrast, as a production of relations through the transformation of formulas according to determinate laws ([3], p. 93). Given the structure of the characteristic language, he had assumed such a calculus to be able to determine the truth of propositions by mechanically comparing their concepts.

It should be clear from Schröder's review that in his critique of the Fregean symbolism he took the notion of characteristic language largely in Leibniz's sense. Frege's logic was no such language for him because it failed to construct concepts out of simples by means of a few determinate operations. It is less clear why he thought Fregean logic was an approximation to a Leibnizian *calculus ratiocinator*. Perhaps, he had in mind Frege's insistence that his symbolism was suitable for testing the validity of inferences.

But how did Frege employ the notion of a characteristic language? We will see that his understanding of the term departed in certain important respects from Leibniz's and Schröder's. But it is in any case fairly clear that he used the Leibnizian sense of calculus when he called Boolean algebra a calculus. For he meant by that, first of all, that it was a device for carrying out calculations. In his eyes Boole had been concerned with a limited, practical task: the design of a *technique* that would allow the systematic solution of logical problems ([11], p. 12). Boole had shown how one could solve problems of class and propositional logic through mechanical procedures performed on algebraic symbols. Jevons, who had followed Boole in this respect, had even invented a machine for that purpose ([11], p. 35).

Frege did not want to dispute the significance of such technical improvements in logic. On the contrary, he conceived of his own system as also providing such a technique for problem-solving, such a calculus. But he also thought that there was more to logic than calculation. Boolean logic, he wrote, "represents only part of our thinking; the whole can never be carried out by a machine or be replaced by a purely mechanical activity" ([11], p. 35). In Boolean algebra, he granted, one could indeed draw conclusions out of premises by means of a mechanical calculating procedure, but he was also convinced that a complete logic required more. What he has in mind by that criticism is illuminated by his remark in the preface to the *Begriffsschrift* that the very invention of his notation has advanced logic, for that invention had been made possible only through conceptual analysis, and not through calculative reasoning.

In Schröder's eyes Fregean logic fell short of being a characteristic language because it did not aim at building up complex concepts and judgments out of simple ones by means of a few determinate operations. His critique of Frege was certainly at this point in accord with the Leibnizian conception of a characteristic language. But he had failed to see that Frege was, in another sense, closer to Leibniz. Frege thought that the Booleans had concentrated entirely on one aspect: the mechanical, synthetic procedures by which one can manipulate given elements. That was evidently one aspect of Leibniz's conception of the logical symbolism.

But even for Leibniz that had not been the whole matter. For the question is how the elements to be manipulated are given to us. Do they present themselves to us immediately and as the elements they are? Or do we require extensive labor to find them? Leibniz was clearly of the second opinion and he had therefore spoken of a complementarity of analytic and synthetic methods. Before we can calculate with our elements we must have them, before we can mechanically operate in our characteristic language we must have constructed it. But this process of discovery and invention cannot itself be a process of mechanical reasoning, of calculative manipulation.

We can describe the required process of analysis as a transition from one language to another, a transition from our everyday language to the characteristic language. Even if every operation within a given language can be understood as a process of mechanical reasoning, the transition from one language to another cannot be of this kind. It requires insight and understanding. A proposition in the first language will appear to us with a certain structure. Analysis of that proposition involves a translation of it into another language in which that proposition is assigned a new structure. But that move is possible only if our understanding of the proposition is not limited to a grasp of the structure through which the proposition is presented to us, if it does not merely consist in the ability to mechanically manipulate the elements of that structure. The construction of a logical symbolism demands not only calculative reasoning, but also and first of all conceptual, philosophical analysis.

It helps, at this point, to go back to the Trendelenburg essay on which both Schröder and Frege drew in this debate. In it Trendelenburg agrees with Leibniz that signs are indispensible for thinking, but that natural language is logically deficient. The problem, Trendelenburg writes, is that in natural language the connection between a sign and the content which it conveys is brought about only *by association*, it is a merely psychological bond. "Only to a small extent is there an internal relation between the sign and the content of the signified idea" ([25], p. 3). Because of this shortcoming there arises the idea of a language which brings "the shape of the sign in direct contact with the content of the concept" (ibid.). But, Trendelenburg points out, the construction of a language in which the link between the sign and the concept is *logical*, rather than merely psychological, requires philosophical insight. He draws attention in this context to Descartes's warning that "the invention of such a language depends on the true philosophy" ([25], p. 8). And he concludes that the efforts of Leibniz and others after him on behalf of a logical language have generally remained incomplete because the necessary philosophical analysis of concepts has not yet been achieved.

