Notre Dame Journal of Formal Logic
Volume 33, Number 4, Fall 1992

Book Review

Francisco A. Rodriguez-Consuegra. *The Mathematical Philosophy of Bertrand Russell: Origins and Development*. Birkhäuser Press, Boston, 1991. 256 pages.

This book is a probing new account of the origins and development of Russell's early philosophy of mathematics. Beginning from Moore and Russell's transition away from neo-Hegelian Idealism, Rodriguez-Consuegra examines the many manuscripts that remain from Russell's early efforts to write a treatise on mathematics. In its first chapter, the book discusses Russell's early encounters with the works of Dedekind, Cantor, and Conturat, as well as Whitehead's "universal algebra". The chapter continues with the hint that Bradley's philosophy benefited Russell and discusses the impact of Moore's view that relations are external, his theory of judgment, and his attack on the subject-predicate pattern of the Idealists. Taking up Russell's stalled foundational work before his fateful meeting with Peano in 1900, chapter two attempts to set out the salient features of Russell's 1899 *Fundamental Ideas and Axioms of Mathematics* and the 1898 *Analysis of Mathematical Reasoning*. Two difficult and involuted works, the author makes his way by giving pre-eminence to Moore. The emphasis on relations is again present together with an appearance of Russell's "Principle of Abstraction". This sets the stage for the remainder of the five-chapter book. Subsequent chapters take up the following two themes: the evolution of Russell's views on relations and his views on the Principle of Abstraction. The two are traced in the context of Russell's enlightenment by Peano and Peano's disciples, through the fruition of his early studies in the 1903 *Principles of Mathematics*, and indeed all the way to the 1910 *Principia Mathematica*. The book challenges scholars to reexamine much of the prevailing understanding of many technical and philosophical aspects of Russell's early Logicism.

1 The author gives a valuable discussion and chronicle of the important influence of Peano's *Formulaire de Mathématique* (1894–1905). The book details Peano's many innovations: the notion of "formal implication", marking the difference between "real" (free) and "apparent" (bound) variables; the distinction

Received September 8, 1992

of elementhood (with its characteristic symbol,"∈") from subclasshood, and a singleton class from its single member; Peano's constructions of rational numbers, real numbers, the notion of "limit", and definitions in point-set theory; and, of course, the Peano Postulates. Peano's disciples are also discussed. We are witness to the fact that the work of Peano's school, e.g., Pieri, Burali-Forti, Padoa, and the discussion of such figures as Cantor, Dedekind, and Weierstrass in the *Formulaire*, presented the most contemporary work on the axiomatization of the various branches of pure mathematics, including Real Analysis and Geometry. Indeed, the author shows that Russell's progression from Logic to Arithmetic and onward to the constructions needed for Analysis and then Geometry parallels that of Peano and his disciples.

Now this chronicle is quite significant in itself. No scholar can now rest content with a passing characterization of Peano as merely offering Russell a formal notation. Peano, and his disciples, offered a vast resource of ideas and constructions for pure mathematics—a resource that undoubtedly brought Russell to Logicism. But the book aims at much more. Russell never tired in showing his indebtedness to Peano and his school. In 1900 Russell and Whitehead attended *The International Congress on Philosophy* in Paris. It was, as Russell later recalled in *My Mental Development*, "the most important year of my intellectual life". But Russell claimed that the lack of a developed logic of relations and the reliance on definition by abstraction, by postulates, and by recursion were responsible for the school's view that "the various branches of Mathematics have various indefinables by means of which the remaining ideas of the said subjects are defined" (*Principles*, p. 112). In the 1901 "Mathematics and the Metaphysicians", Russell explained that Peano's primitive mathematical notions, "zero", "natural number", and "successor", "can be explained by means of the notion *relation* and *class*; but this requires the Logic of Relations, which Professor Peano has never taken up" (quoted from *Mysticism and Logic*, p. 62).

Rodriguez-Consuegra calls Russell's claim to task. Russell criticized Peano's adoption of Peirce's notion of an "ordered couple" and rejected his conception of relations as classes of couples. The book rejoins that the Peano/Peirce conception is really a forerunner of the "extensional view" of relations (p. 148). The Wiener-Kuratowski definition of "ordered pair" vindicates the extensional view. Russell's criticism was the result of mistakenly believing that since Moore is correct that relations are external (rather than internal as in neo-Hegelianism), relations must be taken in intension (p. 183). Further, the book claims that Russell eventually broke with Moore, and in *Principia* we witness relations in intension abandoned—a "victory of the Peanian approach" (p. 195).

