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Monadic Π_1^1-Theories of Π_1^1-Properties

KEES DOETS*

Abstract Axiomatizations are provided for the monadic universal second-order theories of: scattered orderings, well-orderings, complete orderings, the ordering of the natural numbers, of the reals, and of well-founded trees. Proofs employ the Ehrenfeucht-Fraïssé-game.

Summary For some Π_1^1-statements $\forall R\phi(R)$, results of the following type are proved: Suppose that a monadic Π_1^1-sentence $\forall X_1 \ldots \forall X_k \psi(X_1, \ldots, X_k)$ is a consequence of $\forall R\phi(R)$, then the first-order sentence $\psi(U_1, \ldots, U_k)$ is already a consequence of the *first-order schema* corresponding to $\forall R\phi(R)$, which requires $\phi(R)$ only for R which are (parametrically) first-order definable in the language of $\psi(U_1, \ldots, U_k)$. Cases considered here are: scattered orderings, well-orderings, complete orderings, models of order type ω, of order type λ, and well-founded trees. The method of proof uses the Ehrenfeucht-Fraïssé-game.

1 Introduction Some natural axioms of a number of theories are of the second-order (Π_1^1) form $\forall R\phi(R)$, where ϕ is a first-order predicate and R is a second-order variable. For instance, the *induction principle* of arithmetic, the *completeness* of the reals, Zermelo's *Aussonderungsaxiom*, and the Fraenkel-Skolem *replacement axiom* in set theory are of this type. As to *first-order versions* of these principles, the natural option is to require $\phi(R)$ not for *all R* but for *parametrically first-order definable R* only, thus replacing the second-order axiom by its corresponding first-order *schema*.

Obviously, the new theory will have models not allowed by the old one (by the Löwenheim-Skolem-Tarski Theorem for instance) and hence it may turn out to be strictly weaker than its second-order companion. For instance, second-

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order arithmetic is categorical, hence it implies first-order sentences beyond the scope of the first-order induction schema.

On the other hand, if the language is restricted sufficiently, conservation may occur. This paper contains a number of examples of this. They all concern theories of (partial) orderings, in which conservation is proved with respect to monadic Π_1^1-sentences. The method of proof consists in showing how to transfer counterexamples to a Π_1^1-sentence on a "nonstandard" model to a standard model.

To be more precise (cf. the condition of Theorem 1.2(ii) below), I will prove results of the following type. Let L be a first-order language containing $<$ and unary predicates $U_1, \ldots, U_k$. Let $\mathfrak{M} = (M,<,U_1,\ldots,U_k)$ be a model of the *first-order L-schema* corresponding to the Π_1^1-property $\forall R\phi(R)$; i.e., each $R \subset M$ which is first-order definable on $\mathfrak{M}$ satisfies $\phi(R)$ on $\mathfrak{M}$. Then for each L-sentence $\psi(U_1,\ldots,U_k)$ true in $\mathfrak{M}$ there is a model $(N,<)$ of the original Π_1^1-sentence $\forall R\phi(R)$ satisfying $\exists X_1 \ldots \exists X_k \psi(X_1,\ldots,X_k)$.

To explain how this is done we need the notion of n-equivalence. Models are called *n-equivalent* (denoted by $\equiv^n$) iff they satisfy the same first-order sentences of quantifier-rank $\leq n$. In the sequel, our languages will be finite and do not contain operation symbols. Under these circumstances, we have the following:

1.1 Lemma *Up to logical equivalence, there are only finitely many first-order formulas of quantifier-rank $\leq n$ in the free variables $x_0,\ldots,x_{k-1}$ in each language.*

Proof: Induction with respect to n. For $n = 0$, notice that there are only finitely many atomic formulas in these variables and use disjunctive normal forms. For the induction step, choose a finite set Σ of formulas of quantifier-rank $\leq n$ in the free variables $x_0,\ldots,x_k$ such that every such formula has an equivalent in Σ. Now, consider disjunctive normal forms over "atoms" $\forall x_k\phi$ and $\exists x_k\phi$ where $\phi \in \Sigma$.

It follows that n-equivalence has only finitely many equivalence classes in each language of the type considered. This is the main important fact used below.

The *n-characteristic* σ of a model $\mathfrak{M}$ is the conjunction of all sentences of quantifier-rank $\leq n$ valid in $\mathfrak{M}$. Thus, $\mathfrak{N} \models \sigma$ iff $\mathfrak{N} \equiv^n \mathfrak{M}$. n-equivalence has been neatly characterized by Ehrenfeucht using his game (for more on this, cf. [3], pp. 93–96, 247–252, and 359–361).

In the following basic theorem Σ is a set of first-order sentences in a language L and $\forall R\phi(R)$ is a Π_1^1-sentence over L. Let $X_1,\ldots,X_k$ be new unary relation-symbols and $L_k = L \cup \{X_1,\ldots,X_k\}$. ($L_k$-) *definably-$\phi$* is, by definition, the set of universal closures of L_k-formulas obtained from ϕ by replacing each occurrence of $R(t_1,\ldots,t_m)$ by some fixed L_k-formula $\eta(t_1,\ldots,t_m)$ (taking measures to avoid the clashing of variables). Thus, L_k-definably-ϕ intuitively requires $\phi(R)$ only when R is (parametrically) first-order definable in the language L_k.

The union over all k of these schemata is the *first-order schema corresponding to $\forall R\phi(R)$*.

1.2 Theorem *The following two conditions are equivalent:*
(i) *for each first-order formula $\psi = \psi(X_1, \ldots, X_k)$ in the language L_k: if $\Sigma + \forall R\phi(R) \models \forall X_1 \ldots \forall X_k \psi$, then $\Sigma + L_k$-definably-$\phi \models \psi(X_1, \ldots, X_k)$;*
(ii) *each model $(\mathfrak{M}, U_1, \ldots, U_k)$ of $\Sigma + L_k$-definably-ϕ has an n-equivalent satisfying $\Sigma + \forall R\phi(R)$ for each n.*

Proof: (i) $\Rightarrow$ (ii): Let $(\mathfrak{M}, U_1, \ldots, U_k) \models \Sigma + L_k$-definably-$\phi$ have the n-characteristic $\tau(X_1, \ldots, X_k)$. We want a model of $\Sigma + \forall R\phi(R) + \exists X_1 \ldots \exists X_k \tau$. If such a model does not exist, then $\Sigma + \forall R\phi(R) \models \forall X_1 \ldots \forall X_k \neg\tau$; hence by (i), $\Sigma + L_k$-definably-$\phi \models \neg\tau(X_1, \ldots, X_k)$, contradicting the assumptions on $(\mathfrak{M}, U_1, \ldots, U_k)$.

(ii) $\Rightarrow$ (i): Assume $\Sigma + \forall R\phi(R) \models \forall X_1 \ldots \forall X_k \psi$ and let $(\mathfrak{M}, U_1, \ldots, U_k)$ be a model of $\Sigma + L_k$-definably-ϕ. By (ii), there is an n-equivalent satisfying $\forall R\phi(R) + \Sigma$ where we take n to be the quantifier rank of ψ. By assumption, this n-equivalent satisfies $\forall X_1 \ldots \forall X_k \psi$, hence $\psi(X_1, \ldots, X_k)$ is also satisfied, so $(\mathfrak{M}, U_1, \ldots, U_k)$ must satisfy this formula as well.

The Σ of the theorem below will always be finite. Therefore we may require that the n-equivalent of (ii) satisfies $\forall R\phi(R)$ only, without invalidating the truth of (ii) $\Rightarrow$ (i): simply let n be at least the maximum of quantifier ranks of formulas in Σ.