Frege's 1882 essay "On the Scientific Justification of the *Begriffsschrift*" reveals to what extent his early conception of the role of his logical symbolism is modelled on Trendelenburg's views. With Trendelenburg he expresses the hope that the invention of such a symbolism will make possible the further "development of reason" and thereby the further advance of science ([8], p. 89). But that invention demands more than calculative reasoning, it demands that we learn to free ourselves from "the physical and psychological conditions of reason" on which ordinary language keeps us dependent ([8], p. 87). We require a system of signs from whose strict logical form the objective content of our thoughts cannot escape, one which reflects the character of the concept itself, one which expresses a subject-matter directly, rather than mediated through our subjective feelings (ibid.). The construction of such a symbolism demands then, first of all, a conceptual and philosophical analysis.

In Frege's eyes the Booleans considered logical operations as mechanical ultimately because they assumed concepts to be pre-existent and ready-made and judgments to be composed from them by aggregation. That assumption, he thought, linked the Booleans closely to the traditional conception of logic that derived from Aristotle. In contrast, he argued that his own achievements in logic were "due to the fact that I have moved further away from Aristotelian logic" ([11], p. 15).

It was from the classical view of the relation of judgment and concept that Frege wished to distance himself most energetically. "In contrast to Boole," he wrote, "I begin with judgments and their contents and not with concepts. . . . The formation of concepts I let proceed from judgments" ([11], p. 16; cf. also [6A], p. 101). This principle of priority, in fact, constitutes the true center of his critique of Boolean logic. That logic is a mere calculus for him because of its inattention to that principle, while his own logic approximates a characteristic language because of its reliance on it.

The crucial importance of the priority principle for Frege's understanding of his own logic has not so far been generally recognized. What is better known is the context principle ("Words have meaning only in the context of a sentence") that Frege made one of three fundamental assumptions in *The Foundations of Arithmetic* [5], his next major piece of writing after the essay "On Boole's Calculating Logic and the *Begriffsschrift*". But the context principle is not explicitly reaffirmed after 1884 while the priority principle is restated as late as 1919 (cf. [11], p. 253). The context principle is, furthermore, only a logical consequence of the priority principle. If an asserted sentence has meaning by expressing a judgment and if to say that the words constituting the sentence have meaning is to say that they express concepts, then given that judgments precede concepts, it follows that sentence meanings precede word meanings. The context principle is, in other words, merely a linguistic version of the priority principle.[8]

Granted the centrality of the priority principle for Frege's assessment of his own logic and its difference from Boolean algebra I want to consider now the implications of that principle. Having done so I want to look at the wider context into which that principle fits by considering the purpose for which Frege constructed his logic.

4 Two epistemologies The priority principle has a number of distinct functions for Frege which he does not explicitly distinguish in his essay on "Boole's Calculating Logic and the *Begriffsschrift*". In order to understand more clearly how Frege saw himself to be different from Boole and his followers it may be useful for us to separate the epistemological and ontological function of the principle from its strictly logical function.

The epistemological and ontological import of the principle becomes clear from Frege's endorsement of a statement by the nineteenth century English linguist A. H. Sayce who declared that "the whole sentence . . . is the only possible unit of thought; subject and object are as much correlated as the positive and negative poles of the magnet".[9]

Frege's reference to Sayce's statement reveals that the priority principle implies for him a certain view of how we get to know concepts and hence what kind of reality concepts have. Concepts must not be considered as given independently of the judgments in which they occur. He holds that simple concepts and relations "originate together with the first judgment in which they are ascribed to things" ([11], p. 17). Concepts are always reached through the splitting up of judgments, through analysis; they are not given separately and the judgment is not composed out of previously given constituents. In 1882 he wrote to Carl Stumpf in just this sense: "I do not think that the formation of concepts can precede judgment, for that would presuppose the independent existence of concepts; I rather imagine that the concept originates in the analysis of a judgeable content" ([12], p. 101).[10]

Such an emphasis on the priority of judgments over concepts links Frege to the Kantian tradition in logic. Kant himself had considered it his greatest achievement in logic to have seen beyond the traditional view of judgments as mere composite concepts. Concepts, he had said, presuppose judgments since "the only use which the understanding can make of these concepts is to judge by means of them" ([15], p. A68). And he had concluded that "concepts, as predicates of possible judgments, relate to some representation of a not yet determined object" ([15], p. A69). The two sentences are worth recalling because Frege says in altogether similar words in his critique of Boole that a concept "is nothing complete, but only a predicate of a judgment, for which a subject is still lacking" ([11], p. 17). He also says that consequently a sign for a property never occurs in the *Begriffsschrift* "without at least indicating a thing which might have the property, the sign for a relation never without indicating things that might stand in that relation" (ibid.).

