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Pleasant Ideals

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Abstract We introduce the concept of a *pleasant ideal* on a regular, uncountable cardinal κ, where an ideal I is pleasant if and only if it is closed under I-diagonal unions. We provide some examples and examine ideal operators related to pleasantness. We show an ideal is normal if and only if it is pleasant and extends the nonstationary ideal if and only if it is pleasant and selective.

When one is introduced to ideals and filters on cardinals, the idea of a normal ideal figures very prominently. In this paper we will examine a somewhat weaker property based on the size of the range of regressive functions.

Preliminaries Our set theoretic notation is (we hope) standard. The axiom of choice is assumed throughout so a cardinal is identified with the set of its ordinal predecessors. The letters κ and λ will be reserved for cardinals, while α, β, etc. will represent ordinals.

An ideal on a regular uncountable cardinal κ is a collection of subsets of κ that is closed under subset and finite union. All of our ideals will contain all singletons and be $<\kappa$-complete (closed under unions of size less than κ), and thus all of our ideals will extend $I_\kappa \equiv \{X \subseteq \kappa \mid |X| < \kappa\}$. If I is an ideal on κ, then I^* will denote the dual filter $\{Y \subseteq \kappa \mid \kappa - Y \in I\}$ and I^+ will be the co-ideal $\{Y \subseteq \kappa \mid Y \notin I\}$. If I is an ideal on κ and $A \subseteq \kappa$, then the set $I \restriction A \equiv \{X \subseteq \kappa \mid X \cap A \in I\}$ is also an ideal on κ.

If $A \subseteq \kappa$ and $f: A \to \kappa$, f will be called *regressive* if $f(\alpha) < \alpha$ for $\alpha \in A - \{0\}$. *Weakly regressive* functions will have $f(\alpha) \leq \alpha$. For $n \in \omega$, f^n will denote f composed with itself n times. If $B \subseteq A$ then $f[B]$ is $\{\gamma \mid (\exists \beta \in B)(\gamma = f(\beta))\}$. If I is an ideal on κ, f will be called I-small if $f^{-1}(\{\xi\}) \in I$ for every $\xi < \kappa$.

An ideal I is normal if I is closed under diagonal unions: if $X_\alpha \in I$ for each $\alpha < \kappa$, then $\nabla_{\alpha<\kappa} X_\alpha \equiv \{\xi < \kappa \mid (\exists \alpha < \xi)(\xi \in X_\alpha)\} \in I$. We will also work with diagonal unions with restricted index sets: $\nabla_{\alpha \in Q} X_\alpha \equiv \{\xi < \kappa \mid (\exists \alpha < \xi)(\alpha \in Q \wedge \xi \in X_\alpha)\}$. For example, normal ideals are closed under all diagonal unions, while pleasant ideals will be shown to be closed under diagonal unions indexed

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by sets in the ideal. The nonstationary ideal $NS_\kappa \equiv \{A \subseteq \kappa \mid (\exists \text{ a club } C \text{ in } \kappa)(A \cap C = \emptyset)\}$ is a normal ideal and it is contained in every normal ideal on κ. An ideal I is subnormal if it is a subset of a normal ideal.

Fodor's Theorem (see Fodor [2]) relates normal ideals and regressive functions. He proves that I is normal if and only if there is no I-small regressive function with domain in I^+. In Baumgartner et al. [1] it is shown that I is subnormal if and only if there is no I-small regressive function with domain in I^*.

1 Pleasant ideals By Fodor's Theorem, we know that if an ideal I is normal, then there is no regressive I-small function with domain in I^+. We will look at a more generous class of ideals by allowing such functions to exist while still restricting their behavior. In particular, we will demand that if f is I-small and regressive and $A \in I^+$, then $f[A] \in I^+$. Such ideals will be called pleasant.

Definition 1.1 An ideal I on κ is said to be *pleasant* if for every $A \in I^+$ and every regressive I-small $f: A \to \kappa$, $f[A] \in I^+$. An ideal which fails to be pleasant will be called *unpleasant*.

Notice that pleasantness is equivalent to demanding that if $A \in I^+$ then $f[A] \in I^+$ for weakly regressive I-small functions f.