In what follows, results of type (ii) are proved. According to the theorem, this shows that, in the context of Σ, *the first-order schema corresponding to $\forall R\phi(R)$ suffices to prove all monadic Π^1_1-consequences of this second-order statement.* (Actually, Theorem 4.9 below does a little better.)

In particular, when 1.2(i) or (ii) holds, the monadic Π^1_1-theory of $\Sigma + \forall R\phi(R)$ is recursively enumerable (cf. [1] for decidability results in this context).

All models encountered here will have the form $\mathfrak{M} = (M, <, U_1, \ldots, U_k)$, where $<$ orders M and $U_1, \ldots, U_k \subset M$. If $X \subset M$, then $\mathfrak{X}$ (and sometimes X as well) denotes the submodel of $\mathfrak{M}$ with universe X. $I \subset M$ is an *interval* if $x < y < z$ and $x, z \in I$ imply $y \in I$; notations like (x, z) and $[x, z)$ denote specific intervals as usual.

If I is an ordered set and $\mathfrak{m}$ a function on I associating a model $\mathfrak{m}(i)$ with every $i \in I$, we may form the *ordered sum* $\Sigma_{i \in I} \mathfrak{m}(i)$, being the model obtained from the $\mathfrak{m}(i)$ by gluing (disjoint copies of) them one after the other according to the ordering of I. Formally, $\Sigma_{i \in I} \mathfrak{m}(i)$ can be defined as the model with universe $\bigcup_{i \in I} (m(i) \times \{i\})$ with the ordering defined by: $(a, i) < (b, j)$ iff $i <_I j$, or: $i = j$ and $a <_i b$ (here, $<_i$ and $<_I$ denote the orderings of $\mathfrak{m}(i)$ and I); and if U^i_n is the n-th unary relation of $\mathfrak{m}(i)$ $(1 \leq n \leq k)$ then $U_n = \bigcup_{i \in I} (U^i_n \times \{i\})$ is the corresponding one of the ordered sum.

A *condensation* of an ordered model $\mathfrak{M}$ is a partition of $\mathfrak{M}$ into intervals. Any condensation P of $\mathfrak{M}$ inherits an ordering from $\mathfrak{M}$ by putting, for $p, q \in P$: $p < q$ iff for some $a \in p$ and $b \in q$ (equivalently, iff for all $a \in p$ and $b \in q$) $a < b$. Hence, a condensation P of $\mathfrak{M}$ is nothing but a way of writing $\mathfrak{M}$ as an ordered sum, $\mathfrak{M} = \Sigma_{p \in P} p$.

If the condensation P is induced by the equivalence $\sim$ (such equivalences are called *congruences* by some), we write $P = M/\sim$.

1.3 Lemma *Let R be any transitive binary relation on the ordered model $\mathfrak{M}$. Define $\sim\ =\ \sim_R$ by: $a \sim b$ iff one of the following holds:*
(i) $a = b$
(ii) $a < b$ *and for all* c, d *such that* $a \leq c < d \leq b$: cRd
(iii) $b < a$ *and for all* c, d *such that* $b \leq c < d \leq a$: cRd.
Then $\sim$ *induces a condensation.*

Proof: The proof is straightforward.

All condensations used in the sequel are defined in this fashion.

1.4 Lemma *If for all $i \in I$ $\mathfrak{m}(i) \equiv^n \mathfrak{m}'(i)$, then $\Sigma_{i\in I}\mathfrak{m}(i) \equiv^n \Sigma_{i\in I}\mathfrak{m}'(i)$.*

Proof: It is straightforward to describe a winning strategy for the second player in the Ehrenfeucht n-game between these sums under the condition given.

The following generalization of Lemma 1.4 is needed in Section 4.

1.5 Lemma *Suppose that I and J are ordered sets and that $\mathfrak{m}$ and $\mathfrak{m}'$ associate ordered models $\mathfrak{m}(i)$, respectively $\mathfrak{m}'(j)$, to each $i \in I$, respectively $j \in J$, such that:*

(*) $(I, \{i \mid \mathfrak{m}(i) \models \sigma\})_{\sigma\in\Sigma} \equiv^n (J, \{j \mid \mathfrak{m}'(j) \models \sigma\})_{\sigma\in\Sigma}$ *where Σ is the set of n-characteristics. Then $\Sigma_{i\in I}\mathfrak{m}(i) \equiv^n \Sigma_{j\in J}\mathfrak{m}'(j)$.*

Proof: Use the Ehrenfeucht game-technique. If the first player chooses, say, $a \in \Sigma_i m(i)$, the second player locates the $i \in I$ for which $a \in m(i)$, then uses (*) to find a j corresponding to i; in particular, $\mathfrak{m}'(j) \equiv^n \mathfrak{m}(i)$, and a countermove is readily found; etc.

1.6 Examples where 1.2(ii) fails

1. (Due to van Benthem.) Consider the Π_1^1 statement $\forall X\phi(X)$ in the language of $<$ where $\phi(X)$ means that X and its complement cannot both be cofinal. Obviously, every ordered model of $\forall X\phi(X)$ has a greatest element. On the other hand, the first-order schema corresponding to ϕ does not imply this. A countermodel is $(\omega, <)$: notice that each *definable* set here is either *finite* or *cofinite*. (*Proof:* Using games, it is easy to verify by induction on n that orderings of type ω and $\omega + \zeta$ are n-equivalent for all n. Now, let $\psi(x)$ be any formula in the free variable x. If no $a \in \zeta$ satisfies ψ in $\omega + \zeta$ then $\exists y\forall x(y < x \rightarrow \neg\psi)$ holds in $\omega + \zeta$; therefore it holds in ω and the set defined by ψ in ω must be finite. On the other hand, if some $a \in \zeta$ satisfies ψ in $\omega + \zeta$ then every $a \in \zeta$ satisfies ψ in $\omega + \zeta$—this is so because for each pair $a, b \in \zeta$ there is an automorphism h of $\omega + \zeta$ such that $ha = b$. Hence, $\exists y\forall x(y < x \rightarrow \psi)$ holds in $\omega + \zeta$; therefore, it holds in ω and the set defined by ψ in ω must be cofinite.)

2. In theories defining a pairing the restriction to *monadic* languages is only apparent and results like ours can fail badly. We mentioned the case of arithmetic; also, each model of set theory certainly is *definably well-founded*, nevertheless such models need not have a well-founded n-equivalent for n large enough: well-founded models have standard integers, therefore they are arithmetically correct; but, Gödel sentences *are* arithmetical.

2 Monadic Π_1^1*-theory of scattered orderings* A linear ordering $\mathfrak{M} = (M,<)$ is called *scattered* if it does not embed the ordering $(\mathbb{Q},<)$ of the rationals.

$\mathbb{Q}$ embeds *every* countable ordering; in particular, it embeds ω^*. It follows that every well-ordering is scattered. I shall need the following lemma:

2.1 Lemma *A scattered ordered sum of scattered orderings is scattered.*

Proof: Suppose $\mathbb{Q} \subset \Sigma_{i \in I} m(i)$. If some $\mathbb{Q} \cap m(i)$ contains at least two rationals, it contains the interval between them and so $m(i)$ cannot be scattered. Hence, sending $p \in \mathbb{Q}$ to the $i \in I$ for which $p \in m(i)$ embeds $\mathbb{Q}$ in I, a contradiction.

There is more than one way to formalize scatteredness into a Π_1^1-statement, and not every formalization is a good one from our point of view.

