66
Notre Dame Journal of Formal Logic
Volume 27, Number 1, January 1986

Solving Functional Equations at Higher Types; Some Examples and Some Theorems

RICHARD STATMAN*

The solvability of higher type functional equations has been studied by a number of authors. Roughly speaking the literature sorts into four topics: constructive solvability (e.g., Gödel [5], Scott [7]); solvability in all models, i.e., unification (e.g., Andrews [1], Statman [8] and [9]); solvability in models of A.C. (e.g., Church [2], Friedman [4]); and the solvability of special classes of equations (e.g., Scott [7]). In this note we shall consider yet a fifth topic, namely, the solvability of functional equations in extensions of models.

Our main result is the no counterexample theorem. This theorem equates the unsolvability of E in every extension of $\mathfrak{A}$ with the solvability of some other $\tilde{E}$ in $\mathfrak{A}$. The theorem can be iterated and applied to λ theories (in extended languages) as well as to models. Thus, it can be used to explain, in a general way, a phenomenon well illustrated by the case of $\lambda\sqcup$.

$\lambda\sqcup$ is the theory of upper semilattices of monotone functionals. $\lambda\sqcup$ has the property that each of its models can be extended to solve all the fixed point equations

$$Mx = x \ .$$

This is a simple consequence of a Scott-type completion argument. It is also an immediate corollary to the no counterexample theorem.

We adopt for the most part the notation and terminology of [8] and [9].

Types τ have the form $\tau(1) \to (\ldots(\tau(t) \to 0)\ldots)$.

If $\mathcal{S}$ is a set of objects (terms, functionals, etc.), $\mathcal{S}^\tau$ is the set of all members of $\mathcal{S}$ of type τ.

*Support for this paper was provided by NSF grant MCS 8301558. The author would also like to thank the referee for his useful comments and corrections.

Received February 28, 1983; revised August 2, 1984

If Σ is a set of constants $\Lambda(\Sigma)$ is the set of all terms with constants from Σ. $\bar{\Lambda}(\Sigma)$ is the set of all closed members of $\Lambda(\Sigma)$. $\Lambda = \Lambda(\phi)$.
M, N range over $\bar{\Lambda}$.
B, C, K, I, W, S are the usual combinators.
$\mathfrak{A}$, $\mathfrak{B}$, . . . range over models of the typed λ calculus.
$\mathfrak{A}^*$ is the result of adjoining infinitely many indeterminates of each type to $\mathfrak{A}$.
$\mathcal{R} = \mathcal{R}_\tau \subseteq \underbrace{\mathfrak{A}^\tau \times \ldots \times \mathfrak{A}^\tau}_{n}$ is logical if $\mathcal{R}(\Phi_1, \ldots, \Phi_n) \Leftrightarrow \forall \Psi_1 \ldots \Psi_n\, \mathcal{R}(\Psi_1, \ldots, \Psi_n) \rightarrow \mathcal{R}(\Phi_1 \Psi_1, \ldots, \Phi_n \Psi_n)$
(this is, of course, no restriction on $\mathcal{R}_0$).

Definition *Functional equations* $E = E(\vec{y}, \vec{x})$ have the form

$$M\vec{y}\vec{x} = N\vec{y}\vec{x}$$

where $M, N \in \bar{\Lambda}$. By λ abstraction this is perfectly general. Given the parameters $\vec{y}$, we wish to solve for $\vec{x}$.

Example 1. Solvability in all models for all choices of parameters: In this case we may assume $\vec{y} = \phi$ [9]. Then E is solvable in all models if and only if $M\vec{x}$ and $N\vec{x}$ are unifiable.

Functional equations are closed under conjunction (see [4]).

Example 2. (Dezani) Invertibility: $BMx = I \wedge BxM = I$ is solvable in all models if and only if M is a hereditary permutation [3]. A solution, if it exists, is given by $\lambda y \underbrace{M(\ldots (M}_{n} y) \ldots)$ for some n.

Functional equations are in general not closed under negation; however, in one important case, they are.

Example 3. Models of A.C. ([4]): Let $\mathfrak{A}$ be a model of A.C. and let $\vec{\Phi} \subseteq \mathfrak{A}$. Then either $E(\vec{\Phi}, \vec{x})$ is solvable in $\mathfrak{A}$ or

$$\lambda \vec{x} z \vec{x}(M\vec{\Phi}\vec{x}) = \lambda uv\, u \wedge \lambda \vec{x} z \vec{x}(N\vec{\Phi}\vec{x}) = \lambda uv\, v$$

is solvable in $\mathfrak{A}$ (so $E(\vec{\Phi}, \vec{x})$ is not solvable in any extension of $\mathfrak{A}$).

