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Embedding Brouwer Algebras in the Medvedev Lattice

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Abstract We prove various results on embedding Brouwer algebras in the Medvedev lattice. In particular, we characterize the finite Brouwer algebras that are embeddable in the Medvedev lattice.

1 Introduction The following definition is fundamental throughout the paper:

Definition 1.1 Let $\mathfrak{L} = \langle L, \vee, \wedge, 0, 1 \rangle$ be a distributive lattice with 0,1 and let $\leq$ be the partial ordering relation of $\mathfrak{L}$. Then $\mathfrak{L}$ is a *Brouwer algebra* if $\mathfrak{L}$ can be given a binary operation $\rightarrow$ such that, for every $a, b, c \in L$,

$$b \leq a \vee c \Leftrightarrow a \rightarrow b \leq c.$$

(Notice that this is equivalent to saying that the set $\{c \in L : b \leq a \vee c\}$ has a least element and this least element equals $a \rightarrow b$.)

Also, we say that a distributive lattice with 0,1 is a *Heyting algebra* if the dual of $\mathfrak{L}$ is a Brouwer algebra (for details on Heyting algebras see e.g. Balbes and Dwinger [1]). Heyting algebras are often called pseudo-Boolean algebras (see e.g. Rasiowa [10]). In the remainder of the paper, we will often use without further comment the fact that every finite distributive lattice with 0,1 is a Brouwer algebra (also, a Heyting algebra).

Now, let $\mathfrak{M}$ be the Medvedev lattice (see Medvedev [7] and Rogers [11]). In Sorbi [13] we show that $\mathfrak{M}$ is not a Heyting algebra. On the other hand, it is known ([7]; see also [11], Theorem 13.XXIV, for a proof) that $\mathfrak{M}$ is a Brouwer algebra. In this paper we show that as a Brouwer algebra $\mathfrak{M}$ is in fact a fairly rich one, by proving various embedding results. In particular, we obtain a characterization of the finite Brouwer algebras that are embeddable in $\mathfrak{M}$, thus extending a similar embedding result proved in Skvortsova [12]. Among the consequences of this result is also a proof (see Corollary 2.8 below) of the fact that the set of identities of $\mathfrak{M}$ (in the sense of [11], §13.7, i.e. the propositional

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formulas which are valid in $\mathfrak{M}$, as defined later in this section) coincides with the set of theorems of the propositional logic obtained by adding the axiom scheme $\neg\alpha \vee \neg\neg\alpha$ to the intuitionistic propositional calculus (this result was essentially stated by Medvedev in [8], Theorem 2; a proof is contained in the aforementioned paper by Skvortsova [12]), contrary to the mistaken attribution to Medvedev, made in [11], of the coincidence of these identities with the theorems of the intuitionistic propositional calculus.

Except for the few changes and additions listed below, our notations are the same as in [11] (in particular §13.7), to which the reader is referred also for any unexplained notations and terminology used in this paper. As is customary in the literature, the operations of least upper bound and greatest lower bound in a lattice—thus in $\mathfrak{M}$ as well—are denoted by the symbols $\vee$ and $\wedge$, respectively. In this paper therefore, in reference to $\mathfrak{M}$ the symbols $\vee$ and $\wedge$ are interchanged with respect to the notation of [11], §13.7. Thus, given any degrees of difficulty A and B, $A \vee B$ denotes the least upper bound of A and B, and $A \wedge B$ denotes the greatest lower bound of A and B; given any functions f and g, by $f \vee g$ we denote the function h such that $f(x) = h(2x)$ and $g(x) = h(2x + 1)$, for every $x \in \omega$ (ω is the set of natural numbers); given $x \in \omega$ and a function f, $x * f$ denotes the function h such that $h(0) = x$ and $h(y + 1) = f(y)$, for all y; given mass problems $\mathfrak{A}$ and $\mathfrak{B}$, we let $\mathfrak{A} \vee \mathfrak{B} = \{f \vee g : f \in \mathfrak{A}\ \&\ g \in \mathfrak{B}\}$ and $\mathfrak{A} \wedge \mathfrak{B} = \{0 * f : f \in \mathfrak{A}\} \cup \{1 * g : g \in \mathfrak{B}\}$; given a mass problem $\mathfrak{A}$ and $x \in \omega$, it is convenient to let $x * \mathfrak{A} = \{x * f : f \in \mathfrak{A}\}$, thus $\mathfrak{A} \wedge \mathfrak{B} = 0 * \mathfrak{A} \cup 1 * \mathfrak{B}$. If I is a finite set and $\{\mathfrak{A}_i : i \in I\}$, $\{f_i : i \in I\}$ are collections of mass problems and functions, respectively, then the expressions $\bigvee_{i \in I} \mathfrak{A}_i$, $\bigwedge_{i \in I} \mathfrak{A}_i$, $\bigvee_{i \in I} f_i$ always refer to some fixed listing of the elements of I: for instance, if $i_0, \ldots, i_n$ is a listing of I then $\bigvee_{i \in I} \mathfrak{A}_i = (\cdots(\mathfrak{A}_{i_0} \vee \mathfrak{A}_{i_1}) \vee \cdots) \vee \mathfrak{A}_{i_n})$. Of course, the degrees of difficulty of the mass problems $\bigvee_{i \in I} \mathfrak{A}_i$ and $\bigwedge_{i \in I} \mathfrak{A}_i$, as well as the Turing degree of $\bigvee_{i \in I} f_i$, are independent of the choice of the listing of I.