2.2 Example Let δ express that $<$ is a dense ordering containing at least two elements. $\phi(X)$ is the formula obtained from $\neg\delta$ by relativizing quantifiers to membership in (the set) X. Clearly, $\mathfrak{M}$ is scattered iff it satisfies $\forall X \phi(X)$. Here is an example of a model $\mathfrak{M} = (M,<,X,Y,Z)$ which is *definably-*ϕ but has *no* scattered 3-equivalent. Partition $\mathbb{Q}$ into dense subsets R, S, T and put $\mathfrak{M} = \Sigma_{q \in \mathbb{Q}} M_q$, where $M_q = (\mathbb{Z},<,X^q,Y^q,Z^q)$ and $X^q = \mathbb{Z}$ if $q \in R$; X^q is empty otherwise; similarly, $Y^q = \mathbb{Z}$ or $\varnothing$ depending on whether $q \in S$ or not; and $Z^q = \mathbb{Z}$ or $\varnothing$ depending on whether $q \in T$. Notice that each interval M_q of $\mathfrak{M}$ is a set of indiscernibles of $\mathfrak{M}$ (use automorphisms of M_q); hence, if A is a *definable* set of $\mathfrak{M}$, either $A \cap M_q = \varnothing$ or $M_q \subset A$. Therefore, no nonempty definable set of $\mathfrak{M}$ is densely ordered and it follows that $\mathfrak{M}$ is definably-ϕ. On the other hand, the fact that $\mathfrak{M}$ satisfies sentences such as $(\forall x \in X)(\forall y \in Y)(x < y \rightarrow (\exists z \in Z)(x < z < y))$ shows that no 3-equivalent of $\mathfrak{M}$ can be scattered.

A "good" formalization of scatteredness should avoid this counterexample.

2.3 Lemma *An ordering is scattered iff it has no densely ordered condensation.*

Proof: Only if: Use the axiom of choice. *If:* Suppose that $\mathbb{Q} \subset M$. Define $\sim$ by way of 1.3 where aRb iff $a < b$ and $(a,b) \cap \mathbb{Q}$ is finite. It is easy to see that $\sim$ induces a dense condensation.

The (dyadic!) Π_1^1-characterization of scatteredness contained in this lemma is a "good" one according to the following theorem, where we call a model *definably scattered* if no *definable* equivalence partitions $\mathfrak{M}$ into a dense ordering of intervals. Notice that the model of 2.2 is *not* definably scattered in this sense.

2.4 Theorem *If* $\mathfrak{M}$ *is definably scattered, then it has scattered n-equivalents for each n.*

Proof: I use what Rosenstein [3] calls a *condensation argument*, which originated with Hausdorff. Define $\sim$ in the fashion of 1.3 with aRb meaning that (a,b) has a scattered n-equivalent (if $a < b$). By 2.1, R is transitive. Hence, $\sim$ induces a condensation by 1.3. Moreover, $\sim$ is *definable*: there are only *finitely* many n-characteristics; let Γ be the (finite) set of n-characteristics belonging to scattered models. Then (c,d) has a scattered n-equivalent iff $\mathfrak{M} \models \bigvee_{\tau \in \Gamma} \tau^{(c,d)}$, where $\tau^{(c,d)}$ is obtained from τ by relativizing quantifiers to membership in (c,d). It is now clear that $\sim$ can be defined as well.

Claim 1 *Each equivalence class has a scattered n-equivalent.*

Proof: Let I be an equivalence class and $a \in I$.

(i) I has a greatest element b. Then $a \sim b$ and $I^{\geq a} = \{x \in I | a \leq x\} = [a,b]$ has a scattered n-equivalent by definition.

(ii) If not, choose a sequence $a_0 = a < a_1 < \ldots < a_\xi < \ldots$ $(\xi < \alpha)$ cofinal in I. Each $(a_\xi, a_{\xi+1})$ and hence each $[a_\xi, a_{\xi+1})$ has a scattered n-equivalent A_ξ. Hence $I^{\geq a} = \Sigma_{\xi<\alpha}[a_\xi, a_{\xi+1})$ has the n-equivalent $\Sigma_{\xi<\alpha}A_\xi$ by 1.4 which, by 2.1, is scattered.

Argue similarly for $I^{<a} = \{x \in I | x < a\}$; so $I = I^{<a} + I^{\geq a}$ has a scattered n-equivalent.

Claim 2 *The induced ordering of the equivalence classes is dense.*

Proof: Suppose that $I < J$ are equivalence classes and no equivalence class is between I and J. Let $a \in I$ and $b \in J$, and suppose that $a \leq c < d \leq b$. Then (c,d) has a scattered n-equivalent: if $c,d \in I$ or $c,d \in J$ this is clear, and if $c \in I$ and $d \in J$ we know from the previous proof that $I^{>c}$ and $J^{<d}$ have scattered n-equivalents; but, $(c,d) = I^{>c} + J^{<d}$. Therefore, $a \sim b$, a contradiction.

Since $\mathfrak{M}$ is definably scattered, $\sim$ cannot have more than *one* equivalence class: M itself. Consequently, $\mathfrak{M}$ must have a scattered n-equivalent by the first claim.

2.5 Remark By 2.2 and 2.3, we have two Π^1_1-formalizations of scatteredness; however, the first-order schema corresponding to the second one (definable scatteredness) is strictly stronger than the first-order schema belonging to the first.

2.6 Corollary *The monadic Π^1_1-theory of scattered orderings is recursively enumerable.*

Proof: Use 2.4 and 1.2.

3 Monadic Π^1_1-theory of ω and of the class of finite orderings It is clear what it means for an ordered set to *satisfy complete induction* when there is a least element and every element has an immediate successor. *Definable induction* requires that every *definable* set containing the least element and closed under immediate successors contains every element. *Complete induction* is the usual Π^1_1-instrument transforming a suitable set of first-order axioms of quantifier rank ≤ 3 into a categorical description of the order type ω. *Definable induction* does not come close to this, but it suffices for the monadic Π^1_1-theory:

3.1 Theorem *If* (1) $(M,<) \equiv^3 (\omega, <)$ *and* (2) $\mathfrak{M} = (M,<,X_1,\ldots,X_k)$ *satisfies definable induction, then $\mathfrak{M}$ has n-equivalents of order type ω for every n.*

Proof: by the Löwenheim-Skolem Theorem, we may assume $\mathfrak{M}$ to be countable. Define $X = \{a \in M | \forall b < a([b,a)$ has a *finite* n-equivalent$)\}$. Just as in the case of $\sim$ in the proof of 2.4, it is easy to see that X is a *definable* set. Trivially, X contains the least element of $\mathfrak{M}$. Also, X is closed under immediate successors: if S is a finite n-equivalent of $[b,a)$ and c is the immediate successor of a then it is clear that the ordered sum $S + \{a\}$ is the required finite n-equivalent of $[b,c)$. By definable induction then, $X = M$. Let a_0 be the least element of $\mathfrak{M}$ and choose $a_0 < a_1 < a_2 < \ldots$ cofinal in M (which we have assumed to be countable!). Choose a finite n-equivalent S_i of $[a_i, a_{i+1})$ for each i. Then $S = \Sigma_i S_i$ is the required n-equivalent of order type ω.

Virtually the same proof works for the class of *finite* ordered models.

Notice that a linear ordering $(M,<)$ is finite if it contains a least and a greatest element, every nonmaximal element has an immediate successor and *restricted induction* is satisfied, which says that every set containing the least element and closed under immediate successors (insofar as they exist) contains the greatest element as well. (Of course other characterizations work as well.) Restricted induction brings along its first-order companion: *definable restricted induction.*

3.2 Theorem *If the linearly ordered model* $\mathfrak{M}$
(1) *has a least and a greatest element and every nonmaximal element has an immediate successor, and*
(2) *satisfies definable restricted induction,*
then $\mathfrak{M}$ *has finite n-equivalents for all n.*

Proof: Begin as in the proof of 3.1. Definable restricted induction now shows X to contain the greatest element b of $\mathfrak{M}$. Thus, $[a,b)$ has a finite n-equivalent and so does $[a,b] = \mathfrak{M}$, as required.