Definition $\mathfrak{B} \supseteq \mathfrak{A}$ is called *functionally complete* (over $\mathfrak{A}$) if no extension of $\mathfrak{A}$ solves more functional equations, with parameters from $\mathfrak{A}$, than $\mathfrak{B}$.

Example 4. Upper semilattices of monotone functionals: Let $\perp \in 0$ ($\perp_\tau \equiv \lambda x_1 \ldots x_t \perp$) and $\sqcup \equiv \sqcup_\tau \in \tau \rightarrow (\tau \rightarrow \tau)$ (for greater clarity we shall infix $\sqcup$). Let $\lambda\sqcup$ be the following equations:

$$\begin{aligned} \perp \sqcup I &= I \\ \lambda x x \sqcup x &= I \\ \lambda xy x \sqcup y &= \lambda xy y \sqcup x \\ \lambda xyz x \sqcup (y \sqcup z) &= \lambda xyz (x \sqcup y) \sqcup z \\ \lambda xy x \sqcup y &= \lambda xyz (xz) \sqcup (yz) \\ \lambda xyz (xy) \sqcup x(y \sqcup z) &= \lambda xyz x(y \sqcup z) \ . \end{aligned}$$

Let $\mathfrak{A}$ be a model of $\lambda\sqcup$. Then the fixed point equation

$$Mx = x$$

is solvable in every functionally complete extension of $\mathfrak{A}$. For if Φ, $\Psi \in \mathfrak{A}$ define $\Phi \sqsubseteq \Psi \Leftrightarrow \Phi \sqcup \Psi = \Psi$, then $\sqsubseteq$ is a partial order and a logical relation. Thus we can apply a Scott-type completion argument.

Given E, for z_0, $z_1 \in 0$ define $\tilde{E}_n \equiv \tilde{E}_n(z_0 z_1 \vec{y}, \vec{u})$ to be

$$\begin{aligned} \lambda\vec{x}z_0 &= \lambda\vec{x}\,u_1\vec{x}\,(M\vec{y}\vec{x})(N\vec{y}\vec{x}) \wedge \\ \lambda\vec{x}\,u_1\vec{x}\,(N\vec{y}\vec{x})(M\vec{y}\vec{x}) &= \lambda\vec{x}\,u_2\vec{x}\,(M\vec{y}\vec{x})(N\vec{y}\vec{x}) \wedge \\ &\vdots \\ \lambda\vec{x}\,u_n\vec{x}\,(N\vec{y}\vec{x})(M\vec{y}\vec{x}) &= \lambda\vec{x}z_1 \ . \end{aligned}$$

A solution to $\tilde{E}_n(ab\vec{\Phi}, \vec{u})$ for $a \neq b$ is called a *counterexample* to $E(\vec{\Phi}, \vec{x})$. $E(\vec{\Phi}, \vec{x})$ is said to be *no counterexample interpretable* in $\mathfrak{A}$ if for each $a \neq b$ and n, $\tilde{E}_n(ab\vec{\Phi}, \vec{u})$ has itself a counterexample in $\mathfrak{A}$. The reader might wish to compare these notions to [6].

Example 5. The fixed point equation $Mx = x$: The fixed point equation is no counterexample interpretable in any model of $\lambda\sqcup$. For put $M^n = \underbrace{M(\ldots(M}_{n}\bot)\ldots)$, and suppose $\mathfrak{A}$ is a model of $\lambda\sqcup$ and $\vec{\Psi}$ is a solution of $\tilde{E}_n(ab, \vec{u})$. We have $a = \Psi_1 M^1 M^2 M^1 \sqsubseteq \Psi_2 M^2 M^3 M^2 \sqsubseteq \Psi_3 M^3 M^4 M^3 \sqsubseteq \ldots = b$ so $a \sqsubseteq b$. Symmetrically $b \sqsubseteq a$ so $a = b$. The rest is simply an exercise in the definition. Observe that this gives an alternative proof of Example 4.