The relation of reducibility between mass problems is denoted by $\leq$; consequently, given mass problems $\mathfrak{A}$ and $\mathfrak{B}$, we have that $\mathfrak{A} \leq \mathfrak{B}$ if there exists a recursive operator Ψ such that $\Psi(\mathfrak{B}) \subseteq \mathfrak{A}$; $\mathfrak{A} \equiv \mathfrak{B}$ means that $\mathfrak{A} \leq \mathfrak{B}$ and $\mathfrak{B} \leq \mathfrak{A}$: $[\mathfrak{A}]$ is the equivalence class of $\mathfrak{A}$ under $\equiv$, i.e., $[\mathfrak{A}]$ is the degree of difficulty of $\mathfrak{A}$. 0 denotes the least element of $\mathfrak{M}$ and $\mathbf{1}$ denotes the greatest element of $\mathfrak{M}$.

Let $({}^{\omega}\omega)^*$ be the set of partial functions from ω into ω; the operation $\vee$ already defined on total functions can be extended in an obvious way to partial functions; likewise, given $i \in \omega$ and a partial function ϕ, the symbol $i * \phi$ has the obvious meaning. Given any finite initial segment $\tilde{f}$, $\mathrm{lh}(\tilde{f})$ denotes the length of $\tilde{f}$. The set of finite initial segments will be denoted by *Fis*.

Let **Pord** be the category of partial orders; we shall be interested also in the following subcategories of **Pord**: the category **Dltt** of distributive lattices; the category $\mathbf{Dltt}_{01}$ of distributive lattices with 0 and 1; the category **Brw** of Brouwer algebras. Given any category C, a *C-embedding* is a monomorphism of C (for terminology, see MacLane [6]); the class of objects of C is denoted by $ob(C)$. Let *Form* denote the set of formulas of a standard propositional language with denumerably many propositional letters and connectives $\vee, \wedge, \rightarrow, \neg$. Given any Brouwer algebra $\mathfrak{L} = \langle L, \vee, \wedge, 0, 1\rangle$, a mapping $V: Form \rightarrow \mathfrak{L}$ is a *valuation* if for all $\alpha, \beta \in Form$, we have: $V(\alpha \vee \beta) = V(\alpha) \wedge V(\beta)$; $V(\alpha \wedge \beta) = V(\alpha) \vee V(\beta)$; $V(\alpha \rightarrow \beta) = V(\alpha) \rightarrow V(\beta)$; $V(\neg\alpha) = V(\alpha) \rightarrow 1$: it may be appropriate to remark that in these equations the symbols $\vee, \wedge, \rightarrow$ denote, in the left side, propositional connectives and, in the right side, Brouwer algebra operations; notice also the correspondence of the connectives $\vee, \wedge$ with the operations $\wedge, \vee$, respectively. A formula α is *valid in* $\mathfrak{L}$ if, for every valuation $V: Form \rightarrow \mathfrak{L}$, $V(\alpha) = 0$. Let $Th(\mathfrak{L}) = \{\alpha : \alpha \text{ is valid in } \mathfrak{L}\}$ and let *Int* denote the (set of theorems of the) intuitionistic propositional calculus (see e.g. [10], §IX.1): it is well-known that $Int \subseteq Th(\mathfrak{L})$.

2 Embedding Brouwer algebras in $\mathfrak{M}$ Since $\mathfrak{M}$ is a Brouwer algebra, according to Definition 1.1 the type of $\mathfrak{M}$ can be enriched with a binary operation $\rightarrow$ satisfying, for every degree of difficulty A, B, and C,

$$B \leq A \vee C \Leftrightarrow A \rightarrow B \leq C.$$

We need the following

Definition 2.1 Given any function f, let $\mathcal{B}_f = \{g : g \nleq_T f\}$ and $B_f = [\mathcal{B}_f]$.

Lemma 2.2 (1) *For every function* f, B_f *is both join-irreducible and meet-irreducible*; (2) *for any two functions* f *and* g, $f \leq_T g$ *if and only if* $B_f \leq B_g$.

Proof: (1) That B_f is meet-irreducible follows from Dyment, [4], Corollary 2.9, since the mass problem $\mathcal{B}_f$ satisfies: $(\forall x \in \omega)(\forall g \in \mathcal{B}_f)[x * g \in \mathcal{B}_f]$. As to show that B_f is join-irreducible, notice that the set of degrees of difficulty $\{C : C < B_f\}$ is a principal ideal generated by $B_f \wedge [\{f\}]$; indeed, $B_f \wedge [\{f\}] < B_f$: moreover, if $\mathcal{C}$ is a mass problem such that $\mathcal{C} < \mathcal{B}_f$, then $\mathcal{C} \nsubseteq \mathcal{B}_f$, and thus there exists $g \in \mathcal{C}$ such that $g \leq_T f$; hence $\mathcal{C} \leq \{f\}$ and therefore $[\mathcal{C}] \leq B_f \wedge [\{f\}]$.

(2) Immediate, as $f \leq_T g$ if and only if $\mathcal{B}_g \subseteq \mathcal{B}_f$.

Lemma 2.3 *Let* $\{X_j : j \in J\}$, $\{Y_v : v \in V\}$ *be finite collections of finite subsets of* $\{B_f : f \in \omega^\omega\}$. *Then* $\bigvee_{j \in J}(\wedge X_j) \rightarrow \bigvee_{v \in V}(\wedge Y_v) = \vee\{\wedge Y_v : v \in V \;\&\; (\forall j \in J)[\wedge Y_v \nleq \wedge X_j]\}$.

Proof: Let $\{X_j : j \in J\}$ and $\{Y_v : v \in V\}$ be as in the statement of the theorem; then for every $j \in J$ and $v \in V$ there exist finite sets of functions $\{f_i^j : i \in I_j\}$ and $\{g_u^v : u \in U_v\}$ such that $X_j = \{B_{f_i^j} : i \in I_j\}$ and $Y_v = \{B_{g_u^v} : u \in U_v\}$. It is convenient to assume that the sets I, J and I_j $(j \in J), U_v$ $(v \in V)$ are finite subsets of ω.