Theorem 1.2
(a) *An ideal I on κ is pleasant if and only if I is closed under I-diagonal unions; in other words if $Q \in I$ and $(\forall \alpha \in Q)(X_\alpha \in I)$, then $\nabla_{\alpha \in Q} X_\alpha \in I$.*
(b) *An ideal I on κ is normal if and only if for every $A \in I^*$ and every weakly regressive I-small $f: A \to \kappa$, $f[A] \in I^*$.* $(*)$

Proof: (a) If $\nabla_{\alpha \in Q} X_\alpha \in I^+$, consider the following regressive function on $\nabla_{\alpha \in Q} X_\alpha$:

$$f(\beta) = \text{some } \alpha \in \beta \cap Q \text{ such that } \beta \in X_\alpha.$$

Then every $f^{-1}(\{\xi\})$ is a subset of X_ξ, so f is I-small, but $f[(\nabla_{\alpha \in Q} X_\alpha)] \subseteq Q$, so I is not pleasant. On the other hand, if I is not pleasant, fix a regressive, I-small function f and $A \in I^+$ such that $f[A] \in I$. Let $X_\alpha = f^{-1}(\{\alpha\})$ for each α, and then $\nabla_{\alpha \in f[A]} X_\alpha = A$, which shows I is not closed under I-diagonal unions.

(b) The proof of 1.2b will consist of a series of three lemmas.

Lemma 1.2.1 *If I is normal then $(*)$ holds.*

Proof: Since I is normal and f is I-small, we know $\{\alpha \in A \mid f(\alpha) < \alpha\} \in I$. Therefore $\{\alpha \in A \mid f(\alpha) = \alpha\} \in I^*$ and $(*)$ holds.

Lemma 1.2.2 *Suppose $(*)$ holds for an ideal I, suppose $B \in I^+$, and suppose $f: B \to \kappa$ is I-small and weakly regressive. Then $f[B] \cap B \in I^+$. (And thus if $(*)$ holds for an ideal I then I is pleasant.)*

Proof: If not, we may assume (by intersecting B with $\kappa - (f[B] \cap B)$) that $f[B] \cap B = \emptyset$. Define $g: \kappa \to \kappa$ by

$$g(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in B \\ \alpha & \text{otherwise.} \end{cases}$$

It is easy to check that g is weakly regressive and I-small, but $g[\kappa] \cap B = \varnothing$, so $g[\kappa] \notin I^*$ and condition $(*)$ fails.

Lemma 1.2.3 *If an ideal I satisfies $(*)$, then I is normal.*

Proof: Assume I satisfies $(*)$ but is not normal. By 1.2.2, we know that I is pleasant. Since I is not normal, find $A \in I^+$ and an I-small, regressive function $f: A \to \kappa$. Extend f to $g: \kappa \to \kappa$ such that g is the identity function on $\kappa - A$. Clearly g is I-small and weakly regressive.

By well-foundedness, we know that for each $\alpha \in A$ there exists $n \in \omega$ such that $g^n(\alpha) \in \{0\} \cup (\kappa - A)$. Let $A_0 = \{0\} \cap A$ and for each $n \in \omega$, $n > 0$, let $A_n = \{\alpha \in A \mid g^n(\alpha) \in \{0\} \cup (\alpha - A) \text{ and } g^{(n-1)}(\alpha) \in A - \{0\}\}$. Clearly $A = \bigcup_{n \in \omega} A_n$. Since the ideal I is $<\kappa$-complete and $A \in I^+$ we know that some $A_k \in I^+$. But $A_0 \in I$ by construction, and for $k > 0$ $g[A_k] \cap A_k = \varnothing$, and so by Lemma 1.2.2 each $A_k \in I$, which is a contradiction. Thus our assumption is wrong and any ideal I satisfying $(*)$ must be normal.

Clearly every normal ideal on κ is pleasant. The converse, however, is not true.

Proposition 1.3 *I_κ is a pleasant ideal.*

Proof: Suppose Q, X_α are given with each bounded in κ. Since κ is regular, $|Q| < \kappa$ and I_κ is $<\kappa$-complete, $\bigcup_{\alpha \in Q} X_\alpha \in I_\kappa$, so certainly $\nabla_Q X_\alpha \in I_\kappa$.