The No Counterexample Theorem
*$E(\vec{\Phi}, \vec{x})$ is solvable in an extension of $\mathfrak{A} \Leftrightarrow$
$E(\vec{\Phi}, \vec{x})$ has no counterexample in $\mathfrak{A}$.*

Proof: Clearly it suffices to show $\Leftarrow$, so assume that $E(\vec{\Phi}, \vec{x})$ has no counterexample in $\mathfrak{A}$. Let $T_1 \equiv M\vec{\Phi}$ and $T_2 \equiv N\vec{\Phi}$. Define $\sim$ on $\mathfrak{A}^* \times \mathfrak{A}^*$ by $\Phi_1 \sim \Phi_2 \Leftrightarrow \exists\Psi\, \Phi_1 = \Psi(T_1\vec{x})(T_2\vec{x}) \wedge \Phi_2 = \Psi(T_2\vec{x})(T_1\vec{x})$. Let $\approx$ be the transitive closure of $\sim$. Since $\sim$ is reflexive and symmetric, $\approx$ is an equivalence relation.

(1) $\sim$ is a logical relation.

In particular, if $y \notin \vec{x}$ and $\Phi_1 \sim \Phi_2$ then $\lambda y\Phi_1 \sim \lambda y\Phi_2$. Thus

(2) $\approx$ is a logical relation.

Thus, as in Example 8 of [13], putting $[\Phi] = \{\Psi\colon \Phi \approx \Psi\}$ and $[\Phi][\Psi] = [\Phi\Psi]$, we obtain a model $\mathfrak{B} = \{[\Phi]\colon \Phi \in \mathfrak{A}^*\}$. Moreover, the map $\Phi \mapsto [\Phi]$ is a total homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{B}$.

Now suppose Φ_1, $\Phi_2 \in \mathfrak{A}$, $\Phi_1 \neq \Phi_2$, and $\Phi_1 \approx \Phi_2$. By (2) there exists a, $b \in \mathfrak{A}°$, $a \neq b$, and $a \approx b$. Thus, there exists $a_1, \ldots, a_{n-1} \in \mathfrak{A}°$ such that $a \sim a_1 \sim \ldots \sim a_{n-1} \sim b$. Hence, there exists $\Psi_1, \ldots, \Psi_n \in \mathfrak{A}^*$ such that

$$\begin{aligned} a &= \Psi_1(T_1\vec{x})(T_2\vec{x}) \wedge \\ a_1 &= \Psi_1(T_2\vec{x})(T_1\vec{x}) \wedge \\ a_1 &= \Psi_2(T_1\vec{x})(T_2\vec{x}) \wedge \\ &\;\;\vdots \\ a_{n-1} &= \Psi_n(T_1\vec{x})(T_2\vec{x}) \wedge \\ b &= \Psi_n(T_2\vec{x})(T_1\vec{x}) \;. \end{aligned}$$

Thus

$$\begin{aligned} \lambda\vec{x}a &= \lambda\vec{x}(\lambda\vec{x}\Psi_1)\vec{x}(T_1\vec{x})(T_2\vec{x}) \wedge \\ \lambda\vec{x}(\lambda\vec{x}\Psi_1)\vec{x}(T_2\vec{x})(T_1\vec{x}) &= \lambda\vec{x}(\lambda\vec{x}\Psi_2)\vec{x}(T_1\vec{x})(T_2\vec{x}) \wedge \\ &\;\;\vdots \\ \lambda\vec{x}(\lambda\vec{x}\Psi_n)\vec{x}(T_2\vec{x})(T_1\vec{x}) &= \lambda\vec{x}b \;. \end{aligned}$$

So $\tilde{E}_n(ab\vec{\Phi}, \vec{x})$ is solvable in $\mathfrak{A}^*$, therefore it is solvable in $\mathfrak{A}$. This is a contradiction.

Thus $\mathfrak{A} \subseteq \mathfrak{B}$.

Finally, $[\vec{x}]$ is a solution of $E(\vec{\Phi}, \vec{x})$ in $\mathfrak{B}$.

Corollary 1 *$E(\vec{\Phi}, \vec{x})$ is solvable in every functionally complete extension of $\mathfrak{A}$ $\Leftrightarrow$ $E(\vec{\Phi}, \vec{x})$ is no counterexample interpretable in $\mathfrak{A}$.*

Corollary 2 *$E(\vec{\Phi}, \vec{x})$ is solvable in some extension of $\mathfrak{A}$ $\Leftrightarrow$ it is solvable in some total homomorphic image of $\mathfrak{A}^*$ which extends $\mathfrak{A}$.*

Corollary 3 *$M\vec{x} = N\vec{x}$ is not solvable in any model $\Leftrightarrow$ for some n*

$$\begin{aligned} \lambda\vec{x}(\lambda xyx) &= \lambda\vec{x}u_1\vec{x}(M\vec{x})(N\vec{x}) \wedge \\ \lambda\vec{x}u_1\vec{x}(N\vec{x})(M\vec{x}) &= \lambda\vec{x}u_2\vec{x}(M\vec{x})(N\vec{x}) \wedge \\ &\;\;\vdots \\ \lambda\vec{x}u_n\vec{x}(N\vec{x})(M\vec{x}) &= \lambda\vec{x}(\lambda xyy) \end{aligned}$$

is solvable in every model.

