Notre Dame Journal of Formal Logic
Volume 33, Number 1, Winter 1992

A Note on Philip Kitcher's Analysis of Mathematical Truth

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Abstract Philip Kitcher presents an attractive view of mathematical reality with an attending stipulational account of mathematical truth in *The Nature of Mathematical Knowledge*. However, if Kitcher's analysis of mathematical statements is correct, then some statements which are referentially true cannot satisfy his stipulational truth conditions. Thus, Paul Benacerraf's objection that stipulational theories of truth do not introduce genuine notions of truth finds Kitcher as a mark.

Philip Kitcher presents an attractive view of mathematical reality in *The Nature of Mathematical Knowledge* [2]. However, the attending stipulational account of mathematical truth is susceptible to a criticism articulated by Paul Benacerraf: Stipulational theories of truth provide truth conditions for mathematical statements which can be known to obtain. However, such theories do not introduce a genuine notion of *truth* because they do not employ the concepts of reference and satisfaction, and these notions are integral to our conception of truth (Benacerraf [1], pp. 678–679). Because the satisfaction of referential truth conditions is central to our conception of truth, a *minimal* requirement of a stipulational analysis of mathematical truth is that all and only those mathematical statements which satisfy the proposed stipulational truth conditions also satisfy referential truth conditions. This is why, for example, provability fails as a truth condition: All provable mathematical statements satisfy referential truth conditions, but not conversely.

Now Kitcher interprets the truth of mathematical statements stipulationally, which provides truth conditions known to obtain: A mathematical statement is true if it is the logical consequence of conventional definitions. Then, after analyzing mathematical statements as universally quantified conditionals, Kitcher attempts to show that: (i) referential explanations of truth can be enhanced by a stipulational account of reference, and (ii) stipulational truth conditions can be connected with referential truth conditions ([2], pp. 69, 141). Although (i) is

Received October 19, 1989; revised February 15, 1991

plausible, (ii) is not, for if Kitcher's analysis of mathematical statements is correct, then some statements which are referentially true cannot satisfy his stipulational truth conditions. Thus, Benacerraf's objection has force.

1 Consider Kitcher's Mill-arithmetical analysis of the theorem that there is no greatest prime. Stipulate by definition the primitive predicates $P_1, \ldots, P_n$ and primitive relations $R_1, \ldots, R_k$ of arithmetic such that our actual collecting and segregating activities approximately conform to the definitions. So, for some i between 1 and n, P_i is the primitive predicate '$\times$ is a one operation', and for some j between 1 and k, R_j is the primitive relation '$\times$ is a successor operation of y'. Let the conjunction of the axioms of Mill Arithmetic be represented as '$\mathrm{Mill}(P_i(\ldots \times \ldots), R_j(\ldots \times \ldots))$', where P_i and R_j are the primitive predicates and relations, respectively. Finally, let '$T(P_i(\ldots \times \ldots), R_j(\ldots \times \ldots))$' be the Mill-arithmetical analogue of the theorem that there is no greatest prime. Of course, the theorem is a logical consequence of the postulates of Mill Arithmetic, so by the deduction theorem:

$$(1)\quad (\ldots \times \ldots)(\mathrm{Mill}(P_i(\ldots \times \ldots), R_j(\ldots \times \ldots)) \\ \rightarrow T(P_i(\ldots \times \ldots), R_j(\ldots \times \ldots))).$$

Now Kitcher understands all mathematical statements to be universally quantified conditionals like (1), which makes the *implicit* universal quantifier in the arithmetical theorem '$T(P_i(\ldots \times \ldots), R_j(\ldots \times \ldots))$' *explicit*. Now (1) is a logical truth, but it follows by virtue of how the primitive predicates and relations were initially defined by stipulation. So in this sense (1) is stipulationally true, that is, true in virtue of the meanings of the primitives. In general, then, mathematics explores the consequences of a system of definitions which we are to understand as an implicit specification of the ideal operations of an ideal agent ([2], pp. 112–116).

Consider Kitcher's first claim that referential explanations of truth can be enhanced by a stipulational account of reference. On his view, stipulations determine which referential relations hold by fixing referents. Because a referential explanation of the truth of a statement can be enhanced by an analysis of how the extensions of the component expressions are determined, this much of Kitcher's exposition is on target simply because *any* appropriate theory of reference will enhance explanations of referential truth. Thus, the causal theory of natural kind terms "enhances" the referential explanation of the truth of "Acids are proton donators". But, whereas such accounts of reference fixing may enhance referential explanations of truth, they are ancillary to the Tarskian analysis which—quite frankly—gets along nicely without them.