An argument similar to the proof of 1.2.3 provides us with the fact that not every ideal on κ is pleasant.

Proposition 1.4 *If I is pleasant then I is subnormal.*

Proof: If I is pleasant but not subnormal, let $A \in I^*$ and $f: A \to \kappa$ be regressive and I-small. Extend f to g as in the proof of 1.2.3, and again each $A_k \in I$ while $\bigcup_{k \in \omega} A_k \in I^+$, a contradiction.

Proposition 1.5 *A subnormal ideal need not be pleasant.*

Proof: Fix a stationary $A \subseteq \kappa$ such that $\kappa - A$ is also stationary and write A as the union of κ A_α's, with each A_α stationary, $A_\alpha \cap (\alpha + 1) = \varnothing$, and the A_α's pairwise disjoint. Let $\hat{I}$ be the $<\kappa$-complete ideal generated by $NS_\kappa \cup \{A_\alpha\}_{\alpha<\kappa}$. We claim that $A \notin \hat{I}$. If $A \in \hat{I}$, then $A \subseteq B \cup (\bigcup_{\alpha<\xi} A_\alpha)$ for some nonstationary set B and some $\xi < \kappa$. But $A - \bigcup_{\alpha<\xi} A_\alpha$ is certainly stationary, as it includes $A_{\xi+1}$, which provides a contradiction.

But now we can show that $\hat{I}$ is not pleasant. Notice that if $\beta \in A$ then there is some $g(\beta) < \beta$ such that $\beta \in A_{g(\beta)}$. Since $g^{-1}(\{\xi\}) = A_\xi$, g is $\hat{I}$-small and regressive. Define

$$f(\beta) = \begin{cases} \text{least successor ordinal} \geq g(\beta) & \text{if } \beta \in A \\ \beta & \text{otherwise.} \end{cases}$$

Clearly f is weakly regressive, and since $f^{-1}(\{\xi\}) \subseteq g^{-1}(\{\xi\}) \cup g^{-1}(\{\xi - 1\}) \cup \{\xi\}$, f is $\hat{I}$-small. But $A \in \hat{I}^+$ and $f[A] \subseteq \{\beta \mid \beta \text{ is a successor ordinal}\} \in \hat{I}$, so $\hat{I}$ is not pleasant.

Now to show that not every subnormal ideal is pleasant, let $\tilde{I}$ be the normal ideal generated by $NS_\kappa \cup \{A\}$, then $\hat{I} \subseteq \tilde{I}$ and $\hat{I}$ is subnormal but not pleasant.

The construction outlined above provides many examples of extensions of NS_κ that are not pleasant. Alternatively, one can take any non-normal ideal I, and since I does not satisfy condition (*) of 1.2(b) there is an I-small weakly regressive $g: \kappa \to \kappa$ and $C \in I^*$ such that $g[C] \notin I^*$. Find $A \in I^+$ such that $(g[C]) \cap A = \varnothing$ and let $B = A \cap C$. Then g is weakly regressive and $I \restriction A$-small, $B \in (I \restriction A)^+$, but $g[B] \in (I \restriction A)$, and so $I \restriction A$ is not pleasant. To summarize,

Observation 1.6 Every non-normal ideal I has an unpleasant extension $I \restriction A$.

We will now examine a method for extending an ideal to a pleasant ideal, and show that not every pleasant ideal is normal.

Definition 1.7 If I is an ideal on κ, let $P(I)$ be the smallest (not necessarily proper) pleasant ideal extending I. $P(I)$ is called the *pleasant closure* of I.

In Proposition 1.9 we will show that we can construct $P(I)$ by iterating closure under diagonal unions indexed by sets in I. Lemma 1.8 is easy to prove by chasing the definitions.