Example 6. Consistency ([11]): $M = N$ is false in every nontrivial model $\Leftrightarrow$ $uM = \lambda xyx \wedge uN = \lambda xyy$ is solvable in every model.

Definition Functional equations of the form $M\vec{x} = \lambda xyx$ or equivalently $M\vec{x} = \lambda xyy$ are called *isolated*. Functional equations of the form $M\vec{x} = N$ are *semi-isolated*.

Example 7. Semi-isolated functional equations: $M\vec{x} = N$ has a solution in all models $\Leftrightarrow$ it has a λ definable (possibly with a type 0 parameter) solution in $\mathcal{P}_n$ for all sufficiently large n ([11]). Here n depends only on N.

Example 5 continued: Let Σ consist of $\perp$, $\sqcup$, and constants $F \in \underbrace{0 \to (\ldots(0}_{n} \to 0)\ldots)$ for various n. For $T \in \bar{\Lambda}(\Sigma)$ put $T^n \equiv \underbrace{T(\ldots(T}_{n}\perp)\ldots)$. We shall show that $Tx = x$ is solvable in every model of $\lambda\sqcup$ if and only if $\lambda\sqcup \vdash T^{n+1} = T^n$ for some n. Note that, since the axioms of $\lambda\sqcup$ are typically ambiguous [12], the corresponding result follows for the typed λ calculus.

For this it suffices to construct a universal model of $\lambda\sqcup$ in which $\subseteq$ is locally finite. With a little more care we can construct such a model which is generated by its 1-section. In this model $\subseteq$ is not only locally finite but also recursive. The decidability of the word problem for $\lambda\sqcup$ follows immediately.

As a preliminary we need some simple results about the first-order theory of upper semilattices with smallest element and monotone functions. Consider the first-order language with the constant $\perp$, function symbols F of various arities, the binary function symbol $\sqcup$ (infixed), and the binary relation symbol $\subseteq$. Let $\mathfrak{J}$ be the following set of sentences:

$$\forall x\ x \sqcup x \subseteq x$$
$$\forall x\quad \perp \subseteq x$$
$$\forall x \forall y\quad x \sqcup y \subseteq y \sqcup x$$
$$\forall x \forall y \forall z\quad x \sqcup (y \sqcup z) \subseteq (x \sqcup y) \sqcup z$$
$$\forall x \forall y \forall z\quad (x \sqcup y) \sqcup z \subseteq x \sqcup (y \sqcup z)$$
$$\forall x \forall y\quad x \subseteq x \sqcup y$$
$$\forall x \forall y \forall z\quad x \subseteq y \wedge y \subseteq z \rightarrow x \subseteq z$$
$$\forall xy \forall uv\quad x \subseteq u \wedge y \subseteq v \rightarrow x \sqcup y \subseteq u \sqcup v$$
$$\forall x_1 \ldots x_n \forall y_1 \ldots y_n\ x_1 \subseteq y_1 \wedge \ldots \wedge x_n \subseteq y_n \rightarrow Fx_1 \ldots x_n \subseteq Fy_1 \ldots y_n.$$

Obviously, if we define $x = y \leftrightarrow x \subseteq y \wedge y \subseteq x$ then $=$ is a congruence. For what follows it is convenient to think of the terms of $\mathfrak{J}$ as independent of the association of $\sqcup$'s and the order of arguments of $\sqcup$'s. This is harmless since $\sqcup$ is associative and commutative.

We write $a \subseteq b$ if a is a subterm of b. $a \preccurlyeq b \Leftrightarrow \exists c \subseteq b\ \mathfrak{J} \vdash a \subseteq c$. a is in normal form if

$a \equiv \perp$,
$a \equiv Fa_1 \ldots a_n$ where each a_i is in normal form, or
a $\equiv F_1 a_{11} \ldots a_{1n_1} \sqcup \ldots \sqcup F_m a_{m1} \ldots a_{mn_m}$ where each $F_i a_{i1} \ldots a_{in_i}$ is in normal form and $i \neq j \rightarrow \mathfrak{J} \nvdash F_i a_{i1} \ldots a_{in_i} \subseteq F_j a_{j1} \ldots a_{jn_j}$.