Let $X = \bigvee_{j \in J}(\wedge X_j)$, $Y = \bigvee_{v \in V}(\wedge Y_v)$, and $Z = \vee\{\wedge Y_v : v \in V \;\&\; (\forall j \in J)[\wedge Y_v \nleq \wedge X_j]\}$.

Let also $\mathcal{X} = \bigvee_{j \in J}(\bigwedge_{i \in I_j} \mathcal{B}_{f_i^j})$: thus $X = [\mathcal{X}]$. Clearly $Y \leq X \vee Z$; we want to show that Z is the least element C such that $Y \leq X \vee C$. To this end, it suffices to show that, for every $v \in V$,

$$(\forall j \in J)[\wedge Y_v \nleq \wedge X_j] \Rightarrow (\forall C)[\wedge Y_v \leq X \vee C \Rightarrow \wedge Y_v \leq C].$$

So, let $v \in V$ be such that $(\forall j \in J)[\wedge Y_v \nleq \wedge X_j]$ and let $\mathcal{C}$ be a mass problem such that $\bigwedge_{u \in U_v} \mathcal{B}_{g_u^v} \leq \mathcal{X} \vee \mathcal{C}$. The assumptions on $v \in V$ allow us to conclude that

$$(\forall j \in J)(\exists i \in I_j)\left[\bigwedge_{u \in U_v} \mathcal{B}_{g_u^v} \nleq \mathcal{B}_{f_i^j}\right].$$

Hence,

$$(*) \qquad (\forall j \in J)(\exists i \in I_j)(\forall u \in U_v)\,[\mathfrak{B}_{g_u^v} \not\leq \mathfrak{B}_{f_i^j}].$$

Since, by distributivity, $\mathfrak{X} \equiv \wedge\{\bigvee_{j\in J}\mathfrak{B}_{f^j_{\xi(j)}} : \xi \in \prod_{j\in J} I_j\}$ (where $\prod_{j\in J} I_j$ is the cartesian product), we conclude by (*) and Lemma 2.2(1) (as $B_{g_u^v}$ is join-irreducible) that there exists $\xi \in \prod_{j\in J} I_j$ such that

$$(\forall u \in U_v)\left[\mathfrak{B}_{g_u^v} \not\leq \bigvee_{j\in J} \mathfrak{B}_{f^j_{\xi(j)}}\right].$$

Choose such a $\xi \in \prod_{j\in J} I_j$. Since $\bigwedge_{u\in U_v}\mathfrak{B}_{g_u^v} \leq \mathfrak{X} \vee \mathfrak{C}$ we also have that $\bigwedge_{u\in U_v}\mathfrak{B}_{g_u^v} \leq (\bigvee_{j\in J}\mathfrak{B}_{f^j_{\xi(j)}}) \vee \mathfrak{C}$. On the other hand, it is easy to see that $\bigwedge_{u\in U_v}\mathfrak{B}_{g_u^v} \equiv \bigcup_{u\in U_v}(u * \mathfrak{B}_{g_u^v})$; therefore the mass problem $(\bigvee_{j\in J}\mathfrak{B}_{f^j_{\xi(j)}}) \vee \mathfrak{C}$ is reducible to the mass problem $\bigcup_{u\in U_v}(u * \mathfrak{B}_{g_u^v})$.

Claim *Let F_0, F_1, D, Z be degrees of difficulty such that $F_0 \wedge F_1 \leq D \vee Z$ and D contains a mass problem $\mathfrak{D}$ such that $(\forall \tilde{f} \in Fis)(\forall f \in \mathfrak{D})\,[\tilde{f} * f \in \mathfrak{D}]$. Then there exist degrees of difficulty Z_0, Z_1 such that $Z_0 \wedge Z_1 = Z$ and $F_i \leq D \vee Z_i$ $(i \in \{0,1\})$.*

Proof of Claim: (The claim also follows from [12], Lemma 6.) Let F_0, F_1, D, Z be degrees of difficulty as in the statement of the claim; let $\mathfrak{F}_0 \in F_0$, $\mathfrak{F}_1 \in F_1$, $\mathfrak{D} \in D$, $\mathfrak{Z} \in Z$ be mass problems and suppose that $(\forall \tilde{f} \in Fis)(\forall f \in \mathfrak{D})\,[\tilde{f} * f \in \mathfrak{D}]$. We will suppose that $D \neq 1$ (i.e. $\mathfrak{D} \neq \emptyset$), otherwise the claim is trivial. Let Ψ be a recursive operator such that $\mathfrak{F}_0 \wedge \mathfrak{F}_1 \leq \mathfrak{D} \vee \mathfrak{Z}$ via Ψ. For every $u \in \omega$, let $\mathfrak{Z}_u = \{h \in \mathfrak{Z} : (\exists \tilde{f} \in Fis)\,[\Psi(\tilde{f} \vee h)(0)$ is defined & $\Psi(\tilde{f} \vee h)(0) = u]\}$. Given any set A, let $\mathrm{even}(A) = \{(2x, y) : \langle 2x, y\rangle \in A\}$ and $\mathrm{odd}(A) = \{(2x+1, y) : \langle 2x+1, y\rangle \in A\}$. Given any finite single-valued set D, let $\tilde{D}$ be the least finite initial segment $\tilde{f}$ (in the lexicographical ordering of Fis) such that $D \subseteq \mathrm{graph}(\tilde{f})$. Finally, given any partial functions ϕ and ψ, let us say that ϕ and ψ are *not compatible* if there is some $i \in \omega$ on which ϕ and ψ are both defined and $\phi(i) \neq \psi(i)$. Let the r.e. set W define (through the corresponding enumeration operator, see [11], §9.8) the recursive operator Ψ and let $\{W^s : s \in \omega\}$ be a finite recursive approximation to W, such that $W^{s+1} - W^s$ is at most a singleton.