We can contrast this epistemological viewpoint with the one adopted by Schröder. According to the long epistemological introduction to Schröder's *Algebra of Logic* human understanding begins with the mental representation (*Vorstellung*) of particular objects. Such representation is made more precise through the invention of names for individual things. We can, then, "separate and more or less completely isolate in our mind certain elements of the representation of a concrete thing" ([21], p. 57). Schröder explains that "when the isolation (separation) does not succeed completely, we call the represented object an abstract and its name an abstract name" ([21], p. 57). Finally, he says: "We tend to name such things with the same common names which resemble each other with respect to certain marks. . . . As a result there occurs in the mind a peculiar psychological process which culminates in the fact that we connect a 'concept' with the common name" ([21], p. 81). It is thus through mental abstraction that we proceed from representations of concrete objects and from their names to concepts and concept expressions (cf. [21], p. 82). Schröder admits that the fact that concepts play a role in judgments is, on this account, derivative and subsidiary (cf. [21], pp. 97f).

Such philosophical assumptions directly determine Schröder's approach to the construction of his formal theory. He begins it with a calculus of areas of a manifold as an "auxiliary discipline" ([21], p. 157). Such a manifold is for him any totality of elements and any arbitrary subset of a manifold counts for him as an area. Schröder argues that the calculus of areas which results can be interpreted in a number of ways, for the variable letters which are initially meant to stand for areas can also be interpreted as standing for classes, i.e., "species of individuals, in particular, also concepts considered extensionally" ([21], p. 160). Furthermore, we can equally interpret them to mean "concepts considered intensionally, specifically also representations", as well as "judgments, assertions, statements", and even inferences, functional equations, algorithms, and whole calculi (ibid.). The auxiliary discipline of the calculus of areas can thus be applied to the specifically logical notions of concept, judgment, and inference "in that one simply carries out a change in the interpretation of the signs" (ibid.). It is clear then that Schröder's theory builds on the philosophical assumption that our knowledge is to begin with a knowledge of objects and, then, a knowledge of arbitrary groupings of such objects. Furthermore it assumes that classes (as extensions of concepts) and concepts (as intensions) are derivative in our understanding from areas and individual objects and that judgments and inferences are, once again, derivative from classes and concepts.

Quine has recently called the Fregean principle of "the semantic primacy of sentences" a milestone of empiricism ([18], p. 70), i.e., one of the points "where empiricism has taken a turn for the better" ([18], p. 67). I am sure that Frege would have looked upon such an assessment with surprise; for whatever contribution he made to this turn, he was certainly not himself an empiricist. On the contrary, the introduction to the *Foundations of Arithmetic* which had announced the principle concludes with the wish that empiricists might take the opportunity of his book "to examine afresh the principles of this theory of knowledge" ([5], p. xi).

It was Schröder, rather than Frege, who was taking the empiricist side in this dispute. Frege, on the other hand, thought of himself as the anti-empiricist and consciously associated himself with the Kantian tradition.[11] In doing so he was, of course, opposing both the atomism of the empiricists as also that of Leibniz. And because he distanced himself at this point from Leibniz he had to distance himself also to some extent from Leibniz's conception of the characteristic language, whereas Schröder with his atomistic views could stay closer to it.

5 Two kinds of logic The priority principle has for Frege not only epistemological and ontological significance, it also implies certain methodological standards in logic itself. But here again it helps to distinguish two things which he does not keep sufficiently separate, for the priority principles implies, on the one hand, a certain methodology of how one should go about setting up a logical symbolism and, on the other hand, certain assumptions about what kind of concepts should be expressible within the symbolism. I have discussed the first issue above in the section on calculus and characteristic language. We need, therefore, to consider here only how the principle bears on the internal construction of the symbolism.

Frege was, in fact, convinced that the priority principle had guided him at two points in the construction of his logic. One was in the discovery of the function-argument analysis of judgments and the other in his analysis of general propositions.

In his eyes the traditional subject-predicate account of judgments was closely associated with the compositional view of meaning which the priority principle rejects. In order to escape from that tradition he wants us to speak a new language of function and argument and this replacement is meant to be more than a change in terminology and more than a technical convenience, for the new language is meant to be in tune with priority principle. He writes in the *Begriffsschrift*: "In a first draft of my formula language I let myself be misled by the example of [everyday] language to *compose* judgments out of subject and predicate" ([6], p. 13; my emphasis). And with a significant change of verb he adds that in the final version of his logic a content is "*segmented* into function and argument" ([6], p. 23; my emphasis).

While the language of subject and predicate suggests that judgeable contents have a single, unique structure; the function-argument language is meant to show us that the same content can be analyzed in various ways. It is this newly gained flexibility that makes the function-argument analysis of judgments so appealing and so useful in the formation of new concepts.