The following facts are easily verified:

(1) $\mathfrak{J} \vdash Fa_1 \ldots a_n \subseteq a \sqcup b \rightarrow \mathfrak{J} \vdash Fa_1 \ldots a_n \subseteq a \vee \mathfrak{J} \vdash Fa_1 \ldots a_n \subseteq b$.
(2) $\mathfrak{J} \vdash Fa_1 \ldots a_n \subseteq Gb_1 \ldots b_m \rightarrow F \equiv G \wedge$ for $1 \leq i \leq n\ \mathfrak{J} \vdash a_i \subseteq b_i$.
(3) $\mathfrak{J} \nvdash Fa_1 \ldots a_n \subseteq \perp$.
(4) For each a there is a unique normal b such that $\mathfrak{J} \vdash a = b$.
(5) If $a \leq b \wedge \mathfrak{J} \vdash b \subseteq c$ then there exists $d \subseteq c$ such that $\mathfrak{J} \vdash a \subseteq d$.
(6) If a is normal and $b \lneq a$ then $\mathfrak{J} \nvdash a \subseteq b$.
(7) $\preccurlyeq$ is transitive, reflexive, and $a \preccurlyeq b \wedge b \preccurlyeq a \Leftrightarrow \mathfrak{J} \vdash a = b$. Let $\mathfrak{J}_a = \mathfrak{J} \cup \{b \subseteq c\colon b, c \not\preccurlyeq a\}$.
(8) $\mathfrak{J}_a \vdash b \subseteq a \Rightarrow \mathfrak{J} \vdash b \subseteq a$.

Let $\mathfrak{A}$ be the free model of $\mathfrak{J}$. We consider $\mathfrak{A}$ modulo $=$ as an algebra with $x \subseteq y \Leftrightarrow x \sqcup y = y$. Since $\mathcal{P}_\omega$ contains all functions on its ground domain $\mathcal{P}_\omega^\circ$, we may assume that this algebra $\subseteq \mathcal{P}_\omega$, and that its domain is $\mathcal{P}_\omega^\circ$. Let $\mathfrak{M}$ be its Gandy hull in $\mathcal{P}_\omega$ ([11]; briefly, to build $\mathfrak{M}$ take all elements of $\mathcal{P}_\omega$ λ-definable from parameters in $\mathcal{A}$, and "collapse" the result to an extensional model of genuine set theoretic functionals).

Now define $\Phi \sqcup \Psi = \lambda z(\Phi z) \sqcup (\Psi z)$ and define a logical relation $\subseteq = \underset{\tau}{\subseteq}$ on by $\Phi \underset{0}{\subseteq} \Psi \Leftrightarrow \Phi \sqcup \Psi = \Psi$.

Claim $\Phi \subseteq \Psi \Leftrightarrow \Phi \sqcup \Psi = \Psi$.

Proof: By induction on types. The basis case is by definition so we proceed to the induction step. Suppose $\Phi \subseteq \Psi$. By induction hypothesis $X \subseteq X$, for X of lower type, so $\Phi X \subseteq \Psi X$. Thus by induction hypothesis $\Phi X \sqcup \Psi X = \Psi X$. Thus $\Phi \sqcup \Psi = \Psi$ by definition of $\sqcup$. Now suppose $\Phi \sqcup \Psi = \Psi$. Now $\subseteq$ is a partial ordering of its field. In addition, by the construction of $\mathfrak{M}$, $\bot \subseteq \bot$, $F \subseteq F$ and $\sqcup \subseteq \sqcup$ so by the fundamental theorem of logical relations ([13]) $\forall x\, x \subseteq x$. Thus $\subseteq$ is a partial ordering. Suppose $X_1 \subseteq X_2$. We have $\Phi X_2 \sqcup \Psi X_2 = (\Phi \sqcup \Psi)X_2 = \Psi X_2$ so by induction hypothesis $\Phi X_2 \subseteq \Psi X_2$. But $\Phi X_1 \subseteq \Phi X_2$ so $\Phi X_1 \subseteq \Psi X_2$. Hence $\Phi \subseteq \Psi$.

In particular, $\mathfrak{M} \models \lambda\sqcup$, i.e., $\lambda \sqcup \vdash T_1 \subseteq T_2 \Rightarrow \mathfrak{M} \models T_1 \subseteq T_2$. We shall prove the converse.