Moreover, the book continues, a careful investigation of Russell's "Principle of Abstraction"—which the book labels the "philosophical axis" of *Principles* (p. 154)—fails to provide Russell with any technical criticism of Peano's work. The author says the principle evolved through three stages, and he enumerates their roles in Russell's work (p. 192). Contending that its occurrence in *Principles* was the outcome of philosophical concerns imposed by Moore (p. 183), he says that its utter "uselessness" (p. 202) became apparent to Russell in *Principia*. "The final destiny of the principle of abstraction", he remarks, "was the (secret) admission by Russell to have been pursuing an illusion . . ." (p. 203). Continuing this theme, the book comes to the conclusion that the crucial segment of Russell's 1901 "Sur la logique des relations", containing the codiscovery of Frege's theory of cardinal number, was itself a development of an idea set forth in the 1901 *Formulaire*—albeit rejected therein. Left to its Principle of Abstraction, Russell's article is regarded as a "failed attempt at a nominal construction of cardinals" (p. 152).

2 The book's examination of the contributions of Peano and his disciples to the foundations of mathematics is admirable. But its interpretations (above) are absolutely astounding. All, I'm afraid, are based on serious misunderstandings of (A) Russell's views on relations, and (B) Russell's Principle of Abstraction. In what follows, we shall set out the author's arguments more carefully and offer a critical discussion.

(A) Russell considered the notion of an ordered "couple" logically odious. What sort of *single* entity, after all, is a "couple"? He felt that Peano's conception of relations as classes of couples was nothing but a holdover from the algebraic methods of Peirce—methods which worked with cumbersome Boolean operations rather than formal implication. Indeed, the conception of relational statements as predications involving couples arose from a failure to free logic from the paradigm of subject-predicate. One puts "$(x;y)$ is an R" and then, conflating predication with membership, "$(x;y) \in R$". But with the apparatus of formal implication the obscure "couple" is entirely avoidable for the logical form "x bears R to y."

Now the Wiener-Kuratowski definition of an "ordered pair" as a certain class avoids logical problems with the notion of a "couple" (Wiener, 1912; Kuratowski, 1920). But the book is surely misleading when it says that this is a "victory for the Peanian approach". To be sure, Peano's innovation of "formal implication" together with the device of an ordered couple (bracketing the logical problems later resolved) suffices to simulate a logic of relations adequate for arithmetic purposes. But Peano never put the two devices together in the appropriate way. Peano's definitions involving relational notions such as that of a function are flawed in just the way Russell says in "Sur la logique des relations". Russell's paper was to improve Peano's method by giving proper definitions of relational notions. That this could have been done without dropping the notion of a "couple" is irrelevant.

The book is equally misleading in speaking of the use of ordered couples as an "extensional notion of a relation", as if this notion "eliminates" relations in intension. The Wiener-Kuratowski definition does not exclude Russell's Logicism—a thesis which holds that the logic of concepts and relations (in intension), i.e., the logic of predication, is fundamental. Nor does it exclude Frege's similar notion that the predicational form "xRy" is fundamental and not the notion of a class of ordered couples. (The book erroneously allies Frege's theory of *Werthverläufe* with the approach of Peano and Peirce (p. 148).) For Frege and Russell a class is the extension of a concept; a relation-in-extension is the extension of a relation.

Principia, of course, does not adopt the Wiener-Kuratowski definition. Whitehead and Russell give an account that shows (through their treatment of class symbols as incomplete symbols) how to proxy the notion of the extension of a polyadic propositional function. To be sure, there is a debate about the nature of propositional functions in *Principia*. Are relations to be identified with polyadic propositional functions? Quine says "yes", but suppose not. There is, nevertheless, an ontology of relations in intension in *Principia*. Relations occur in facts such as '*a*-in-the-relation-R-to-*b*' and in judgment itself. Indeed, "Knowledge by acquaintance and knowledge by description" (1911) as well as *Problems of Philosophy* (1912) catalog a long list of relations in intension.