Frege says of it, therefore, already in the *Begriffsschrift* that "one can easily see that the viewing of a content as a function of an argument leads to the formation of [new] concepts" ([6], p. 7). It is to this new analysis of judgments that he ascribes the discovery of his quantifier analysis of general judgments. Considering the two sentences "The number 20 is representable as the sum of four squares" and "Every positive integer is representable as the sum of four squares", he writes that we discover that "the expression 'every positive integer' does not . . . by itself yield an independent idea, but acquires a meaning only in the context of a sentence" ([6], p. 23). He considers it a mistake of traditional accounts of general judgments that they assumed expressions like "every positive integer" to have independent meaning, a belief fostered by the compositional, subject-predicate account of judgments.

In Frege's eyes both Aristotelian and Boolean logic assumed that "the formation of concepts through abstraction is the fundamental logical operation and [that] judging and inferring are brought about through direct or indirect comparison of the extensions of these concepts" ([11], p. 15). For the Boolean, he said, the formation of new concepts is always a question of new (conjunctive or disjunctive) combinations of previously given concepts. "With this form of concept-formation," he wrote, "one must presuppose as given a system of concepts or, metaphorically speaking, a network of lines. In this the new concepts are really already contained" ([11], p. 34). Given any set of classes C we can, in fact, by means of Boolean operations, define a set of minimal classes whose boundaries are entirely composed of the boundaries of the classes in C. Every other class definable in terms of C in Boolean algebra can now be described as a sum of minimal classes definable in C. That shows that every newly definable class will share its boundaries with the classes in C or, as Frege puts it, that any newly defined concept in Boolean algebra is already contained in the network of given boundary lines.

Frege thought that in a complete logic there would also have to be a method of concept-formation that could generate scientifically fruitful concepts with completely new boundaries [Ibid]. That method, he believed, was given in the quantifier notation. The quantifier notation, far from being a minor thing as Schröder had argued, was, in fact, at the heart of what made his own logic superior to Boolean algebra. By means of quantification we can define wholly new classes whose boundaries will not coincide with any part of the boundaries of any previously given classes. Frege drew, in this context, attention to his definitions of the notions of the continuity of a function, of a limit, and of that of following in a series. Such definitions exemplified for him the advantage of his own logic over Boolean algebra.

There remains a matter I have not yet discussed, but one that deserves our attention. In the letter to Stumpf from which I have previously quoted, Frege also writes: "I do not think that for every judgeable content there is only one way in which it can fall apart, or that one of the possible ways can always demand priority" ([12], p. 101). A few years later he was to illustrate the first part of the remark by saying that "it is not impossible that one way of analysing a given thought should make it appear as a singular judgment; another, as an existential judgment; and a third, as a universal judgment" ([10], p. 49). That possibility, he makes clear, is linked to the function-argument account of judgments. But what still requires our attention in the letter to Stumpf is the additional remark that we should not think that one of the possible analyses can always demand priority. In his *Notebooks* Wittgenstein argues compellingly that the doctrine that there are absolute simples is equivalent to the claim that our judgments (thoughts, propositions) have a single ultimate analysis (cf. [28], p. 63). Frege's denial of the priority of any particular analysis of a judgment, if that is what he is denying, would imply then the rejection of simple elements as the absolute endpoints of analysis. But that conclusion must be counterbalanced by the observation that he recognizes simple concepts and relations in the essay on Boole of which he says, however, that they are given with the first judgment in which they are ascribed to a thing. The assumption of simples is thus meant to be reconciled with the priority principle ([11], p. 17). But whether and how that reconciliation can be achieved remains to my mind uncertain.[12]

6 Two purposes Frege characterizes the difference between the two kinds of logic also by saying that Boolean algebra is a "pure" or "abstract" logic that lacks a "content" ([6A], p. 97 and [11], p. 12). Sarcastically he notes that it deals with problems which "for the most part seem to have been invented to be solved by its formulas" ([6A], p. 97). Of his own logic he says, on the other hand, that it has a definite purpose. "In it I had my eye from the start on *the expression of a content.* The goal of my efforts is a *lingua characterica* to begin with for mathematics, not a *calculus* restricted to pure logic" ([11], p. 12; Frege's own emphases). He aims at the construction of a language in which mathematical notions can be expressed more precisely than in ordinary language and in which we can strictly determine the grounds on which mathematical truths rest.