3.3 Examples The following models show that we cannot strengthen the conclusions of 3.1 and 3.2 to: $\mathfrak{M}$ has an *elementary* equivalent (i.e., a model n-equivalent with $\mathfrak{M}$ for all *n simultaneously*) which has order type ω (respectively which is finite). For the second, let $\mathfrak{M}$ have order type $\omega + \omega^*$ and let the X_i be empty. For the first, consider $\mathfrak{M} + \mathfrak{N}$ where $\mathfrak{M}$ is the previous model and $\mathfrak{N}$ has order type ω—but $X_0 = N$ this time. The bigger n is, the longer an n-equivalent of $\mathfrak{M}$ has to be (namely, at least $2^n - 1$) and hence the larger the first element of X_0 in an n-equivalent of $\mathfrak{M} + \mathfrak{N}$ is.

3.4 Remark The direct method of proof of 3.1 and 3.2 works for the class of well-ordered models too; however, I shall derive that result from the corresponding one for the order-complete models.

3.5 Remark The monadic Π^1_1-theories of ω and the class of finite orderings are recursively enumerable. (Compare 2.6.)

4 Monadic Π^1_1-theory of complete orderings, of well-orderings, and of the reals The ordering $(M,<)$ is *complete* if each nonempty set with an upper bound has a *least* upper bound (a *sup*). Hence, $\mathfrak{M}$ is called *definably complete* if this holds for *definable* sets.

4.1 Theorem *If* $\mathfrak{M}$ *is definably complete, it has complete n-equivalents for each n.*

Before proving 4.1, here are an example and a corollary.

4.2 Example The following model shows that it is impossible to strengthen the conclusion of 4.1 to requiring an elementary equivalent of $\mathfrak{M}$.

Choose rationals $q_0 < q_1 < q_2 < \ldots$ and $r_0 > r_1 > r_2 > \ldots$ such that $\lim q_i = \lim r_i$ is *irrational*; take $A = \{q_i | i \in \mathbb{N}\} \cup \{r_i | i \in \mathbb{N}\}$ and consider $\mathfrak{M} = (\mathbb{Q},<,A)$. For each n, the models $(\mathbb{R},<,\{1,\ldots,m\})$ for $m \geq 2^n - 1$ are n-equivalents of $\mathfrak{M}$. On the other hand, suppose that $(N,<,B)$ is a complete elementary equivalent of $\mathfrak{M}$. It follows that B has order type $\omega + \alpha$ for some α: N must contain a sup of the first ω elements of B. However, $\mathfrak{M}$ lacks an element which is a limit of A's—a contradiction.

$\mathfrak{M}$ is (*definably*) *well-ordered* if each nonempty subset of M (which is parametrically first-order definable on $\mathfrak{M}$) has a least element.

The following trivial lemma may look surprising, as *completeness* usually is considered only in the context of *dense* orderings.

4.3 Lemma *$\mathfrak{M}$ is (definably) well-ordered iff it is (definably) complete, has a least element, and every nonmaximal element has an immediate successor.*

Proof: Suppose that $\emptyset \neq X \subset M$ and X has no minimum. Put $Y = \{y \in M | \forall x \in X(y < x)\}$. Y is definable if X is definable. Since the least element of $\mathfrak{M}$ must be in Y, Y is nonempty; moreover, every $x \in X$ is an upper bound of Y. Thus, Y has a sup y. If $y \in Y$, the immediate successor of y is minimal in X. Hence, $y \notin Y$. But then y must be minimal in X, a contradiction.

4.4 Corollary *If $\mathfrak{M}$ is definably well-ordered, it has well-ordered n-equivalents for each n.*

Proof: Notice that 4.3 defines *well-order* as *completeness* plus a quantifier rank 3 statement. By 4.3, $\mathfrak{M}$ is definably complete. Thus, let $m = \max(n,3)$ and take $\mathfrak{N}$ to be a complete m-equivalent of $\mathfrak{M}$ by 4.1. By 4.3 again, $\mathfrak{N}$ is the model required.

We say that an ordered sum $\Sigma_{i \in I} m(i)$ is *completely* ordered if the ordering of I is complete.

4.5 Lemma

(i) *Completely ordered sums of complete orderings with endpoints are complete.*

(ii) *Well-ordered sums of complete orderings with least elements are complete.*

Proof: (i): Let $X \subset \Sigma_{i \in I} m(i)$ have an upper bound in $m(i_0)$. Then $J = \{j | X \cap m(j) \neq \emptyset\}$ has the upper bound i_0. Let $j = \sup J$. *Case 1:* $j \in J$. Then max $m(j)$ is an upper bound for $X \cap m(j)$ and $\sup X = \sup(X \cap m(j))$. *Case 2:* $j \notin J$. Then $\sup X = \min m(j)$. (ii): Similar.

Proof of 4.1: Define $\sim$ in the fashion of 1.3 with aRb meaning: $a < b$ and (a,b) has a complete n-equivalent.

Notice that R is transitive. Hence, $\sim$ induces a condensation by 1.3. (N.B. this would not have been so obvious in case we had defined $x \sim y$ to mean that (x,y) had a complete n-equivalent only.)

Furthermore, $\sim$ is definable: compare the proof of 2.4. Hence, the equivalence classes are definable as well.

Claim 1 *Each equivalence class with an upper (lower) bound has a greatest (respectively least) element and each equivalence class has a complete n-equivalent.*

Proof: Let I be an equivalence class and $a \in I$. If I has no upper bound, choose $a_0 = a < a_1 < a_2 < \ldots < a_\xi < \ldots$ $(\xi < \alpha)$ cofinal in I. Choose a complete n-equivalent N_ξ of $[a_\xi, a_{\xi+1})$ for each $\xi < \alpha$. Then $\Sigma_{\xi<\alpha} N_\xi$ is a complete n-equivalent of $\Sigma_\xi [a_\xi, a_{\xi+1}) = \{x \in I \mid a \leq x\} = I^{\geq a}$. If I has an upper bound, it must have a sup s by definable completeness. I claim that $s \in I$ (and so, s is the maximum of I, $I^{\geq a} = [a,s]$ and hence (a,s) and, therefore, $I^{\geq a}$ as well, has a complete n-equivalent by definition). For if not choose $a_0 = a < a_1 < a_2 < \ldots < a_\xi < \ldots$ cofinal in I again to show that (a,s) has a complete n-equivalent as before; a similar argument will show that $a \sim s$.

Much the same goes for the other half $I^{<a} = \{x \in I \mid x < a\}$ of I, and so the claim has been proved.

Claim 2 *The induced ordering on the class $M/\sim$ of equivalence classes is dense.*

Proof: Suppose that $I < J$ are neighbors in $M/\sim$. Then $a = \sup I$ and $b = \inf J$ are neighbors in $\mathfrak{M}$; moreover, $a \in I$ and $b \in J$. Hence, (a,b) is empty; therefore, $a \sim b$—a contradiction.

If there is but *one* equivalence class we are done. So, assume not. The rest of the proof works towards a contradiction.