Subclaim 1 $\mathfrak{Z} \equiv \mathfrak{Z}_0 \wedge \mathfrak{Z}_1$.

Proof of Subclaim 1: Certainly $\mathfrak{Z} \leq \mathfrak{Z}_0 \wedge \mathfrak{Z}_1$, since each $\mathfrak{Z}_u$ is a subset of $\mathfrak{Z}$, and thus $\mathfrak{Z} \leq \mathfrak{Z}_u$.

Let us show the converse. Define

$$\begin{aligned}
W' = \{\langle\langle x, y\rangle, z\rangle : [&x = 0 \;\&\; (\exists s)(\exists w)\,[D_w \text{ single-valued} \;\&\; y \in \{0,1\} \;\&\\
&\langle\langle 0, y\rangle, w\rangle \in W^s \;\&\; \mathrm{odd}(D_w) \subseteq \{(2x+1, y) : \langle x, y\rangle \in D_z\} \;\&\\
&(\forall t < s)(\forall \langle\langle i, j\rangle, k\rangle \in W^t)\,[i = 0 \;\&\; j \in \{0,1\} \;\&\\
&D_k \text{ single-valued} \Rightarrow \mathrm{odd}(D_k),\ \mathrm{odd}(D_w) \text{ not compatible}]]] \text{ or}\\
&[x > 0 \;\&\; D_z = \{\langle x-1, y\rangle\}]\}.
\end{aligned}$$

Clearly W' is r.e.; also, it is not difficult to see that W' defines a recursive operator Ψ' such that, for every $h \in \mathfrak{Z}$, $\Psi'(h) = i * h$ for a suitable $i \in \{0,1\}$.

Such a number i exists since the mass problem $\mathfrak{D}$ is nonempty and, thus, if $h \in \mathfrak{Z}$ then, for some $f \in \mathfrak{D}$, $\Psi(f \vee h)(0)$ is defined; thus for some $\tilde{f} \in Fis$, $\Psi(\tilde{f} \vee h)(0)$ is defined and $\Psi(\tilde{f} \vee h)(0) \in \{0,1\}$. On the other hand, we have that $(\forall x)(\forall \phi \in ({}^{\omega}\omega)^*)\,[x > 0 \Rightarrow \Psi'(\phi)(x) = \phi(x-1)]$. Thus $\mathfrak{Z}_0 \wedge \mathfrak{Z}_1 \leq \mathfrak{Z}$ via Ψ' as desired.

Subclaim 2 *For every $i \in \{0,1\}$, $\mathfrak{T}_i \leq \mathfrak{D} \vee \mathfrak{Z}_i$.*

Proof of Subclaim 2: Let $i \in \{0,1\}$ be given. Define $W'' = \{\langle\langle x,y\rangle,z\rangle :$ $(\exists s)(\exists w)\,[\langle\langle 0,i\rangle,w\rangle \in W^s$ & D_w single-valued & $\mathrm{odd}(D_w) \subseteq \mathrm{odd}(D_z)$ & $(\forall t < s)(\forall \langle\langle j,i\rangle,k\rangle \in W^t)\,[j = 0$ & D_k single-valued $\Rightarrow$ $\mathrm{odd}(D_k, \mathrm{odd}(D_u)$ not compatible$]$ & $(\exists u)\,[\langle\langle x+1,y\rangle,u\rangle \in W$ & $\{(j,k) : \langle 2j,k\rangle \in D_u\} \subseteq \tilde{D}_w \cup \{(j,k) : j \geq \mathrm{lh}(\tilde{D}_w)$ & $\langle 2(j - \mathrm{lh}(\tilde{D}_w)),k\rangle \in D_z\}$ & $\mathrm{odd}(D_u) \subseteq \mathrm{odd}(D_z)]]\}$.

Clearly, W'' is r.e.: it is not difficult to see that W'' defines a recursive operator Ψ'' whose behavior can be informally described as follows: given a function h, if, say, $h = f \vee g$, then Ψ'' selects some finite initial segment $\tilde{f}$ such that $\Psi(\tilde{f} \vee g)(0)$ is defined and equals i, if such a $\tilde{f}$ exists; then, for every $x \in \omega$, $\Psi''(f \vee g)(x) = \Psi((\tilde{f} * f) \vee g)(x+1)$; otherwise $\Psi''(f \vee g)$ is the empty function. Clearly, if $f \vee g \in \mathfrak{D} \vee \mathfrak{Z}$, then $\Psi''(f \vee g) \in \mathfrak{T}_i$.

The proof of the claim is complete, taking $Z_0 = [\mathfrak{Z}_0]$, $Z_1 = [\mathfrak{Z}_1]$.

Let us now return to the proof of Lemma 2.3. We observe that, by the Claim, since the mass problem $\bigvee_{j\in J} \mathfrak{B}_{f^j_{\xi(j)}}$ satisfies the property of the mass problem $\mathfrak{D}$ in the statement of the Claim, there exist mass problems $\mathfrak{C}_u (u \in U_v)$, such that $\mathfrak{C} \equiv \bigwedge_{u\in U_v} \mathfrak{C}_u$, and, for every $u \in U_v$, $\mathfrak{B}_{g^v_u} \leq (\bigvee_{j\in J} \mathfrak{B}_{f^j_{\xi(j)}}) \vee \mathfrak{C}_u$. Since each $\mathfrak{B}_{g^v_u}$ belongs to a join-irreducible degree of difficulty (Lemma 2.1(1)), we deduce that, for every $u \in U_v$, $\mathfrak{B}_{g^v_u} \leq \mathfrak{C}_u$ and, thus, $\bigwedge_{u\in U_v} \mathfrak{B}_{g^v_u} \leq \mathfrak{C}$, as desired.