The presence of such a purpose will, of course, not guarantee the superiority of Frege's undertaking, but it gives it right away a philosophical character which is so evidently absent from the work of Boole and Schröder. Frege's first motivation had been the question of the epistemic status of mathematical truths. Were they a priori or empirical, were they analytic or synthetic? Those were the question he had begun with. It was the publication of (the first edition of) Lotze's book on logic in 1876 in which its author had argued that arithmetic is merely an extended logic that had aroused Frege's interest in the subject (cf. [23]). Having made it his own he constructed his new symbolism with that reductionist thesis in mind. More single-minded than most philosophers he turned that thesis into the fountainhead of almost all of his ideas about logic, language, meaning and knowledge (cf. also [24]).

Frege's reductionist program fits into a more global picture of human knowledge of which he gives us ever so often a glimpse in his writings. In it we view knowledge as an integrated whole, a totality of elements standing in precise structural relations to each other, a structure with a hierarchy in which some parts are more fundamental than others. Not a unique view, perhaps, but still one that can guide a whole program of constructive conceptual activity. It is certainly the view that is evident in the critique of Boolean algebra. Here he emphasizes again and again that this algebra leaves the logic of primary and the logic of secondary propositions without an organic link, unintegrated or even unrelated, that it fails to give proper expression to the forms of thought, that it does not "melt together" the mathematical and the logical symbolism into a single whole, that it perverts the real situation by not showing that logic is the foundation of arithmetic.

Given our ordinary algebraic equations, Boole argues, we can interpret the letters usually taken to indicate numbers as standing for classes, we can interpret addition and multiplication respectively as the operations of class product and sum, and we can take the numerals "1" and "0" to stand for the universe of discourse and the empty class respectively. In this way, ordinary algebra can be re-interpreted as a class logic. That logic can in turn be interpreted as a propositional logic if we take the letters to stand for classes of moments at which propositions are true and the numerals "1" and "0" as representing the class of all moments and the class of no moments respectively. Hence, class logic is primary and propositional logic secondary for the Booleans.

Frege considers this objectionable on two grounds. First because it conflicts with the priority principle according to which judgments precede concepts (and their associated classes). For him propositional logic is primary and fundamental; everything else has to be built upon that foundation.[13] But he is equally dissatisfied with the Boolean account because it leaves class and propositional logic unrelated to each other. He writes: "In contrast to Boole I reduce the primary propositions to the secondary. . . . In this way, I believe, I have produced in an easy and appropriate manner an organic link between the two parts" ([11], pp. 17f). Where Boole had treated the two parts of his logic as two separate interpretations of the same algebra, Frege's aim is to "produce the whole in one piece" ([11], p. 14).

In order to facilitate calculation the Booleans had used algebraic symbols to express logical relations. Frege considers that an inappropriate choice. He writes: "Someone who demands that the relations of signs should be as far as possible in accord with the facts will always consider it a perversion of the real situation for logic to borrow its symbols from arithmetic. The subject-matter of logic is correct thinking and that means the foundations of arithmetic as well" ([11], p. 12).

The Booleans, on the other hand, had tended to a somewhat different view of the relation of logic to mathematics. Boole, for instance, had maintained in the *Laws of Thought* that "the ultimate laws of Logic are mathematical in their form" ([1], p. 11). This was not because he believed logic to be reducible to the science of number, but because he considered both of them to rely on "general principles founded in the very nature of language" and on a resulting "agreement in processes" ([1], p. 6). Beyond that point, he was sure: "The two provinces of interpretation [logic and algebra] remain apart and independent, each subject to its own laws and conditions".

Such convictions express themselves in the very construction of Boolean algebra. It is a calculus cast in algebraic notation, capable of a variety of independent interpretations. In agreement with this conception Boole had chosen negation, conjunction, and disjunction as the basic logical operations, since they corresponded most closely to the operations of algebra.

Frege also believed that logic could profit from the use of mathematical techniques. He had, after all, called his *Begriffsschrift* a language "modelled upon the formula language of arithmetic" ([6], p. 1). But he had constructed his logical symbolism specifically in order to show that arithmetic was only an extended logic. "There arises the task," he wrote, "to set up signs for logical relations in such a way that they are suitable to melt together with the mathematical formula language so that at least for one area they form a complete *Begriffsschrift*. This is the point from which my little pamphlet takes off" ([11], p. 14).

Given that purpose it should be evident why Frege set out to construct a symbolism in which both logic and arithmetic could be expressed within a single interpretation, why he gave his logic an axiomatic and deductive form, rather than an algebraic one, why he considered the conditional—with its close relations to the inference relation—as the more basic logical operation, rather than disjunction and conjunction with their close similarity to the algebraic operations of addition and multiplication. We have finally reached the point where we can form a comprehensive picture of Frege's assessment of Boolean algebra. If Frege did not consider Boolean logic a *lingua characterica*, i.e. an appropriate notation, that was, in the end, due to the fact that he did not believe that the Boolean system of notation gives a proper intuitive representation of the forms of thought and of the structure of human knowledge. A proper logic would, in Frege's eyes, have to be built on the priority principle, reflect the primacy of propositional logic over class logic, show that logic is the foundation of arithmetic, and facilitate the integration of various kinds of knowledge into one symbolism with a single interpretation.