The following argument is taken from [3] (Theorem 7.17, p. 117). Choose a complete n-equivalent $\tau(I)$ for each I with $I \in M/\sim$ in such a way that $T = \{\tau(I) \mid I \in M/\sim\}$ is finite (this is possible by 1.1). Now, if for some $\sigma \in T$, $\{I \in M/\sim \mid \tau(I) = \sigma\}$ is *not* dense in the ordering of $M/\sim$, there must be a proper interval $C_0 \subset M/\sim$ such that no $I \in C_0$ has $\tau(I) = \sigma$. Repeating this argument (first with C_0 and $T \setminus \{\sigma\}$ etc.) using induction on the finite cardinal $|T|$, one ultimately arrives at the following:

Claim 3 *There is a proper (open) interval D of $M/\sim$ and a set $\Sigma \subset T$ such that* (i) *every $I \in D$ has $\tau(I) \in \Sigma$, and* (ii) *if $\sigma \in \Sigma$ then $\{I \in D \mid \tau(I) = \sigma\}$ is dense in D.*

The contradiction aimed for is contained in the next claim.

Claim 4 *D has but one element.*

Proof: Suppose that $a, b \in \bigcup D$ and $a < b$. We need to show that (a,b) has a complete n-equivalent. Suppose that $a \in I$, $b \in J$. If $I = J$, there is nothing to prove. Let E be the interval (I,J) in D. Now, $(a,b) = I^{>a} + \bigcup E + J^{<b}$; therefore it suffices to show that these components have complete n-equivalents. For $I^{>a}$ and $J^{<b}$, this is already known (cf. the proof of Claim 1). Therefore, it remains to show that $\bigcup E$ has such an n-equivalent as well.

First, notice that Claim 3 remains valid if we replace D by E. Now, construct a complete n-equivalent $\mathfrak{N}$ of the submodel $\bigcup E = \Sigma_{I \in E} I$ of $\mathfrak{M}$ as follows: let h: $\mathbb{R} \to \Sigma$ be any partition of $\mathbb{R}$ into $|\Sigma|$ classes $\{x \in \mathbb{R} \mid h(x) = \sigma\}$ $(\sigma \in \Sigma)$, each of which is dense in $\mathbb{R}$ and put $\mathfrak{N} = \Sigma_{x \in \mathbb{R}} h(x)$.

By 4.5 and Claim 1, $\mathfrak{N}$ is completely ordered. It remains to show that $\mathfrak{N}$ is n-equivalent to $\bigcup E$.

First, notice that the models $(E,<,\{I \in E \mid \tau(I) = \sigma\})_{\sigma\in\Sigma}$ and $(\mathbb{R},<,\{x \in \mathbb{R} \mid h(x) = \sigma\})_{\sigma\in\Sigma}$ (with $|\Sigma|$ unary relations each) are partially isomorphic and *a fortiori* n-equivalent. (The argument for dense orderings is well-known; the extra structure involved here—partitions into $|\Sigma|$-many dense sets—does not complicate it terribly much.) The result now follows from 1.5.

This completes the proof of 4.1.

The most prominent type of (dense) complete ordering is λ, the order type of the set of reals. The following example shows that we cannot strengthen the conclusion of 4.1 by requiring the n-equivalent to be of type λ, under the assumption that the ordering of $\mathfrak{M}$ is dense.

4.6 Example For $x \in \mathbb{R}$, let $m(x) = ([0,1],<,\varnothing)$ if x is rational, and $m(x) = ([0,1],<,[0,1])$ otherwise. Consider $\mathfrak{M} = \Sigma_{x\in\mathbb{R}}\, m(x)$. $\mathfrak{M}$ has the complete order type $(1 + \lambda + 1)\cdot\lambda$ (cf. 4.5), so it certainly is definably complete. On the other hand, the proof of Lemma 4.8 below shows that it lacks a 5-equivalent of order type λ: each complete 5-equivalent of $\mathfrak{M}$ has a definable equivalence splitting the model in an uncountable number of proper intervals—contradicting the *Suslin property* of $\mathbb{R}$. Hence, the Suslin property of $\mathbb{R}$ *contributes* to its monadic Π_1^1-theory.

The following definition, suggested by 4.6, isolates this contribution:

4.7 Definition $\mathfrak{M}$ has *property I* if each densely ordered condensation of $\mathfrak{M}$ has a dense set of singletons.

4.8 Lemma *Models of order type λ and, more generally, all complete orderings with the Suslin property, have property I.*

Proof: Suppose that P is a densely ordered condensation of a Suslin ordering. Suppose that $p < q$ in P but (p,q) does not contain a singleton. By Suslinity, (p,q) must be countable; hence, it has the order type of the rationals. Therefore, (p,q) has as many bounded sets without *sup* as there are irrationals. Let K be such a set. Then $\bigcup K$ is a bounded set in the original ordering without *sup*.

4.9 Theorem *If $\mathfrak{M}$ is definably-I, definably complete, and densely ordered without endpoints, then it has n-equivalents of order type λ for each n.*

Proof: First, we follow the proof of 4.1 with some slight modifications.

To begin with, we may assume by the Löwenheim-Skolem Theorem that $\mathfrak{M}$ is only countable.

Now, define $\sim$ by the scheme of 1.3 with aRb meaning: $a < b$ and (a,b) has an n-equivalent of order type λ. Again, R is transitive, so $\sim$ induces a condensation by 1.3.

Claim 1 *Each equivalence class has an n-equivalent of one of the following types:* 1, $\lambda + 1$ *(if it begins $\mathfrak{M}$),* $1 + \lambda$ *(if it ends $\mathfrak{M}$),* λ *(if it does both), or* $1 + \lambda + 1$.

Proof: Much as before. Notice that we need to form only *countable* sums since $\mathfrak{M}$ is countable, thus preserving the separability of the models involved, thereby guaranteeing one of the order types required.

Claim 2 *$M/\sim$ is densely ordered.*

Proof: Use the fact that the ordering of $\mathfrak{M}$ is dense.

Again, it suffices to show that M is the only class in $M/\sim$. Suppose it is not.

Claim 3 *There is a proper (open) interval D of $M/\sim$ and a finite set Σ of models of order type either* 1 *or* $1 + \lambda + 1$ *such that*
(i) *every $I \in D$ has an n-equivalent in Σ, and*
(ii) *if $\sigma \in \Sigma$ then $\{I \in D \mid I \equiv^n \sigma\}$ is dense in D.*

Proof: As before.

In order to reach the desired contradiction, and stepping over some obvious details (cf. the proof of 4.1, Claim 4), construct an n-equivalent $\mathfrak{N}$ of $\bigcup D$ of order type λ as follows:

Since $\mathfrak{M}$ is definably-I, $M/\sim$ has a dense set of singletons; hence Σ contains a singleton model τ_0. Take $h\colon \mathbb{R} \to \Sigma$ partitioning $\mathbb{R}$ into $|\Sigma|$ classes $\{x \in \mathbb{R} \mid h(x) = \sigma\}$ ($\sigma \in \Sigma$) each of which is dense and such that $\{x \in \mathbb{R} \mid h(x) = \tau_0\}$ happens to be the set of *irrationals.*

Put $\mathfrak{N} = \Sigma_{x \in \mathbb{R}} h(x)$. By 4.5, $\mathfrak{N}$ is complete as before and it is easy to see that $\mathfrak{N}$ has a countable dense set this time, whence $\mathfrak{N}$ has the order type λ. That $\mathfrak{N} \equiv^n \bigcup D$ follows as before, using 1.5.

4.10 Corollary *Every ordering which has I, is complete, and is densely ordered without endpoints satisfies the monadic Π^1_1-theory of $\mathbb{R}$.*

4.11 Example For each ordinal α, $\lambda + (1 + \lambda) \cdot \alpha$ has the required properties (and differs from λ for $\alpha \geq \omega_1$, since $\omega_1 \not\leq \lambda$).