Definition For $T \in \bar{\Lambda}(\Sigma) = \bar{\Lambda}$ we define the notion of $\lambda \sqcup$ normal form as follows: $T \in \tau$ is in $\lambda\sqcup$ *normal form* if

(a) $T \equiv \lambda x_1 \ldots x_t \bot$
(b) $T \equiv \lambda x_1 \ldots x_t\, F(T_1 x_1 \ldots x_t) \ldots (T_n x_1 \ldots x_t)$ where each T_i is in $\lambda\sqcup$ normal form and $F \in \underbrace{0 \to (\ldots (0}_{n} \to 0) \ldots)$
(c) $T \equiv \lambda x_1 \ldots x_t\, x_i(T_1 x_1 \ldots x_t) \ldots (T_n x_1 \ldots x_t)$ where each T_i is in $\lambda\sqcup$ normal form and $n = t(i)$
(d) $T \equiv \lambda x_1 \ldots x_t\, (T_1 x_1 \ldots x_t) \sqcup \ldots \sqcup (T_n x_1 \ldots x_t)$ where $n > 1$, each T_i is in $\lambda\sqcup$ normal form and each T_i is of type (b) or (c).

It is easy to see that every term has a $\lambda\sqcup$ normal form.

Lemma *Suppose $T_1, T_2 \in \bar{\Lambda}^{\tau}_{\mathfrak{M}}$ and $\lambda \sqcup \nvdash T_1 \subseteq T_2$. Then there exist $U_1 \ldots U_t \in \bar{\Lambda}_{\mathfrak{M}}$ such that $\mathfrak{M} \not\models T_1 U_1 \ldots U_t \subseteq T_2 U_1 \ldots U_t$.*

Proof: We may assume that T_1 and T_2 are in $\lambda\sqcup$ normal form. The proof is by induction on $|T_1| + |T_2|$. We distinguish the following cases:

$T_1 \equiv$ (1) $\lambda x_1 \ldots x_t \bot$
(2) $\lambda x_1 \ldots x_t\, F(T_{11} x_1 \ldots x_t) \ldots (T_{1n} x_1 \ldots x_t)$
(3) $\lambda x_1 \ldots x_t\, (T_{11} x_1 \ldots x_t) \sqcup \ldots \sqcup (T_{1n} x_1 \ldots x_t)$
(4) $\lambda x_1 \ldots x_t\, x_i\, (T_{11} x_1 \ldots x_t) \ldots (T_{1n} x_1 \ldots x_t)$

$T_2 \equiv$ (a) $\lambda x_1 \ldots x_t \bot$
(b) $\lambda x_1 \ldots x_t\, G(T_{21} x_1 \ldots x_t) \ldots (T_{2m} x_1 \ldots x_t)$
(c) $\lambda x_1 \ldots x_t\, (T_{21} x_1 \ldots x_t) \sqcup \ldots \sqcup (T_{2m} x_1 \ldots x_t)$
(d) $\lambda x_1 \ldots x_t\, x_j\, (T_{21} x_1 \ldots x_t) \ldots (T_{2m} x_1 \ldots x_t)$.

Now the cases when $T_1 \equiv$ (1) are impossible and the cases when $T_2 \equiv$ (a) are trivial.

Case $T_1 \equiv (2)$. The cases when $T_2 \equiv$ (b) or $T_2 \equiv$ (d) are immediate or follow directly from the induction hypothesis. Thus we may assume $T_2 \equiv$ (c).

For $1 \le j \le m$ $\lambda \sqcup \nvdash T_1 \subseteq T_{2j}$ so by induction hypothesis there exist $U_{1j} \ldots U_{tj} \in \bar{\Lambda}_{\mathfrak{M}}$ such that $\mathfrak{M} \nvDash T_1 U_{1j} \ldots U_{tj} \subseteq T_{2j} U_{1j} \ldots U_{tj}$. Let H be new and for $1 \le i \le t$ set

$$U_i \equiv \lambda y_1 \ldots y_k\, H(U_{i1} y_1 \ldots y_k) \ldots (U_{im} y_1 \ldots y_k)\ .$$

Suppose $\mathfrak{M} \vDash T_1 U_1 \ldots U_t \subseteq T_2 U_1 \ldots U_t$. Then for some $1 \le j \le m$ $\mathfrak{M} \vDash T_1 U_1 \ldots U_t \subseteq T_{2j} U_1 \ldots U_t$. Thus $\lambda \sqcup \vdash T_1 U_1 \ldots U_t \subseteq T_{2j} U_1 \ldots U_t$. Hence, $\lambda \sqcup \vdash T_1 U_{1j} \ldots U_{tj} = [\lambda u_1 \ldots u_m u_j / H] T_1 U_1 \ldots U_t \subseteq [\lambda u_1 \ldots u_m u_j / H] T_{2j} U_1 \ldots U_t = T_{2j} U_{1j} \ldots U_{tj}$ and $\mathfrak{M} \vDash T_1 U_{1j} \ldots U_{tj} \subseteq T_{2j} U_{1j} \ldots U_{tj}$. This is a contradiction.