7 The possibility of metamathematics The struggle between Schröder and Frege did not remain confined to the period between 1880 and 1882 when Schröder published his review and Frege set out to respond to it. In 1884 Frege used the context of *The Foundations of Arithmetic* to ridicule Schröder's conceptual confusion in his early work on arithmetic. That, in turn, provoked some acerbic remarks about Frege's book in the first volume of Schröder's *Algebra of Logic*. Shortly afterwards, in 1893, Frege began his *Grundgesetze* with a critique of Schröder's conception of classes, a critique which he elaborated a couple of years later into a separate essay. Once again his theme was that a coherent logical theory must derive classes from concepts and that those, in turn, must be understood through the role they play in judgments. It was Schröder's neglect of this fact that had driven him into incoherence. "I believe indeed that the concept logically precedes its extension," Frege wrote, "and I regard as a failure the attempt to rest the extension of the concept (conceived as a class) not on the concept, but on individual objects. On this path one may get to a calculus of areas, but not to a logic" ([10], p. 106).

Jean van Heijenoort remarked some years ago that metamathematics represents the next large step in the history of logic after Russell and Whitehead's *Principia Mathematica*, but that it owes little to that monumental work. Metamathematical considerations are, in fact, entirely absent from *Principia*, just as they seem to play no systematic role in Frege's *Grundgesetze*. They surface first in Löwenheim's 1915 paper *"Über Möglichkeiten im Relativkalkül"* and are there related to Ernst Schröder's work in Boolean algebra—not to the logic of Russell and Frege. van Heijenoort could therefore write that with Löwenheim's paper we have "a sharp break with the Frege–Russell approach to the foundations of logic and a return to, or at least a connection with, pre-Fregean or non-Fregean logic" ([26], p. 328).

He argued specifically that the logicists' conception of their symbolism as a universal language stood effectively in the way of the development of the metamathematical viewpoint. The universe of discourse of such a logic is *the* universe. As a consequence nothing can be, or has to be, said outside of the system and thus any metasystematic questions are ruled out. The Boolean tradition, on the other hand, according to van Heijenoort, treats the logical symbolism as a formal calculus capable of various interpretations with different domains of discourse. That leads to questions of the validity of well-formed formulas in different domains and it is such considerations that underlie Löwenheim's proof that every satisfiable formula is satisfiable in a denumberable domain.

There is no doubt in my mind that van Heijenoort's essay contains valuable historical insights. His distinction of two basically different traditions in modern logic, the algebraist tradition stemming from Boole and the logicist tradition going back to Frege, is certainly fundamental for any account of the development of the discipline and has proved fruitful for understanding the growth of mathematical logic in the twenties. (cf. [14]). It is indeed important to see that systematic work in metamathematics originated in the algebraist and not in the logicist tradition. But, in making the contrast between the two traditions as explicit as possible, van Heijenoort appears to have overdrawn the picture.

One must, for instance, keep in mind that neither Boole nor Schröder engaged in metamathematical investigations. In the *Laws of Thought* Boole assures us, in fact, that the *only* test for "the completeness and the fundamental character of its laws" lies in "the completeness of its system of derived truths" ([1], p. 5). That remark shows no awareness of the possibility of specifically metamathematical investigations. It could equally have come from Frege or Russell.

One must also keep in mind that the views of logicists like Frege, Russell, and the early Wittgenstein were by no means identical. Of those three it was Wittgenstein who formulated the objections to metatheoretical questions most sharply. As far as logic is concerned, the *Tractatus* is in fact summed up in the thesis that there cannot be any (meta)theory of the logic of our language (cf. [27], 6.13). That impossibility is also implied by the puzzling proposition "Logic must take care of itself" (5.473) with which Wittgenstein had begun his *Notebooks* in 1914—a proposition which he had then called "an extraordinarily deep and important insight" ([28], p. 2).

Russell's attitude towards the question was, however, much more ambiguous. Though he himself never developed any systematic metatheory, he nevertheless objected to Wittgenstein's argument against its possibility: "These difficulties suggest to my mind some such possibility as this: that every language has, as Mr. Wittgenstein says, a structure concerning which, *in the language*, nothing can be said, but that there may be another language dealing with the structure of the first language, and having itself a new structure, and that to this hierarchy of languages there may be no limit" ([27], p. xxii).