4.12 Remark Section 17 of [2] contains a Π^1_1-characterization of $(\mathbb{R},<)$.

4.13 Remark The theorems above imply recursive enumerability of the monadic Π^1_1-theories of complete orderings, well-orderings, and $\mathbb{R}$. (Note that these theories are in fact *decidable.* For the first and last one this is due to Gurevich. The decidability of the second one is due to Rabin. Cf. [1].)

5 Monadic Π^1_1-theory of well-founded trees Previous sections dealt with linearly ordered models only; the scope is widened here somewhat to the notion of a tree.

A partially ordered set $\mathfrak{M} = (M,<)$ is called a *tree* iff, for each $m \in M$, the set $m\!\downarrow\, = \{m' \in M \mid m' < m\}$ is linearly ordered.

The Π^1_1-property considered here is well-foundedness: $\mathfrak{M}$ is *well-founded* iff each nonempty set has a minimal element; equivalently, when $\mathfrak{M}$ is a tree: iff each $m\!\downarrow$ is well-ordered.

Definable well-foundedness, of course, restricts this to *definable* sets.

The proof of Theorem 5.1 below can be considered as a paradigm for a method applicable in a variety of situations, where the models considered belong to certain types of partial orderings (trees being the simplest example) and the Π^1_1-property involved can be either well-foundedness, *converse* well-foundedness, or, more generally, some kind of completeness as in Section 4. It did not seem useful, however, to aim for this greater generality here, as a most general result probably does not exist and the generalizations obtained of the result below all appeared to be rather arbitrary.

5.1 Theorem *If $\mathfrak{M}$ is a tree with finitely many extra unary relations which is definably well-founded, then it has well-founded n-equivalents for all n.*

Again, we have the companion result on recursive enumerability of the monadic Π_1^1-theory of well-founded trees.

Before embarking on the proof, we need some surgical terminology on trees and three lemmas.

A *component* of $\mathfrak{M}$ is a maximal connected subset. An element of $\mathfrak{M}$ is minimal iff it is the least element of its component. In particular, components are (first-order parametrically) definable.

Therefore, if $\mathfrak{M}$ is definably well-founded, so are its components. The converse of this holds as well: Piet Rodenburg (by private communication) recently proved—in a setting more general than this one—that the restriction of a definable set to a component must be definable on that component. This result is not used here, however, which makes for some complications in the formulation of the lemmas below.

If $X \subset M$ is *downward closed* (i.e., if $a < b \in X$ implies that $a \in X$) then $M \backslash X$ is *upward closed*, and vice versa.

I shall use the following notations. If $X \subset M$ then $(\mathfrak{M}, X)$ denotes the expansion of $\mathfrak{M}$ obtained by adding X as a new unary relation. If $a \in M$ then $a\uparrow$ equals $\{c \in M | a \leq c\}$. Notice that, somewhat arbitrarily, $a \in a\uparrow$, but $a \notin a\downarrow$.

5.2 Lemma *Suppose that the tree $\mathfrak{M}$ is definably well-founded. If $a < b$ in $\mathfrak{M}$ then, for each n, $((a\uparrow)\backslash(b\uparrow), [a,b))$ has an n-equivalent $(\mathfrak{N}, \beta)$ such that β is well-ordered and all components of $N\backslash\beta$ are definably well-founded (see Figure 1).*

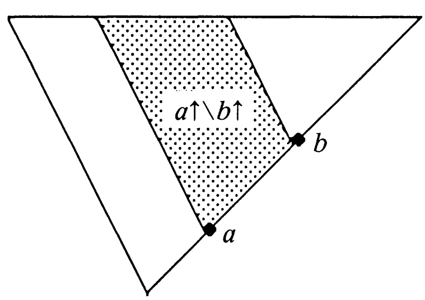

Figure 1.

Thus, $\mathfrak{N}$ can be used as a substitute (within n-equivalence) of the part $(a\uparrow)\backslash(b\uparrow)$ of $\mathfrak{M}$, thereby exchanging $[a,b)$ for the well-ordered β and preserving definable well-foundedness of the rest, *component-wise.* The other two lemmas are similar in spirit. The proof of 5.1 finally will show how to carry out such substitutions repeatedly, thereby eventually arriving at the desired well-founded n-equivalent. To see that such substitutions actually work, the following remark is needed.

Remark In what follows, a lot of cutting and pasting of trees has to be performed. To see that in each case n-equivalence is preserved, the Ehrenfeucht game technique can be applied. The general procedure is as follows. Suppose that $\mathfrak{M}'$ is obtained from $\mathfrak{M}$ by exchanging some part $\mathfrak{N}$ by an n-equivalent $\mathfrak{N}'$. In all cases occurring it will be clear how this exchange-process has to be performed, since the way $\mathfrak{N}$ is "attached" to $\mathfrak{M}\setminus\mathfrak{N}$ will be particularly simple. Let f be the identity-map on $\mathfrak{M}\setminus\mathfrak{N}$.

5.2.1 Lemma *Suppose that, for each partial isomorphism h between $\mathfrak{N}$ and $\mathfrak{N}'$, the union $f \cup h$ is a partial isomorphism between $\mathfrak{M}$ and $\mathfrak{M}'$.* (In applications it always will be rather obvious that this condition is satisfied.) *Then it will be the case that $\mathfrak{M} \equiv^n \mathfrak{M}'$.*

Proof: Consider the n-game between $\mathfrak{M}$ and $\mathfrak{M}'$. The second player wins this if, to answer moves by the first one in either $\mathfrak{N}$ or $\mathfrak{N}'$, he uses a winning strategy for the n-game between these models, and if he copies the first player on $\mathfrak{M}\setminus\mathfrak{N}$.

Proof of 5.2: Let X be the set of $b \in M$ such that for all $a < b$, $(a\uparrow\setminus b\uparrow, [a,b))$ has an n-equivalent of the type desired. The lemma asserts that $X = M$.

Suppose that $X \neq M$. Observe that X is definable: there are only finitely many n-characteristics of models $(\mathfrak{N},\beta)$ such that β is well-ordered and all components of $N\setminus\beta$ are definably well-founded; moreover, that $(a\uparrow\setminus b\uparrow, [a,b))$ satisfies a given characteristic is a first-order property of $(\mathfrak{M},a,b)$. By definable well-foundedness, $M\setminus X$, assumed to be nonempty, has a minimal element b. Suppose $a < b$ is such that a corresponding n-equivalent of the required type does not exist. Obviously, b cannot have an immediate predecessor. Choose $a_0 = a < a_1 < \ldots < a_\xi < \ldots$ $(\xi < \alpha)$ cofinal in $b\downarrow$. By the minimality of b, choose $(N_\xi, \beta_\xi) \equiv^n (a_\xi\uparrow\setminus a_{\xi+1}\uparrow, [a_\xi, a_{\xi+1}))$ such that β_ξ is well-ordered and all components of $N_\xi\setminus\beta_\xi$ are definably well-founded for each $\xi < \alpha$ (see Figure 2).

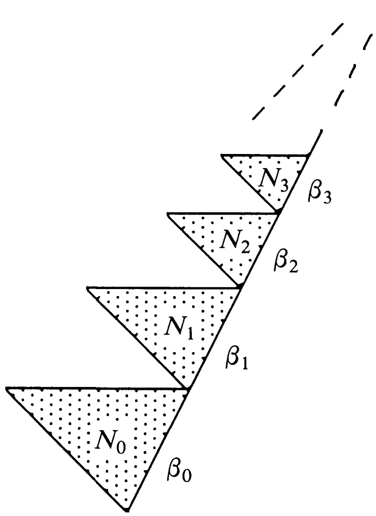

Figure 2.