Lemma 1.8 *If $B_\alpha \subseteq \kappa$, $Q_\beta \subseteq \kappa$ for $\alpha, \beta < \kappa$ and $R \subseteq \kappa$, then*

$$\nabla_{\alpha \in (\nabla_{\beta \in R} Q_\beta)} B_\alpha \subseteq \nabla_{\beta \in R} \left[\nabla_{\alpha \in Q_\beta} B_\alpha \right].$$

Proposition 1.9 *Suppose I is an ideal on κ. Define by recursion on the ordinals*

$$P_0 = I$$

$$P_\lambda = \bigcup_{\beta < \lambda} P_\beta \quad \textit{for limit ordinals } \lambda$$

$$P_{\beta+1} = P_\beta \cup \{X \mid (\exists R \in I)(\exists R\textit{-indexed sequence of sets } B_\alpha) \\ (\textit{each } B_\alpha \in P_\beta \textit{ and } X = \nabla_{\alpha \in R} B_\alpha)\}.$$

Since each $P_\beta \subseteq \mathcal{P}(\kappa)$ there is some ξ such that $P_\xi = P_{\xi+1}$. Call this set P. Then $P = P(I)$.

Proof: To show $P \subseteq P(I)$ it suffices to show that if $P_\alpha \subseteq P(I)$ then $P_{\alpha+1} \subseteq P(I)$. So assume $P_\alpha \subseteq P(I)$ and $X = \nabla_{\beta \in Q} X_\beta \in P_{\alpha+1}$, with Q and X_β in P_α. Since $P(I)$ is closed under $P(I)$-diagonal unions and by hypothesis Q and X_β are all in $P(I)$, $X \in P(I)$, as needed.

To complete the proof we will show that P is a pleasant ideal, by showing that P is closed under P-diagonal unions. This easily implies that P is $<\kappa$-complete. Downward closure follows from the definition.

To prove closure under P-diagonal unions we will induct on the ordinals. In particular we induct on the level where the index set Q is put into P. So, suppose B_α and Q are in P. We want to show $\nabla_{\alpha \in Q} B_\alpha \in P$. Since $Q \in P$ we know for some $\xi + 1$ $Q \in P_{\xi+1} - P_\xi$. Thus there exist $R \in I$ and $Q_\beta \in P_\xi$ (for each $\beta \in R$) such that $Q = \nabla_{\beta \in R} Q_\beta$. By our induction hypothesis we know that P is closed under diagonal unions indexed by R or by Q_β for any $\beta \in R$.

But now everything is easy. We want to show $\nabla_{\alpha \in Q} B_\alpha \in P$. But $\nabla_{\alpha \in Q} B_\alpha = \nabla_{\alpha \in \nabla_{\beta \in R} Q_\beta} B_\alpha \subseteq \nabla_{\beta \in R} [\nabla_{\alpha \in Q_\beta} B_\alpha]$, where the inclusion is justified by Lemma 1.8. But $\nabla_{\beta \in R} [\nabla_{\alpha \in Q_\beta} B_\alpha]$ is in P by the induction hypothesis. Thus P is an ideal which is closed under P-diagonal unions. Therefore P is pleasant.

It is also clear that $P(I)$ is the intersection of all pleasant ideals extending I. The intersection is a subset of $P(I)$ as $P(I)$ is pleasant, and the reverse inclusion follows immediately from 1.9.

Theorem 1.10 *There exist extensions of I_κ that are pleasant but not normal.*

Proof: Naturally, we will construct such an ideal. Let I be the ideal generated by $I_\kappa \cup \{\lambda + 2 \mid \lambda \text{ is a limit ordinal less than } \kappa\}$. Clearly $P(I) \subseteq NS_\kappa$ as NS_κ is a pleasant extension of I. We will show $P(I) \subset NS_\kappa$ and thus that $P(I)$ is not normal.

We claim that if R is a stationary set of limit ordinals, then $R \notin P(I)$ and $R + 1 \equiv \{\beta + 1 \mid \beta \in R\} \notin P(I)$. The first of these claims is clear, as $P(I) \subseteq NS_\kappa$, and the proof of the second is by induction on the levels of $P(I)$.