But it must also be noted that Russell (as well as Frege) rejected the possibility of independence proofs – at least in logic – arguing that it is impermissible to assume, as such proofs seemed to, that a logical axiom was actually false since logic sets the standards of reasoning ([19], p. 15). But this objection surely fails to get at anything substantial since we can easily redescribe the proof procedure without assuming any logical axiom to be false. All we need in order to establish the independence of a proposition P from a set of propositions S is to find a *correlation* of those propositions with two arbitrary values m and n such that P has m correlated to it, whereas every proposition in S and all their derivatives have n correlated to them.

Of more significance are Frege's apparent doubts about the possibility of a semantic theory. His conviction that the difference between functions and objects cannot be described in fully legitimate language ([10], p. 54), that terms like "concept", "relation", "function", "object", and even at times the phrase "the reference" are strictly speaking illegitimate ([11], p. 255), that in a perfect language we would not need the word "true" ([11], p. 252) – all these convictions seem inevitably to lead us to the conclusion that there can never be a semantic theory. And Frege is ready on occasions to embrace such Wittgensteinian sentiments. Thus he writes in 1915: "If our language were logically more perfect, we would perhaps have no further need of logic, or we might read it off from the language" (ibid.).

But such seemingly firm conclusions must be balanced by the further observation that Frege also engages in actual metatheoretical argumentation in *The Basic Laws of Arithmetic* when he sets out to show that every well-formed expression of his symbolism has a reference ([7], pp. 83–89). In addition it appears from evidence related to his correspondence that Löwenheim eventually convinced him of the possibility of an investigation of the logical and mathematical formalism as an uninterpreted calculus. Löwenheim had written to Frege in 1908 to argue for the possibility of a purely formal arithmetic on the basis of considerations from the second volume of the *Basic Laws of Arithmetic.* The correspondence, which lasted for two years, eventually grew to ten letters from Löwenheim and ten responses from Frege. Löwenheim apparently succeeded in convincing Frege and both agreed that their correspondence was sufficiently important to be published. Unfortunately, publication never came about (possibly because of the First World War) and the letters were ultimately lost in the Second World War (cf. [9], pp. 157–161).

van Heijenoort's description of the emergence of metamathematics needs to be corrected also in another respect. In characterizing the difference between the Boolean and the Fregean tradition he draws attention to the fact that Frege called Boolean algebra a *calculus ratiocinator* while he described his own as a *lingua characterica.* But in making that distinction Frege was not defining the difference which van Heijenoort tries to capture with the terms calculus and language.[14]

We have seen that the terms calculus and characteristic language do indeed play a central role in the quarrel between Frege and Schröder. But they do not play the role that van Heijenoort assigns to them. It is not the case that the

Booleans (or at least Schröder as their representative) thought of their logic as a calculus rather than a characteristic language. On the contrary, Schröder just as much as Frege insisted that his logic provided an approximation to a characteristic language whereas the other's did not. We have also seen the reasons why Frege called Boolean algebra a mere calculus and these have little to do with considerations that might facilitate or obstruct the development of metamathematics. Frege called Boolean logic a calculus, first of all, because it is not constructed in the light of the priority principle. The quarrel between him and Schröder was over the question of whether judgments or their elements should be given philosophical, epistemological, or logical priority. That was the point at issue when they debated the question of which system of notation should be considered a characteristic language. Frege called his own logic a characteristic language, precisely because it observes that principle, and furthermore because it gives, as a consequence, primacy to the logic of propositions, because it allows the definition of entirely new concept by means of the quantifier-notation, and because it permits the incorporation of arithmetic into logic.

Frege and Schröder did, of course, also quarrel over the question of whether the logical symbolism should have one fixed interpretation or several alternative ones. But they did not thereby raise the question of alternative interpretations in the sense in which van Heijenoort discusses it—that is, the sense with which we would be most familiar today. They were not arguing over the question of whether we can consider various domains of *individuals* and various *assignments* of subclasses of that domain to the class or concept terms of the symbolism. The alternative interpretations they had in mind were the interpretation of one and the same algebraic notation as an ordinary numerical algebra, as a class logic, and as a propositional calculus. Frege objected to such an approach, not because he was trying to cut short any metatheoretical reasoning, but because he wanted propositional logic, class logic and arithmetic united in a single theory.

Such criticisms detract in no way from the significance of van Heijenoort's article. His essay was the first to raise the question of the origin of metamathematics and to suggest the path on which it should be investigated. But the full story of the growth of metamathematics still remains to be told and more remains to be said about the notions of calculus and language which play so important a role in that story.