Case $T_1 \equiv (3)$. For some $1 \le i \le n$ $\lambda \sqcup \nvdash T_{1i} \subseteq T_2$, so this case follows immediately from the induction hypothesis.

Case $T_1 \equiv (4)$. This case is obvious when $T_2 \equiv$ (b).

Subcase $T_2 \equiv$ (c). Now for $1 \le j \le m$ $\lambda \sqcup \nvdash T_1 \subseteq T_{2j}$, so by induction hypothesis $\exists U_{1j} \ldots U_{tj} \in \bar{\Lambda}_{\mathfrak{M}}$ such that $\mathfrak{M} \nvDash T_1 U_{1j} \ldots U_{tj} \subseteq T_{2j} U_{1j} \ldots U_{tj}$. Let H be new and set

$$U_i \equiv \lambda y_1 \ldots y_k\, H(U_{i1} y_1 \ldots y_k) \ldots (U_{im} y_1 \ldots y_k)\ .$$

We have $T_1 U_1 \ldots U_t \underset{\beta\eta}{=} H(U_{i1}(T_{11} U_1 \ldots U_t) \ldots (T_{1n} U_1 \ldots U_t)) \ldots (U_{im}(T_{11} U_1 \ldots U_t) \ldots (T_{1n} U_1 \ldots U_t))$. The remainder of this case proceeds as in case $T_1 \equiv$ (2) $T_2 \equiv$ (c).

Subcase $T_2 \equiv$ (d). This case is obvious unless $i = j$ (so $n = m$). For some $1 \le l \le n$, $\lambda \sqcup \nvdash T_{1l} \subseteq T_{2l}$ so by induction hypothesis there exist $V_1 \ldots V_{t+k} \in \bar{\Lambda}_{\mathfrak{M}}$ such that $\mathfrak{M} \nvDash T_{1l} V_1 \ldots V_{t+k} \subseteq T_{2l} V_1 \ldots V_{t+k}$. Let H be new and set

$$U_i \equiv \lambda y_1 \ldots y_n\, H(y_l V_{t+1} \ldots V_{t+k})(V_i y_1 \ldots y_n)$$
$$U_r \equiv V_r \text{ if } r \ne i\ .$$

We have $T_1 U_1 \ldots U_t \underset{\beta\eta}{=} H(T_{1l} U_1 \ldots U_t V_{t+1} \ldots V_{t+k})(V_i(T_{11} U_1 \ldots U_t) \ldots (T_{1n} U_1 \ldots U_t))$ and $T_2 U_1 \ldots U_t \underset{\beta\eta}{=} H(T_{2l} U_1 \ldots U_t V_{t+1} \ldots V_{t+k})(V_i(T_{2l} U_1 \ldots U_t) \ldots (T_{2n} U_1 \ldots U_t))$. If $\mathfrak{M} \vDash T_1 U_1 \ldots U_t \subseteq T_2 U_1 \ldots U_t$ then $\mathfrak{M} \vDash T_{1l} U_1 \ldots U_t V_{t+1} \ldots V_{t+k} \subseteq T_{2l} U_1 \ldots U_t V_{t+1} \ldots V_{t+k}$. Hence, $\lambda \sqcup \vdash T_{1l} U_1 \ldots U_t V_{t+1} \ldots V_{t+k} \subseteq T_{2l} U_1 \ldots U_t V_{t+1} \ldots V_{t+k}$. Thus $\lambda \sqcup \vdash T_1 V_1 \ldots V_{t+k} = [\lambda uv\ v/H] T_{1l} U_1 \ldots U_t V_{t+1} \ldots V_{t+k} \subseteq [\lambda uv\ v/H] T_{2l} U_1 \ldots U_t V_{t+1} \ldots V_{t+k} = T_{2l} V_1 \ldots V_{t+k}$. Thus $\mathfrak{M} \vDash T_{1l} V_1 \ldots V_{t+k} \subseteq T_{2l} V_1 \ldots V_{t+k}$. This is a contradiction.