It is clear that $R + 1 \notin I$. Suppose $R + 1 \in P_{\xi+1} - P_\xi$. Then $R + 1 = \nabla_{\alpha \in Z} X_\alpha$, where Z, X_α are in P_ξ and (without loss of generality) the X_α's are pairwise disjoint. Since $R + 1 = \nabla_{\alpha \in Z} X_\alpha$, if $\beta + 1 \in R + 1$ then there is an ordinal $f(\beta) \leq \beta$ such that $\beta + 1 \in X_{f(\beta)}$. The function f is weakly regressive and has domain R. If $\{\beta \mid f(\beta) < \beta\}$ is stationary, then $f(\beta) = \eta$ for each β in a stationary set Q. But then $X_\eta \supseteq Q + 1 \notin P_\xi$ by the induction hypothesis. If, on the other hand, $\{\beta \mid f(\beta) = \beta\}$ is stationary, then as the range of f is a subset of Z, Z must be stationary, contradicting our first claim in the previous paragraph. Thus the function f cannot exist and $R + 1 \notin P(I)$. Thus $P(I)$ is a proper subset of NS_κ and therefore $P(I)$ is not normal.

2 Characterizing normality

In this section we examine the connections among normal, selective, and pleasant ideals. Our main result is the following:

Theorem 2.1 *If I is an ideal on κ, the following are equivalent*:
(a) *I is normal.*
(b) *I is pleasant and extends NS_κ.*
(c) *I is pleasant and selective.*

Proof: Recall that an ideal I on κ is said to be *selective* if for every I-small $f: \kappa \to \kappa$ there is an $A \in I^*$ such that $f \restriction A$ is one to one. It is well-known that (a) implies both (b) and (c).

To show that (c) implies (a), assume that I is selective, pleasant, and not normal. We will derive a contradiction. Since I is not normal there is a weakly regressive I-small function $f: \kappa \to \kappa$ such that $f[\kappa] \notin I^*$. So $D \equiv \{\alpha \mid f(\alpha) < \alpha\} \in I^+$. As I is selective and f is I-small, there is an $A \in I^*$ such that $f \restriction A$ is injective. Consider $\alpha \in D \cap A$. If $f(\alpha) \in A$ and $f(f(\alpha)) = f(\alpha)$ then we have a contradiction as $f(\alpha) \neq \alpha$, but $f \restriction A$ is one to one. Therefore, if $f(\alpha) \in A$ then $f(\alpha) \in D$. So by well-foundedness, for each $\alpha \in D \cap A$ there is a first $n \in \omega$ such that $f^n(\alpha) \notin A$. (Otherwise $\langle f^n(\alpha) \rangle_{n \in \omega}$ is a decreasing sequence in $D \cap A$.) This partitions $D \cap A$ into ω pieces, one of which must be in I^+ by $<\kappa$-completeness. So there is some $k \in \omega$ such that $Z \equiv \{\alpha \in D \cap A \mid f(\alpha) \in A, f^2(\alpha) \in A, \ldots, f^{k-1}(\alpha) \in A, f^k(\alpha) \notin A\} \in I^+$. Since I is pleasant $f^k[Z] \in I^+$, but $f^k[Z] \cap A = \varnothing$, a contradiction.

This leaves us with the task of showing that (b) implies (a). We prove a slightly more general result:

Claim 2.2.1 *Suppose I is pleasant ideal such that I contains an unbounded set Q and the complement of the closure of Q. Then I is normal.*

Proof: Suppose $A_\alpha \in I$ for each $\alpha < \kappa$ and $A = \nabla_{\alpha<\kappa} A_\alpha$. We show $A \in I$. For $\beta \in Q$, let $B_\beta = \bigcup_{\alpha \leq \beta} A_\alpha$. As I is $<\kappa$-complete, each B_β is in I. Let $W = A - \nabla_{\beta \in Q} B_\beta$. We will show $W \in I$, and thus $A \subseteq \nabla_{\beta \in Q} B_\beta \cup W$ is in I, and we are done.

To show $W \in I$, we show W is disjoint from the closure of Q. Suppose η is a limit point of Q and $\eta \in A$. Thus there is some $\alpha < \eta$ such that $\eta \in A_\alpha$. As η is a limit point of Q there is some $\beta \in Q$ such that $\alpha \leq \beta < \eta$, and so $\eta \in B_\beta$. Thus $\eta \in \nabla_{\beta \in Q} B_\beta$ and $\eta \notin W$. This means that W is contained in the complement of the closure of Q, which is in I by hypothesis. Therefore W is in I as needed.

And now to see that any pleasant extension of the nonstationary ideal is normal, we need only notice that the closure of any unbounded set is a club.

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