Now, let $\mathfrak{M}'$ be the sublattice (with 0,1) of $\mathfrak{M}$ generated by the set $\{B_f :$ $f \in \omega^\omega\}$:

Corollary 2.4 *$\mathfrak{M}'$ is a sub-Brouwer algebra of $\mathfrak{M}$.*

Proof: Immediate by Lemma 2.3, as $\mathfrak{M}'$ is closed under the operation $\rightarrow$.

It is not difficult to see that the forgetful functors $\mathbb{U} : \mathbf{Dltt} \rightarrow \mathbf{Pord}$ and $\mathbb{U} : \mathbf{Dltt}_{01} \rightarrow \mathbf{Dltt}$ have left adjoint functors, say $\mathbb{F} : \mathbf{Pord} \rightarrow \mathbf{Dltt}$ and $\mathbb{L}_{01} : \mathbf{Dltt} \rightarrow \mathbf{Dltt}_{01}$, respectively (see [6], Chapter IV, for the category theoretic terminology employed here). Here are useful descriptions of $\mathbb{F}$ and $\mathbb{L}_{01}$: given any partial order $\mathfrak{P} = \langle P, \leq_p\rangle$, let $\mathbb{F}r(P)$ be the free distributive lattice generated by the set P: via identification of generators with the corresponding elements of P, each element of $\mathbb{F}r(P)$ can be represented as $\bigvee_{i\in I}(\bigwedge S_i)$, for some nonempty finite subsets $S_i \subseteq P$, and some finite nonempty set I of indices (see Balbes [1], §V.3). Then $\mathbb{F}(\mathfrak{P})$ is the lattice obtained by dividing $\mathbb{F}r(P)$ modulo the equivalence relation (indeed a lattice-theoretic congruence) generated by the preordering (i.e., reflexive and transitive) relation $\lesssim$ on $\mathbb{F}r(P)$ defined by

$$\bigvee_{v\in V} (\wedge S_v) \lesssim \bigvee_{j\in J} (\wedge T_j) \text{ if } (\forall v \in V)(\exists j \in J)(\forall t \in T_j)(\exists s \in S_v)\,[s \leq_p t].$$

As to $\mathbb{L}_{01}$, given any $\mathfrak{L} \in ob(\mathbf{Dltt})$, simply let $\mathbb{L}_{01}(\mathfrak{L}) = \underset{\sim}{1} \oplus \mathfrak{L} \oplus \underset{\sim}{1}$, where $\underset{\sim}{1}$ denotes the one-element partial order and $\oplus$ is ordinal sum, as in [1], II.1 (see also

[1], Theorem II.5.7). Let also $\mathbb{F}_{01} = \mathbb{L}_{01} \circ \mathbb{F} : \mathbf{Pord} \to \mathbf{Dltt}_{01}$. Clearly, for every $\mathfrak{P} \in ob(\mathbf{Pord})$, $\mathbb{F}_{01}(\mathfrak{P})$ is a Brouwer algebra. Let $\mathfrak{D}_T$ be the partial order of Turing degrees: we have

Corollary 2.5 $\mathbb{F}_{01}(\mathfrak{D}_T)$ *is* **Brw**-*embeddable in* $\mathfrak{M}$.

Proof: Immediate by Corollary 2.4, as $\mathbb{F}_{01}(\mathfrak{D}_T) \simeq \mathfrak{M}'$: indeed, by Lemma 2.3, the function which maps the generator $[f]_T$ into B_f extends to an isomorphism between $\mathbb{F}_{01}(\mathfrak{D}_T)$ and $\mathfrak{M}'$: to show this, simply use the fact that, in the notation of Lemma 2.3,

$$\bigvee_{v \in V} (\wedge Y_v) \leq \bigvee_{j \in J} (\wedge X_j) \Leftrightarrow \bigvee_{j \in J} (\wedge X_j) \to \bigvee_{v \in V} (\wedge Y_v) = 0.$$

We are now ready for the desired characterization of the finite Brouwer algebras which are **Brw**-embeddable in $\mathfrak{M}$. Let $\mathbf{Brw}' = \{\mathfrak{L} \in ob(\mathbf{Brw})$: the least element of $\mathfrak{L}$ is meet-irreducible and the greatest element of $\mathfrak{L}$ is join-irreducible$\}$.

Theorem 2.6 *A finite Brouwer algebra* $\mathfrak{L}$ *is* **Brw**-*embeddable in* $\mathfrak{M}$ *if and only if* $\mathfrak{L} \in \mathbf{Brw}'$.

Proof: The "only if" part follows from the observation that, in $\mathfrak{M}$, $\mathbf{0}$ is meet-irreducible and $\mathbf{1}$ is join-irreducible. Let $\underset{\sim}{2}$ and $\underset{\sim}{3}$ denote the two-chain and the three-chain, respectively. Let $\mathbf{Brw}_J$ be the smallest class of Brouwer algebras such that

(1) $\underset{\sim}{2} \in \mathbf{Brw}_J$;
(2) if $\mathfrak{L} \in \mathbf{Brw}_J$ then $\underset{\sim}{1} \oplus \mathfrak{L} \in \mathbf{Brw}_J$;
(3) $\mathbf{Brw}_J$ is closed under finite products.