From the proof of the lemma we obtain $\lambda \sqcup \vdash T_1 \subseteq T_2 \Leftrightarrow$

$T_1 \equiv (1)$ or
$T_1 \equiv (2)$ and
  $T_2 \equiv$ (b) and $F \equiv G$ and for $1 \le i \le n$ $\lambda \sqcup \vdash T_{1i} \subseteq T_{2i}$ or
  $T_2 \equiv$ (c) and for some $1 \le j \le m$ $\lambda \sqcup \vdash T_1 \subseteq T_{2j}$ or
$T_1 \equiv (3)$ and for $1 \le i \le n$ $\lambda \sqcup \vdash T_{1i} \subseteq T_2$ or

$T_1 \equiv (4)$ and
$T_2 \equiv (c)$ and for some $1 \leq j \leq m$ $\lambda\sqcup \vdash T_1 \subseteq T_{2j}$ or
$T_2 \equiv (d)$ and $i = j$ and for $1 \leq k \leq n$ $\lambda\sqcup \vdash T_{1k} \subseteq T_{2k}$.

Thus we have the

Proposition $\mathfrak{M} \models T_1 \subseteq T_2 \Leftrightarrow \lambda\sqcup \vdash T_1 \subseteq T_2$. *Moreover, in* $\mathfrak{M}$, $\subseteq$ *is locally finite* (*i.e., intervals are finite*) *and recursive.*

Corollary 1 *If* $Tx = x$ *is solvable in every model of* $\lambda\sqcup$ *then for some* n $\lambda\sqcup \vdash T^{n+1} = T^n$.

Proof: If $\mathfrak{M} \models TU = U$ then for all n $\mathfrak{M} \models T^n \subseteq U$. Thus for some n $\mathfrak{M} \models T^{n+1} = T^n$.

Corollary 2 $\lambda\sqcup$ *has the finite model property, i.e., invalid equations have finite countermodels.*

Proof sketch: Construct $\mathfrak{M}_a$ for $\mathfrak{J}_a$ as $\mathfrak{M}$ was constructed for $\mathfrak{J}$ using $\mathcal{P}_n$ for sufficiently large n. There exists a total homomorphism from $\mathfrak{M}$ onto $\mathfrak{M}_a$. In particular $\mathfrak{M}_a \models \lambda\sqcup$. Now apply the proposition for appropriate a.

REFERENCES

[1] Andrews, Peter B., "Resolution in type theory," *The Journal of Symbolic Logic*, vol. 36 (1971), pp. 414–432.

[2] Church, Alonzo, "A formulation of the simple theory of types," *The Journal of Symbolic Logic*, vol. 5 (1940), pp. 56–68.

[3] Dezani-Ciancaglini, M., "Characterization of normal forms possessing inverses in the $\lambda\beta\eta$ calculus," *Theoretical Computer Science*, vol. 2 (1976), pp. 323–337.

[4] Friedman, Harvey, "Equality between functionals," *Springer Lecture Notes in Mathematics*, vol. 453 (1975), pp. 22–37.

[5] Gödel, Kurt, "Über eine bisher noch nicht benutzen Erweiterung des finiten Standpunktes," *Dialectica*, vol. 12 (1958), pp. 280–287.

[6] Jacopini, G., "A condition for identifying two elements of whatever model of combinatory logic," pp. 213–219 in *λ-Calculus and Computer Science Theory*, C. Bohm, ed., *Springer Lecture Notes in Computer Science*, vol. 37, Springer-Verlag, Berlin, 1975.

[7] Scott, Dana, Lectures on a mathematical theory of computation, Oxford University Computer Lab Tech. Monograph PRG 19.

[8] Statman, Richard, "On the existence of closed terms in the typed λ calculus I," pp. 511–534 in *Combinatory Logic, Lambda Calculus and Formalism*, eds., R. Hindley and J. Seldin, Academic Press, New York, 1980.

[9] Statman, Richard, "On the existence of closed terms in the typed λ calculus II," *Theoretical Computer Science*, vol. 15, pp. 329–338.

[10] Statman, Richard, "Embeddings, homomorphisms and λ-definability," manuscript, 1982.

[11] Statman, Richard, "Completeness, invariance and λ-definability," *The Journal of Symbolic Logic*, vol. 47, no. 1 (1981), pp. 297–306.

[12] Statman, Richard, "λ-definable functionals and $\beta\eta$ conversion," *Archiv für Mathematische Logik Grundlagenforschung*, vol. 22 (1983), pp. 1–6.

[13] Statman, Richard, "Logical relations and the typed λ-calculus," to appear in *Information and Control*.

Department of Mathematics
Carnagie-Mellon University
Pittsburgh, Pennsylvania 15213