The model $\Sigma_{\xi<\alpha}(N_\xi,\beta_\xi)$, obtained by gluing the β_ξ one after the other now forms a counterexample to the choice of a and b: to see that this is an n-equivalent of $(a\uparrow\backslash b\uparrow,[a,b))$, apply the remark to $(a\uparrow\backslash b\uparrow,[a,b)) = \bigcup_{\xi<\alpha} (a_\xi\uparrow\backslash a_{\xi+1}\uparrow, [a_\xi,a_{\xi+1}))$ and $\Sigma_{\xi<\alpha}(N_\xi,\beta_\xi)$.

5.3 Corollary *Suppose the tree $\mathfrak{M}$ is definably well-founded and $b \in M$. Then, for each n, $(\mathfrak{M},b)$ has an n-equivalent $(\mathfrak{M}',b')$ such that $b'\downarrow$ is well-ordered and all components of $M'\setminus(b'\downarrow)$ are definably well-founded.*

Proof: If b is minimal in $\mathfrak{M}$, then $(\mathfrak{M},b)$ itself satisfies the stipulations. Otherwise, let a be the least element of $b\downarrow$. By 5.2, $(a\uparrow\backslash b\uparrow,[a,b))$ has an n-equivalent $(\mathfrak{N},\beta)$ with β well-ordered and all components of $N\backslash\beta$ definably well-founded. Replace $(a\uparrow\backslash b\uparrow,[a,b))$ in $\mathfrak{M}$ by $(\mathfrak{N},\beta)$; the result is $\mathfrak{M}'$. In $\mathfrak{M}'$, $b\downarrow = \beta$. Thus, putting $b' = b$ makes $b'\downarrow$ well-ordered. The components of $M'\setminus(b'\downarrow)$ are the ones of $N\backslash\beta$ plus $b\uparrow$ plus the $\mathfrak{M}$-components different from the one containing b (if any); these are all definably well-founded. Finally, $(\mathfrak{M}',b') \equiv^n (\mathfrak{M},b)$ follows from the remark above.

The next lemma is the version of 5.3 with finitely many b's at the same time:

5.4 Lemma *Suppose that the tree $\mathfrak{M}$ is definably well-founded and $A \subset M$ is finite. Then, for each n, $(\mathfrak{M},a)_{a\in A}$ has an n-equivalent $(\mathfrak{M}',a')_{a\in A}$ such that each $a'\downarrow (a\in A)$ is well-ordered and all components of $M'\setminus\bigcup_{a\in A} a'\downarrow$ are definably well-founded.*

Proof: By induction on the number of elements in A. To start with, we have 5.3. For the induction step, choose $a \in A$ and put $B = A\setminus\{a\}$. Apply the inductive hypothesis to $(\mathfrak{M},a)$ and B to obtain $(\mathfrak{M}',a',b')_{b\in B}$ with all $b'\downarrow$ $(b\in B)$ well-ordered and $M'\setminus\bigcup_{b\in B} b'\downarrow$ definably well-founded — *component-wise.*

Case 1: Suppose that for some $b \in B$, $a < b$. Then $a' < b'$, $a'\downarrow$ is well-ordered, $M'\setminus\bigcup_{a\in A} a'\downarrow = M'\setminus\bigcup_{b\in B} b'\downarrow$, and we are done.

Case 2: If not, let C be the component of $M'\setminus\bigcup_{b\in B} b'\downarrow$ containing a'. By 5.3, obtain $(\mathcal{C}',a'') \equiv^n (\mathcal{C},a')$ with $a''\downarrow$ well-ordered and $C'\setminus(a'')\downarrow$ definably well-founded, component-wise. Replace $(\mathcal{C},a')$ in $\mathfrak{M}'$ by $(\mathcal{C}',a'')$ to obtain the desired model $(\mathfrak{M}'',a'',b')_{b\in B}$.

We are now ready for the proof of 5.1.

Proof: Define a sequence of models $\mathfrak{M}_0,\mathfrak{M}_1,\mathfrak{M}_2,\ldots$ and sets $T^0,T^1,T^2,\ldots$ such that:

1. $\mathfrak{M}_0 = \mathfrak{M}$; $T^0 = \varnothing$
2. T^i is a well-founded downward-closed part of $\mathfrak{M}_i$ and every component of $M_i\backslash T^i$ is definably well-founded
3. T^{i+1} (considered as a submodel of $\mathfrak{M}^{i+1}$) is an *end-extension* of T^i

(considered as a submodel of $\mathfrak{M}_i$) (i.e., $T^i \subset T^{i+1}$, and for $a, b \in T^{i+1}$, if $b \in T^i$ and $a < b$ then $a \in T^i$)

4. $(\mathfrak{M}_i, t)_{t \in T^i} \equiv^n (\mathfrak{M}_{i+1}, t)_{t \in T^i}$
5. for all $a \in M_i \backslash T^i$ there is a $b \in T^{i+1}$ such that $(\mathfrak{M}_i, t, a)_{t \in T^i} \equiv^{n-1} (\mathfrak{M}_{i+1}, t, b)_{t \in T^i}$.

$\mathfrak{M}_{i+1}$ and T^{i+1} will be obtained from $\mathfrak{M}_i$ and T^i by replacing $M_i \backslash T^i$ in $\mathfrak{M}_i$ by an n-equivalent with a well-founded initial part (namely, $T^{i+1} \backslash T^i$) preserving definable well-foundedness *component-wise.* This will take care of 2–4. However, T^{i+1} must be big enough so as to satisfy 5. This is achieved in the following manner:

Let C be a component of $M_i \backslash T^i$. Choose $A \subset C$ such that for each $c \in C$ there is an $a \in A$ with $(\mathcal{C}, a) \equiv^{n-1} (\mathcal{C}, c)$ and such that A is finite—this can be done according to 1.1. By 5.4, $(\mathcal{C}, a)_{a \in A}$ has an n-equivalent $(\mathcal{C}', a')_{a \in A}$ with every $a'\!\downarrow$ well-ordered and $C' \Big\backslash \bigcup_{a \in A} a'\!\downarrow$ definably well-founded, component-wise.

$\mathfrak{M}_{i+1}$ is obtained from $\mathfrak{M}_i$ by exchanging $\mathcal{C}$ for $\mathcal{C}'$ and making similar replacements for every other component of $M_i \backslash T^i$. T^{i+1} is T^i plus all the $\bigcup_{a \in A} (a'\!\downarrow \cup \{a'\})$ so encountered.

It is now obvious that 2–5 are satisfied.

Now, put $\mathfrak{N} = \bigcup_i T^i$. By 2–3, $\mathfrak{N}$ is well-founded. I claim that $\mathfrak{N}$ is an n-equivalent of $\mathfrak{M}$. Consider the Ehrenfeucht-n-game between these models. Notice that, as $T^i \subset \mathfrak{M}_i$, in order to win it suffices for the second player to choose his moves in such a way that after his k-th move the sequences $a_0, \ldots, a_{k-1} \in M$ and $t_0, \ldots, t_{k-1} \in N$ have been played such that for all i, if $t_0, \ldots, t_{k-1} \in T^i$ then

$$[*]_i \qquad (\mathfrak{M}, a_0, \ldots, a_{k-1}) \equiv^{n-k} (\mathfrak{M}_i, t_0, \ldots, t_{k-1}).$$

Notice that $i \le j$ and $[*]_i$ imply $[*]_j$ by condition 4 above.

Now, the second player can keep up with this requirement: First, if $k = 0$ then $[*]_0$ holds since $\mathfrak{M}_0 = \mathfrak{M}$.

Next, suppose the players have arrived at a position where $[*]_i$ still is satisfied.