Since a Brouwer algebra $\mathfrak{L}$ is subdirectly irreducible if and only if $\mathfrak{L} \simeq \underset{\sim}{2}$ or $\mathfrak{L} \simeq \underset{\sim}{1} \oplus \mathfrak{L}'$, for some Brouwer algebra $\mathfrak{L}'$ (see e.g. [1], Theorem IX.4.5 or, rather, its dual version, since we are dealing here with Brouwer algebras instead of Heyting algebras), it follows by the Birkhoff subdirect product theorem (see e.g. Burris [2], Theorem 2.8.6) that every finite Brouwer algebra is **Brw**-embeddable in some element of $\mathbf{Brw}_J$. Since $\underset{\sim}{2}$ and $\underset{\sim}{3}$ are clearly **Brw**-embeddable in $\mathfrak{M}$, in order to show the claim is then enough to show that, for every $\mathfrak{L} \in \mathbf{Brw}_J$, $\mathbb{L}_{01}(\mathfrak{L})$ is **Brw**-embeddable in $\mathfrak{M}$ (we use here the fact that every finite Brouwer algebra in $\mathbf{Brw}'$ different from $\underset{\sim}{2}$ and $\underset{\sim}{3}$ has the form $\mathbb{L}_{01}(\mathfrak{L})$, for some $\mathfrak{L}$). To this end, we first show that for every $\mathfrak{L} \in \mathbf{Brw}_J$, there exists a finite partial order $\mathfrak{P}$ such that $\mathbb{L}_{01}(\mathfrak{L})$ is **Brw**-embeddable in $\mathbb{F}_{01}(\mathfrak{P})$. Indeed, $\mathbb{L}_{01}(\underset{\sim}{2}) \simeq \mathbb{F}_{01}(\underset{\sim}{2})$. Moreover, for every $\mathfrak{L} \in \mathbf{Brw}_J$ and every $\mathfrak{P} \in ob(\mathbf{Pord})$, if $\mathbb{L}_{01}(\mathfrak{L})$ is **Brw**-embeddable in $\mathbb{F}_{01}(\mathfrak{P})$, then $\mathbb{L}_{01}(\underset{\sim}{1} \oplus \mathfrak{L})$ is clearly **Brw**-embeddable in $\mathbb{F}_{01}(\underset{\sim}{1} \oplus \mathfrak{P})$. Let now $\mathfrak{L}_0, \ldots, \mathfrak{L}_n \in \mathbf{Brw}_J$ and $\mathfrak{P}_0, \ldots, \mathfrak{P}_n \in ob(\mathbf{Pord})$ be such that, for every $i \leq n$, $\mathbb{L}_{01}(\mathfrak{L}_i)$ is **Brw**-embeddable in $\mathbb{F}_{01}(\mathfrak{P}_i)$; for every $i \leq n$, let $\mathfrak{P}_i = \langle P_i, \leq_i \rangle$. Let $\mathfrak{P}$ be the coproduct, in **Pord**, of the family $\langle \mathfrak{P}_i : i \leq n \rangle$ (for instance, let $\mathfrak{P} = \langle \bigcup_{i \leq n} \{i\} \times \mathfrak{P}_i, \leq \rangle$, where $(i,p) \leq (j,q)$ if and only if $i = j$ and $p \leq_i q$) with coproduct injections $J_i : \mathfrak{P}_i \to \mathfrak{P}$ and let $\mathfrak{T}$ be the set of join-irreducible elements of the cartesian product $\prod_{i \leq n} \mathbb{F}(\mathfrak{P}_i)$. Henceforth, we shall identify generators of $\mathbb{F}(\mathfrak{P}_0), \ldots, \mathbb{F}(\mathfrak{P}_n)$ and $\mathbb{F}(\mathfrak{P})$ with the corresponding elements of $\mathfrak{P}_0, \ldots, \mathfrak{P}_n$ and $\mathfrak{P}$, respectively. Let 0_i denote the least element of $\mathbb{F}(\mathfrak{P}_i)$: it is not difficult to see that $\mathfrak{T} = \{(p_0, \ldots, p_n) : p_i \in \mathfrak{P}_i$ and p_i is join-irreducible in $\mathfrak{P}_i$ and there is at most one $i \leq n$ such that $p_i \neq 0_i\}$; notice also that $\mathfrak{T}$ is a partial order with the induced order. Let us define a function $J: \mathfrak{T} \to \mathbb{F}(\mathfrak{P})$ as follows: given $(p_0, \ldots, p_n) \in \mathfrak{T}$, if $p_i = \wedge X_i$, for every $i \leq n$, where $X_i \subseteq \mathfrak{P}_i$, then let

$$J((p_0, \ldots, p_n)) = \bigwedge_{i \leq n} (\wedge J_i(X_i)).$$

(In defining J, we have used the fact that, for every $i \leq n$, the join-irreducible elements of $\mathbb{F}(\mathfrak{P}_i)$ are exactly those elements having the form $\wedge X$, for some $X \subseteq \mathfrak{P}_i$, as is easily seen using the characterization of the join-irreducible elements of $\mathbb{F}r(P_i)$, for which see e.g. [1], Theorem V, 3.7. Also, J is independent of the choices of the X_i's.)

It is easily checked that J is a **Pord**-monomorphism, as each generator in $\mathbb{F}(\mathfrak{P}_i)$ and $\mathbb{F}(\mathfrak{P})$ is meet-irreducible. Now, in every finite distributive lattice, each element is the join of a unique set of mutually incomparable join-irreducible elements (see e.g. [1], Theorem III.2.2); define $H: \mathbb{L}_{01}(\prod_{i \leq n} \mathbb{F}(\mathfrak{P}_i)) \to \mathbb{F}_{01}(\mathfrak{P})$ by

$$H(x) = \begin{cases} 0 & \text{if } x = 0 \\ \vee J(X) & \text{if } x \in \prod_{i \leq n} \mathbb{F}(\mathfrak{P}_i) \text{ and } x = \vee X, \text{ and } X \text{ consists of mutually} \\ & \qquad \text{incomparable join-irreducible elements} \\ 1 & \text{if } x = 1. \end{cases}$$

We claim that H is a **Brw**-embedding. This is an easy consequence of the following observations:

(a) H maps join-irreducible elements of $\mathbb{L}_{01}(\prod_{i \leq n} \mathbb{F}(\mathfrak{P}_i))$ into join-irreducible elements of $\mathbb{F}_{01}(\mathfrak{P})$, by definition of J;
(b) $\mathfrak{T}$ is closed under the operation $\wedge$ of $\prod_{i \leq n} \mathbb{F}(\mathfrak{P}_i)$.

