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Book Review

Troelstra, A. S. and D. van Dalen. *Constructivism in Mathematics*, Volume 1. *Studies in the Logic and the Foundations of Mathematics*, Volume 121. North-Holland Press, Amsterdam, 1988. 376 pages.

Troelstra, A. S. and D. van Dalen. *Constructivism in Mathematics*, Volume 2. *Studies in the Logic and the Foundations of Mathematics*, Volume 123. North-Holland Press, Amsterdam, 1988. 552 pages.

This is a two-volume work, which the authors intend as an "all-round introduction to constructivism". Constructive mathematics can be approached from at least four directions: philosophy, logic, mathematics, and computer science. Each field has something to contribute to the subject, and the proper understanding of it requires all four. That doesn't make it an easy subject to learn, as some background in all four disciplines is essential for a clear understanding. (The difficulty of acquiring this multi-disciplinary background is why the subject is so often misunderstood!) A proper "all-round introduction" to the subject should provide the necessary background in all four areas. Obviously, this is a difficult task.

In the preface, the authors point out that there are now available many introductory texts and monographs dealing with constructive mathematics in its various forms. Most of these books emphasize one or another of the four aspects. For example, approaching constructive mathematics from the mathematician's viewpoint, with no logic or computer science and very little philosophy, we have *Constructive Analysis* by Bishop (revised version by Bishop and Bridges), and *Varieties of Constructive Mathematics* by Bridges and Richman. Similarly, we have Aberth's and Kusner's books presenting work of the Russian constructivists (who use recursion theory in their mathematics). These works on analysis have been followed by *A Course in Constructive Algebra* by Mines, Richman, and Ruitenberg. On the more philosophical side, but making use of logic, we have *Elements of Intuitionism* by Dummett. On the logical side we have Beeson's *Foundations of Constructive Mathematics*, as well as Troelstra's earlier work (Springer Lecture Notes 344). Finally, there is *Computer Algebra*, by J. Davenport, which represents the practice of constructive mathematics in the

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computer era. A person reading all these works would certainly have an "all-round introduction to constructivism".

The question is, then, can a two-volume set really do an equivalent job? I think not. At least, the books under review are not an "all-round introduction"; instead they are thorough and excellent books on "Constructive Mathematics by and for Logicians". I hope this remark will not be negatively interpreted: as I tried to make clear, it is simply too difficult to write an all-round introduction. The two books under review are carefully put together, contain essential material, are logically organized, contain exercises and notes, and could well be used as textbooks in an advanced course, for students who have already had logic and recursion theory. These two volumes are good and useful books which may help serious students a great deal.

Volume I opens with an introductory chapter setting forth the main differences between the schools of constructive mathematics, and putting them in a proper historical perspective. Chapter 2 then takes up the actual discourse on the subject, and the starting point chosen is an introduction to formal logic: natural deduction systems, Hilbert-type systems, and Kripke semantics. The next two chapters duplicate much material from Troelstra's Springer Lecture Notes 344: formal systems for arithmetic, formalization of recursion theory, realizability, formal systems for choice sequences. Chapter 5 takes up technical questions about the formalization of the real numbers: should we use Dedekind cuts or Cauchy sequences? (Constructively they are not equivalent without some axioms of choice.) In Chapter 6, all this logical development is brought to bear on the mathematics: Bishop's notions of (uniform) continuity and differentiability are explained, and their logical relation to the fan theorem is developed. Fourteen pages are devoted to the mathematics of recursive real numbers, a delightfully bizarre subject, covered in more detail in the books by Aberth and Kušner and in a chapter of Beeson's book.

Volume II opens with a discussion of metric spaces. Bishop (1967) introduced the idea that the constructively appropriate definition of "compact" is "complete and totally bounded". Bishop's work also established, by the breadth of its mathematics, the interesting point that metric spaces suffice for mathematics; one does not really need the generality of topological spaces. The book under review takes the logician's view, that the most interesting result about metric spaces is that the fan theorem (all functions on 2^{N} are uniformly continuous) implies the Heine–Borel property for complete totally bounded spaces. The exposition of this result, like all the formal results in the book, is clear, direct, and to the point.

In Chapter 8, constructive algebra is taken up, and attention is paid to the difficulties in constructivizing classical theorems in group theory, linear algebra, polynomial ring theory, and the fundamental theorem of algebra. Much of the material overlaps *A Course in Constructive Algebra* mentioned above, but that book was not available when the authors were writing, so they can't be blamed for any overlap.

Chapter 9 explains formal systems for intuitionistic arithmetic and finite type theory, and proves the normalization theorem for terms. It also explains the theory APP, an abstract version of recursion theory which is a first-order variant of combinatory logic due to Feferman.

Chapter 10 is on the proof theory of intuitionistic logic. This is where the authors really come into their home territory—they have studied and taught this material for years, and the exposition is masterly. They explain natural deduction systems, sequent calculi, and Beth tableaus, and the connections between them, and prove the normalization theorem. Section 8 of Chapter 10 is particularly pleasing: it presents the "formulas-as-types" isomorphism, in which natural deduction trees are represented as terms, and normalization of proofs becomes normalization of terms. This is the only exposition I have seen which treats all the logical connectives and quantifiers; it is customary to treat only the implicative fragment, which corresponds to simple λ-terms, and ignore the difficult details of the general case. Here one can finally see it all spelled out.

Chapter 11 gives an explanation of Martin-Lof's type theories. These theories are in part responsible for the computer scientists' burst of interest in constructive mathematics, because of their applications in theoretical computer science. There isn't a word in the chapter about those applications. (See Constable et al., *The NuPrl Proof Development System*, and the review of it in *JSL*.) On the other hand, the logical details are impeccably well-done; every subtlety is correctly handled, and there are lots of subtleties in these systems. The chapter ends with a brief explanation of Friedman and Myhill's constructive set theories.

Chapter 12 is on Choice Sequences. Troelstra has previously written a whole book on this subject (1977), and so one is not surprised to find in this chapter a clear review of the main results about the formal theory of choice sequences. The hypothetical reader who has never heard of choice sequences before is going to be in deep water, though, by the time we get to the main metatheorem, which is a formal interpretation guaranteeing that choice sequences can be eliminated in favor of inductive definitions. The logical expertise required even to understand the statement of the theorem is far beyond the level of an "all-round introduction".

The last three chapters, which by page-count are about one-quarter of Volume II, are devoted to the semantics of constructive formal systems. Chapter 13 discusses Beth and Kripke models, with a completeness proof. It presents the Kreisel–Godel result that completeness of intuitionistic predicate calculus implies Markov's principle (and hence is not derivable). It goes on to discuss Heyting algebras. This discussion leads into Chapters 14 and 15, which give an explanation of sheaf models and their application to "higher order arithmetic" HAH. The authors stop short of tackling topos theory, saying it is "beyond the scope of this book". That is understandable, considering the already momentous size of the set, but the students will be well-prepared to take up the existing books on topos theory. In the preface, the authors kindly acknowledge their liberal use of handwritten notes by I. Moerdijk in the preparation of Chapters 13–16.

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