NOTES

1. There were altogether five essays: (1) "Applications of the *Begriffsschrift*", (2) "On the Purpose of the *Begriffsschrift*", (3) "On the Scientific Justification of the *Begriffsschrift*", (4) "Boole's Calculating Logic and the *Begriffsschrift*", and (5) "Boole's Logical Symbolism and my *Begriffsschrift*". Of these, three dealt specifically with Boolean logic, namely (2), (4), and (5). Only the first three were published during Frege's lifetime and only (3) appeared in a major philosophical publication.

2. Schröder's own attempt to show in the review how that modification could be arranged is, however, defective, as Frege was quick to point out (cf. [11], p. 20, note).

3. The translators of Frege's *Posthumous Writings* render the title of this piece unhelpfully as "Boole's Logical Calculus and the Concept-Script", thereby obliterating Frege's attempt to characterize what he considers the essential feature of Boole's logic, namely that it is a purely calculating logic ([11], p. 9). My references to this essay will all be to the English text in [11], though I have retranslated all of the German material quoted.

4. There is, in fact, little reason to think that Boole himself or his *Laws of Thought* were at the center of Frege's attention. His use of the English terms "universe of discourse", "primary", and "secondary propositions" does indicate acquaintance with Boole's text, but how thoroughly had he studied the work? If he had read it with care, would he not have complained of its repeatedly psychologistic formulations according to which logic deals with "the laws of the mind" ([1], p. 4), is a "science of the mind", is a "science of the intellectual powers" (p. 3), etc? Such formulations were anathema to Frege, but he never mentions them in his essay nor does he anywhere later include Boole in his attacks on psychologism.

5. On Frege's (and Schröder's) use of the curious term "lingua characterica" cf. [17].

6. Adolf Trendelenburg who is today known mostly as an Aristotelian scholar was also the author of several important essays on Leibniz's logic at a time when new editions of Leibniz's philosophical and mathematical writings were, once again, focusing interest on that philosopher.

   Both Frege and Schröder were familiar with the essay on the universal characteristic. Frege, in fact, took the term *Begriffsschrift* from it ([22], p. 49). Schröder discusses it extensively in the Introduction to his *Algebra der Logik* (cf. [21], pp. 38ff, 93ff). He agrees, in particular, with Trendelenburg's assessment of the need for a logical language and his account of the relation of ordinary language to such a characteristic language.

7. In a different context Frege was, in fact, ready to grant the existence of common ground between himself and Schröder (cf. [10], p. 106).

8. For a different assessment of the relation of the priority principle to the context principle, cf. Dummett's discussion in [2].

9. Quoted from [4], pp. 426f. Frege's characterization of Sayce's view as *bemerkenswert* has given rise to some confusion. I translate the term as "noteworthy" and take Frege to be endorsing Sayce's conception. This has been denied by Dummett ([2], p. 296) who prefers to translate "*bemerkenswert*" as "remarkable" and goes on to say that it is likely that "what Frege found remarkable was Sayce's unwarranted exaggeration". Such a reading is, however, based on nothing but a misleading translation of the crucial term. The translators of Frege's *Posthumous Writings* chose to render the term even more misleadingly as "extraordinary" ([11], p. 17).

10. The editors of Frege's correspondence identify Anton Marty as the recipient of this letter, but for the reason they themselves give ([12], p. 99) and because Frege may have been acquainted with Stumpf since his student years at Göttingen (cf. [23], p. 342), it seems to me more plausible to consider Stumpf the recipient. Stumpf's own letter to Frege ([12], pp. 171f) may indeed provide the explanation why he passed the letter on to Marty.

11. On the relation of Frege to the neo-Kantian movement, cf. [13].

12. Frege also argues in his essay that his logical connectives have a simpler content than Boole's, thus availing himself once more of a notion of logical simplicity (cf. [11], pp. 35ff).

    It should also be noted that his claim that "not further analysable concepts and relations must have their own simple designations" ([11], p. 17) will hold true only in a fully elaborated characteristic language, not in everyday English or German.

13. In the *Grundgesetze* he effectively abandoned that view and tried to develop propositional and quantificational theory together.

14. van Heijenoort may not have known "Boole's Calculating Logic and the *Begriffsschrift*" when he published his article in 1967 and that may have influenced his interpretation of Frege's use of the terms calculus and characteristic language. All he knew was Frege's 1882 essay "On the Purpose of the *Begriffsschrift*". That essay mentions the distinction only once in a context that suggests (but does not elaborate the claim) that Boolean logic is concerned only with what Frege calls "inferential calculation" (*schlussfolgernde Rechnung*) which is just another way of saying that it is a *calculus ratiocinator*. It is no wonder therefore that van Heijenoort thought that the meaning of the distinction is "most of the time not stated by Frege" and that it "is perhaps not discussed explicitly but nevertheless constantly guides Frege" ([26], p. 324).

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Department of Philosophy
University of California
Berkeley, California 94720