Now, clearly H preserves $\vee$; from (a) and the fact that J is a **Pord**-embedding it follows that H is 1-1 and preserves the operation $\to$; indeed, if $\mathfrak{L}$ is any Brouwer algebra, with partial ordering $\leq_L$, and $X, Y \subseteq \mathfrak{L}$ consist of join-irreducible elements, then we have that $\vee X \leq_L \vee Y$ if and only if $(\forall x \in X)(\exists y \in Y)[x \leq_L y]$ and $\vee X \to \vee Y = \vee\{y \in Y: (\forall x \in X)[y \nleq_L x]\}$; from (a) and (b) it follows that H preserves the operation $\wedge$. Since $\mathbb{L}_{01}(\prod_{i \leq n} \mathfrak{L}_i)$ is **Brw**-embeddable in $\mathbb{L}_{01}(\prod_{i \leq n} \mathbb{F}(\mathfrak{P}_i))$, by composition we get a **Brw**-embedding of $\mathbb{L}_{01}(\prod_{i \leq n} \mathfrak{L}_i)$ into $\mathbb{F}_{01}(\mathfrak{P})$.

To finish off the proof it is now enough to show that, for every finite partial order $\mathfrak{P}$, $\mathbb{F}_{01}(\mathfrak{P})$ is **Brw**-embeddable in $\mathfrak{M}$. But every finite partial order $\mathfrak{P}$ is **Pord**-embeddable in $\mathfrak{D}_T$ and the functor $\mathbb{F}_{01}$ takes **Pord**-monomorphisms into **Brw**-monomorphisms (indeed, it clearly takes **Pord**-morphisms into **Dltt**$_{01}$-morphisms by functoriality; moreover, if $K: \mathfrak{P}_1 \to \mathfrak{P}_2$ is a **Pord**-monomorphism, then $\mathbb{F}_{01}(K)$ maps join-irreducible elements into join-irreducible elements, so we can argue as we did for H to conclude that $\mathbb{F}_{01}(K)$ is a **Brw**-embedding); hence, for every finite partial order $\mathfrak{P}$, $\mathbb{F}_{01}(\mathfrak{P})$ is **Brw**-embeddable in $\mathbb{F}_{01}(\mathfrak{D}_T)$ and, thus, by Corollary 2.5, in $\mathfrak{M}$.

Remark 2.7 Let $\mathfrak{M}^- = \mathfrak{M} - \{\mathbf{0}\}$. It is easy to see that $\mathfrak{M}^-$ is still a Brouwer algebra: indeed, if f is any recursive function, then B_f is the least element of $\mathfrak{M}^-$: call $\mathbf{0}^-$ this least element. Now, given any finite partial order $\mathfrak{P}$, we have that $\mathbb{F}_{01}(\mathfrak{P})$ is **Brw**-embeddable in $\mathfrak{M}^-$: indeed, it suffices to use a **Pord**-embedding $I: \mathfrak{P} \to \mathfrak{D}_T$ such that the least element of $\mathfrak{D}_T$ is not in the range of I; then, by composition, we get a **Brw**-embedding of $\mathbb{F}_{01}(\mathfrak{P})$ into $\mathfrak{M}$ which avoids 0^-. Finally, define $H^-: \mathbb{F}_{01}(\mathfrak{P}) \to \mathfrak{M}^-$ by $H^-(x) = H(x)$ if $x \neq 0$ and $H^-(0) = \mathbf{0}^-$; so $\mathbb{F}_{01}(\mathfrak{P})$ is **Brw**-embeddable in $\mathfrak{M}^-$ via H^-. Now, since $\mathbf{0}^-$ is meet-irreducible (by [4], Corollary 2.9), it follows that a finite Brouwer algebra is **Brw**-embeddable in $\mathfrak{M}^-$ if and only if $\mathfrak{L} \in$ **Brw**$'$.

Corollary 2.8 ([8],[12]) *$Th(\mathfrak{M})$ is the intermediate logic obtained by adding the axiom scheme $\neg\alpha \vee \neg\neg\alpha$ to the intuitionistic propositional calculus.*

Proof: Let *Jan* (after Jankov) be the logic obtained by adding the scheme $\neg\alpha \vee \neg\neg\alpha$ to the intuitionistic propositional calculus. It is shown in Jankov [5] that $Jan = \bigcap\{Th(\mathbb{L}_{01}(\mathfrak{L})) : \mathfrak{L}$ is a finite Brouwer algebra$\}$. Thus, by Theorem 2.6 and the fact that if $\mathfrak{L}_1$ is **Brw**-embeddable in $\mathfrak{L}_2$ then $Th(\mathfrak{L}_2) \subseteq Th(\mathfrak{L}_1)$ (see for instance [10]), it follows that $Th(\mathfrak{M}) \subseteq Jan$.

On the other hand, one trivially checks that $Jan \subseteq Th(M)$, by showing that for every $\alpha \in Form$, the formula $\neg\alpha \vee \neg\neg\alpha$ is valid in $\mathfrak{M}$.

3 The case of the Mučnick lattice Theorem 2.6 shows further similarities, besides those pointed out in [13], between the Medvedev lattice and the Mučnick lattice, as is shown in Fact 3.3 below (a comparative study of these lattices is presented in [13]; see also [4] and Mučnick [9]). We proceed to give the main definitions.