Let us turn now to Kitcher's second claim, that his stipulational account of reference yields an analysis of truth robust enough to provide truth conditions which can be connected with referential truth conditions. Employing Kitcher's analogy, observe that the stipulated definition of an ideal gas which satisfies:

$$(2)\quad (x)(Gx \rightarrow Px \cdot Vx = R \cdot Tx)$$

is an idealization of the actual gases with which we are familiar. Thus, the stipulation of such an ideal gas is well-grounded; if all of the "accidental properties" were removed from actual gases, they would be ideal ([2], p. 117). Now, Kitcher continues, if the extension of the predicate 'Gx' in (2) is taken to be comprised of actual gases, then (2) will be false, for the relation expressed by the consequent is false of actual gases (although it approximates their behavior). Yet (2) is true by virtue of the definition of an ideal gas; it indicates how ideal gases would behave if there were any satisfying the definition ([2], p. 116). But, he continues, this stipulational account of truth "need not bypass the concepts of reference and satisfaction" for we can provide a "perfectly good explanation" of the truth of (2) referentially—it is vacuously true, for there are no ideal gases.

> . . . if we specify the notion of an ideal gas in the standard kinetic-theoretic way, we fix as true the statement that the temperature, volume, and pressure of an ideal gas are related by the equation $PV = RT$. This statement has the logical form [of (2)], and we can provide a perfectly good referential explanation of its truth by pointing out that, because there are no ideal gases, no sequence satisfies the antecedent of the conditional whose closure is [(2)]. ([2], pp. 140–141)

We are to understand the statements of mathematics like (1) to be quantified conditionals analogous to the ideal gas laws. So, the statements of mathematics are vacuously true when referentially construed:

> Statements of [mathematics], like statements of ideal gas theory, turn out to be vacuously true. They are distinguished from the host of thoroughly uninteresting and pointless vacuously true statements . . . by the fact that the stipulations on . . . the ideal agent abstract from accidental limitations of human agents. ([2], p. 117n)

That is, because the axioms are the implicit specification of an ideal agent's operations, the explanation of the referential truth of (1) will be the same as (2). If the extension of '$\mathrm{Mill}(P_i(\ldots \times \ldots), R_j(\ldots \times \ldots)$' is taken to be comprised of our actual collecting and segregating activities, then (1) will be false, because the arithmetical activities characterized by '$T(P_i(\ldots \times \ldots), R_j(\ldots \times \ldots)$' are ideal. No actual agent can perform them, just as no actual gas satisfies the ideal gas laws. However, (1) is true by virtue of the stipulated definition of an ideal agent whose operations satisfy the axioms of Mill Arithmetic. And, as in the case of the ideal gas laws, Kitcher can give a "perfectly good referential explanation" for its truth: (1) is vacuously true, for there are no ideal agents.

But here is the rub. Because mathematical statements are universally quantified conditionals, Kitcher understands them to be referentially true since they are vacuous. Yet, *all* universally quantified conditionals based upon stipulation satisfied by nothing will be vacuously true. So, importantly, the statement:

$$\text{(3)}\quad (\ldots \times \ldots)(\mathrm{Mill}(P_i(\ldots \times \ldots), R_j(\ldots \times \ldots)) \\ \rightarrow -T(P_i(\ldots \times \ldots), R_j(\ldots \times \ldots)))$$

will be *referentially true* like (1). But (3) must be *stipulationally false* if the axioms of Mill Arithmetic are consistent, for (1) satisfies Kitcher's stipulational truth conditions.

Benacerraf argues that a satisfactory combinatorial analysis of mathematical truth must connect the truth conditions known to obtain with referential truth conditions. If the "connection" is successful, then all and only those mathematical statements which satisfy the proposed combinatorial truth conditions will satisfy referential truth conditions. On Kitcher's analysis, statements like (3) which are vacuously true when referentially construed cannot satisfy his stipulational truth conditions. Thus, Kitcher is not successful in making the connection required by Benacerraf, so his stipulational account of mathematical truth is unsatisfactory.

Acknowledgments For comments and advice concerning this, and earlier drafts, I wish to thank Timothy McCarthy, Arthur Melnick, and referees at *Notre Dame Journal of Formal Logic* and *Southern Journal of Philosophy.* Steven Wagner's help and encouragement were most valuable.

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