(a) Let the first player choose $t_k \in N$, say, $t_k \in T^j$. If $j \le i$ then $[*]_i$ provides the second player with an $a_k \in M$ such that $(\mathfrak{M}, a_0, \ldots, a_k) \equiv^{n-k-1} (\mathfrak{M}_i, t_0, \ldots, t_k)$. If $i < j$, simply use $[*]_j$.

(b) Assume that the first player chooses $a_k \in M$. By $[*]_i$, there is a $u \in M_i$ such that $(\mathfrak{M}, a_0, \ldots, a_k) \equiv^{n-k-1} (\mathfrak{M}_i, t_0, \ldots, t_{k-1}, u)$. If, by a stroke of luck, $u \in T^i$, we are done. If not, by condition 5 there is a $t_k \in T^{i+1}$ such that $(\mathfrak{M}_{i+1}, t, t_k)_{t \in T^i} \equiv^{n-1} (\mathfrak{M}_i, t, u)_{t \in T^i}$, in particular, $(\mathfrak{M}_{i+1}, t_0, \ldots, t_k) \equiv^{n-k-1} (\mathfrak{M}_i, t_0, \ldots, t_{k-1}, u)$. Hence, $(\mathfrak{M}, a_0, \ldots, a_k) \equiv^{n-k-1} (\mathfrak{M}_{i+1}, t_0, \ldots, t_k)$; so the second player chooses t_k, thereby ensuring that $[*]_{i+1}$ holds for the resulting sequences.

6 Appendix: Strengthening 2.4 and 4.4 Let $\mathfrak{M}_0$ be the smallest class of order types such that

(1) $1 \in \mathfrak{M}_0$
(2) $\alpha,\beta \in \mathfrak{M}_0 \Rightarrow \alpha + \beta \in \mathfrak{M}_0$
(3) $\alpha \in \mathfrak{M}_0 \Rightarrow \alpha\cdot\omega, \alpha\cdot\omega^* \in \mathfrak{M}_0$.

By 2.1, all types in $\mathfrak{M}_0$ are scattered. By a theorem of Laüchli and Leonard ([3], Theorem 7.9, p. 115) $\mathfrak{M}_0$ contains n-equivalents for each scattered ordering and for all n. Their method of proof shows that the extra unary relations of our models do not spoil this situation:

2.4′ Theorem *If $\mathfrak{M}$ is definably scattered, it has (scattered) n-equivalents with order type in $\mathfrak{M}_0$ for each n.*

Proof: On M, define $\sim$ by way of 1.3 with aRb meaning: $a < b$ and (a,b) has an n-equivalent with order type in $\mathfrak{M}_0$. By (1) and (2), R is transitive, so $\sim$ induces a condensation.

Claim 1 *Each equivalence class has an n-equivalent with order type in $\mathfrak{M}_0$.*

Proof: If the class I is unbounded, choose $a < a_0 < a_1 < a_2 < \ldots$ cofinal in I (by Löwenheim-Skolem, assume $\mathfrak{M}$ is countable). For $i<j$, let $h(i,j)$ be the n-characteristic of $[a_i,a_j)$. By *Ramsey's Theorem*, there is an infinite set $A \subset \mathbb{N}$ and a σ such that if $i<j$ and $i,j \in A$ then $h(i,j) = \sigma$. Let $i = \min A$ and choose $N_0 \equiv^n [a,a_i)$ and $N \models \sigma$ with order types in $\mathfrak{M}_0$. Then $I^{\geq a} \equiv^n N_0 + N\cdot\omega$ and this model has order type in $\mathfrak{M}_0$ by (2) and (3).

The same goes for $I^{<a}$, etc.

As before, $M/\sim$ must be either dense or consist of one class only; since the first alternative cannot obtain, the proof is finished.

Next, let $\mathcal{K}$ be the smallest class of order types such that

(1) $1 \in \mathcal{K}$
(2) $\alpha,\beta \in \mathcal{K} \Rightarrow \alpha + \beta \in \mathcal{K}$
(3) $\alpha \in \mathcal{K} \Rightarrow \alpha\cdot\omega \in \mathcal{K}$.

Clearly, $\mathcal{K} \subset \mathfrak{M}_0$. All types in $\mathcal{K}$ are well-ordered and it is easy to see (using Cantor normal forms) that $\alpha \in \mathcal{K}$ iff $0 < \alpha < \omega^\omega$. $\mathcal{K}$ contains n-equivalents for each well-ordering and for all n (it is easy to see that no smaller class has this property). Again, extra unary relations do not change this state of affairs:

4.4′ Theorem *If $\mathfrak{M}$ is definably well-ordered, it has (well-ordered) n-equivalents with order-type in $\mathcal{K}$ for each n.*

Proof: Define $X = \{a \mid \forall b < a [b,a)$ has an n-equivalent with order type in $\mathcal{K}\}$. X is definable, hence if $X \neq M$ then $M \backslash X$ has a least element a. Pick $b < a$ such that $[b,a)$ has no n-equivalent with type in $\mathcal{K}$. By (1) and (2), a cannot be a successor. Now, choose $b < b_0 < b_1 < \ldots$ cofinal in $[b,a)$ and argue as in the previous proof.

Corollary (Ehrenfeucht – [3], Theorem 6.22, p. 108) $\omega^\omega \equiv (\mathrm{OR}, <)$ *(where* OR *is the class of all ordinals).*

Proof: If I starts an $(n+1)$-game with a move $\alpha \in \mathrm{OR}$, II answers with an n-equivalent β in ω^ω. (Notice that $\alpha\uparrow \cong \mathrm{OR}$ and $\beta\uparrow \cong \omega^\omega$.)

The classes $\mathfrak{M}_0$ and $\mathcal{K}$ were inductively defined by closure properties obtained by looking at what it takes to prove 2.4 and 4.4. In the same way, one may find, e.g., a class $\mathcal{C}$ of order types such that each completely and densely ordered model without endpoints has n-equivalents with types in $\mathcal{C}$; closure properties needed here are

(1) $\lambda \in \mathcal{C}$
(2) $\alpha, \beta \in \mathcal{C} \Rightarrow \alpha + 1 + \beta \in \mathcal{C}$
(3) $\alpha \in \mathcal{C} \Rightarrow (\alpha + 1)\cdot\omega,\ (1 + \alpha)\cdot\omega^* \in \mathcal{C}$
(4) if h: $\mathbb{R} \to \mathcal{C}$ has $h[\mathbb{R}]$ finite and all $\{x \in \mathbb{R} | h(x) = \sigma\}$ $(\sigma \in h[\mathbb{R}])$ dense in $\mathbb{R}$ then $\Sigma_{x\in\mathbb{R}}(1 + h(x) + 1) \in \mathcal{C}$.

NOTE

1. After this paper had been completed my attention was drawn to [1] (by Burgess). It appears that the material below displays the following points of overlap with that work, which precedes mine by at last three years. First, Gurevich already identified Property I (cf. 4.7 below). As a matter of fact, he presents it in its *monadic* universal second-order form using a selector-set (this is similar to the phenomenon of two formulations of scatteredness, cf. 2.5 below). Second, much of the proofs of 4.1 and 4.9 (in particular, the proper way to handle the condensation argument and the use of shufflings) can be found in their paper. Third, there are remarks identical with 4.10 and 4.11. However, their paper supercedes mine with respect to 4.13, as it even shows *decidability* of the theories involved.

REFERENCES

[1] Burgess, J. P. and Y. Gurevich, "The decision problem for linear temporal logic," *Notre Dame Journal of Formal Logic*, vol. 26 (1985), pp. 115–126.

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Mathematisch Instituut
Plantage Muidergracht 24
1018 TL Amsterdam
The Netherlands