Given mass problems $\mathcal{A}, \mathcal{B} \subseteq {}^\omega\omega$, let $\mathcal{A} \leq_w \mathcal{B}$ if $(\forall g \in \mathcal{B})(\exists f \in \mathcal{A})[f \leq_T g]$. Let $\equiv_w$ be the equivalence relation generated by $\leq_w$ and, given any mass problem $\mathcal{A}$, let $[\mathcal{A}]_w$ denote the equivalence class of $\mathcal{A}$ under $\equiv_w$: such equivalence classes are partially ordered by: $[\mathcal{A}]_w \leq_w [\mathcal{B}]_w$ if $\mathcal{A} \leq_w \mathcal{B}$.

Definition 3.1 ([9]) Let $\mathfrak{M}_w = \langle\{[\mathcal{A}]_w : \mathcal{A} \subseteq {}^\omega\omega\}, \leq_w\rangle$. $\mathfrak{M}_w$ is in fact an object of $\mathbf{Dltt}_{01}$ called the *Mučnick lattice.*

$\mathfrak{M}_w$ has the following useful representation. Given any partial order $\mathfrak{P} = \langle P, \leq_p\rangle$, let (see [3]) $\mathbb{H}(\mathfrak{P})$ be the object of $\mathbf{Dltt}_{01}$ given by:

(1) the elements of $\mathbb{H}(\mathfrak{P})$ are the subsets $X \subseteq P$ which are upward closed under $\leq_p$ (i.e., $p \in X \;\&\; p \leq_p q \Rightarrow q \in X$);
(2) given $X, Y \in \mathbb{H}(\mathfrak{P})$, let $X \leq Y$ if $Y \subseteq X$; then $\cup, \cap$ correspond to $\wedge, \vee$ respectively; $\varnothing$ is the greatest element and P is the least element.

Lemma 3.2 $\mathfrak{M}_w \simeq \mathbb{H}(\mathfrak{D}_T)$.

Proof: See [13].

It is known that for every partial order $\mathfrak{P} = \langle P, \leq_p\rangle$, $\mathbb{H}(\mathfrak{P})$ (hence $\mathbb{H}(\mathfrak{D}_T)$ and thus $\mathfrak{M}_w$ by Lemma 3.2) is a Brouwer algebra (under the operation $\to$ given by: for every $X, Y \in \mathbb{H}(\mathfrak{P})$, $X \to Y = \{x \in P : (\forall y \in P)[x \leq_p y \Rightarrow y \notin X$ or $y \in Y]\}$).

Fact 3.3 *A finite $\mathfrak{L} \in$ **Brw** is **Brw**-embeddable in $\mathfrak{M}_w$ if and only if $\mathfrak{L} \in$ **Brw**$'$.*

Proof: The "only if" part follows from the fact that $\mathfrak{M}_w \in$ **Brw**$'$. The converse is an easy consequence, via Lemma 3.2, of the duality between partial orders and

Heyting algebras (hence Brower algebras as well), given in [3]. Here is, however, a more direct proof. As in Theorem 2.6, it is enough to show that, for every $\mathfrak{L} \in \mathbf{Brw}_J$, $\mathbb{L}_{01}(\mathfrak{L})$ is **Brw**-embeddable in $\mathfrak{M}_w$. Given any function f, let $B_f^w = [\mathfrak{B}_f]_w$. The following are easily shown:

(1) for every f, B_f^w is join-irreducible (by an argument similar to that of Lemma 2.2(1));
(2) the mapping: $[f]_T \to B_f^w$ is a **Pord**-monomorphism (as in Lemma 2.2(2)), that preserves (existing) infima.

Thus, let $\mathfrak{L} \in \mathbf{Brw}_J$ be given and let $\mathfrak{P}$ be the partial order of the join-irreducible elements of $\mathfrak{L}$: by definition of $\mathbf{Brw}_J$, it is easy to see that $\mathfrak{P}$ is a lower semilattice. Let $J: \mathfrak{P} \to \mathfrak{D}_T$ be an inf-preserving embedding and, for every $p \in \mathfrak{P}$, choose $f_p \in J(p)$. Finally define $I: \mathbb{L}_{01}(\mathfrak{L}) \to \mathfrak{M}_w$ by:

$$I(x) = \begin{cases} 0 & \text{if } x = 0 \\ \bigvee_{i \in I} B_{f_i}^w & \text{if } x \in \mathfrak{L} \;\&\; x = \vee I, \text{ where } I \subseteq \mathfrak{P} \text{ consists of mutually} \\ & \qquad\qquad\qquad\quad \text{incomparable elements} \\ 1 & \text{if } x = 1. \end{cases}$$

It is not difficult to see that I is a **Brw**-embedding, by arguments similar to those employed for H in the proof of Theorem 2.6.

Corollary 3.4 $\quad Th(\mathfrak{M}_w) = Jan$.

Proof: See proof of Corollary 2.8, using Fact 3.3.

Remark 3.5 (1) Let $\mathfrak{M}_w^- = \mathfrak{M}_w - \{\mathbf{0}_w\}$, where $\mathbf{0}_w$ is the least element of $\mathfrak{M}_w$. It is easily seen that $\mathfrak{M}_w^-$ is still a Brouwer algebra. One can show that a finite Brouwer algebra $\mathfrak{L}$ is **Brw**-embeddable in $\mathfrak{M}_w^-$ if and only if the greatest element of $\mathfrak{L}$ is join-irreducible; indeed, in the proof of Fact 3.3, define the embedding I: $\mathbb{L}_{01}(\mathfrak{L}) \to \mathfrak{M}_w$ starting from a **Pord**-embedding J: $\mathfrak{P} \to \mathfrak{D}_T$ which preserves the least element as well as infima: then I restricts to a **Brw**-embedding I^-: $\mathfrak{L} \oplus \underset{\sim}{1} \to \mathfrak{M}_w^-$.

(2) Corollary 3.4 answers a question, raised in [9], aiming to characterize $Th(\mathfrak{M}_w)$. Of course, answering this question is nowadays trivial, because of the work in [3] and [6], not available at the time of [9].

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