Perhaps Rodriguez-Consuegra just means that relations in intension play no role in the treatment of the symbols for the extensions of relations. But then, to take one of Russell's relations, how do we form the extension of the relation 'resemblance'. Is the propositional function '*x* resembles *y*' distinct from 'resemblance'? There is no inkling in the book that some special interpretation of *Principia*'s propositional functions has been assumed. But even with some special theory, any historically accurate account of propositional functions must make them intensional. Hoping to support his claim that Russell came to abandon relations in intension, Rodriguez-Consuegra quotes (p. 152) from a 1910 letter Russell sent to Jourdain. Russell says that in working out the logic of relations in 1901, "I was largely guided by the belief that relations must be taken in *intension*, which I have since abandoned, though I have not abandoned the notations which it led me to adopt". But the point is that he thought relations could not have extensions. This is just the problem of what sort of single entity an "ordered pair" could be. In getting along without classes altogether, *Principia* shows how to avoid the whole trouble. The intensional viewpoint remains fundamental. In fact, Whitehead and Russell say this explicitly (*Principia*, p. 72).

3 Let us turn to (B). In the 1901 *Formulaire*, Peano explicitly stated that no definition (i.e., no nominal definition) of "cardinal number" can be obtained using only the logical notions "cls", "$\in$", "$\supset$", "*n*", "$=$" (p. 39 §20). He did say, as Kennedy (*Notre Dame Journal of Formal Logic*, vol. 14 (1973), p. 369) points out, that we can avoid definition by postulates of "zero" and "successor" by defining "sameness of cardinality" by abstraction (*Formulaire* 1901, §32). On this alternative approach, only logical notions are required. These seem in tension. But Kennedy rightly explains that Peano viewed either method as being committed to taking the notion of "cardinal number" as a primitive. Abstraction yields only a definition of "sameness of cardinality", not of "cardinal number" itself.

In §32 of the *Formulaire* for 1901, Peano also considered and rejected nominally identifying the cardinal number of a class *C* with the class of all classes similar to *C*. Moreover, in the published French version of Russell's "Sur la logique des relations", there is a reference to the 1901 *Formulaire*. Kennedy attributes both of these "oddities" to Peano's having received Russell's paper in October 1900. Having shelved it to complete the 1901 *Formulaire*, he could have added the citations and also the rejection of Russell's nominal definition. Rodriguez-Consuegra's manuscript digging has uncovered some reasons to doubt Kennedy's view (pp. 157–60). Citations of the *Formulaire* also occur in Russell's own French draft of 1901. Moreover, the draft contains many printer's marks and yet a crucial passage is missing—the very passage ("Meanwhile . . .") which says that we may nominally identify the cardinal number of a class with the class of all classes similar to it. Neither does the "Meanwhile . . ." passage occur in the early 1900 manuscript. The author concludes that the crucial passage was added at printing *after* seeing the 1901 *Formulaire.*

From these data, his analysis of the Principle of Abstraction in early manuscripts, and his claim that Pieri (p. 132) sought and Burali-Forti (p. 133) found a means of providing nominal definitions of Peano's arithmetical primitives, the author comes to some bold conclusions. Peano's alternative method, he says, was based on Burali-Forti's work (e.g., the 1900 "Sur les différentes méthodes logiques pour la définition du nombre réel" delivered at the Paris congress). In the author's view (p. 134), this work "gives nominal *logicist definitions* of the three 'indefinable' terms in Peano's arithmetical system (N_0, 0, successor)." Russell would have known Burali-Forti's work from the Paris congress, and he had seen earlier writings of Burali-Forti's, such as "Les propriétés formales des opérations algébriques" (1899). Russell's Principle of Abstraction in his early 1900 work, however, provided no nominal definition and a reference to Burali-Forti was added after Paris. Hence, the author concludes that it was Russell's contact with Burali-Forti, Pieri, and Peano's 1901 *Formulaire* that made him see how Peano's primitive ideas could be given nominal definitions involving only logical notions (p. 134). He goes on to maintain that the absence of the "Meanwhile . . ." passage is the absence of any evidence in "Sur la logique des relations" that Russell codiscovered Frege's nominal definition of "cardinal number" (p. 157).

Now Pieri and Burali-Forti did indeed criticize definition by abstraction and searched for nominal definitions. But did Burali-Forti succeed in giving "logicist definitions" of Peano's primitives? The book cites references, but the reader is left to perform the task of demonstration. A look at Burali-Forti, however, leaves us wanting. The method (so far as I can tell) endeavors to identify the finite cardinal numbers with a class of operations by beginning from something like primitive recursion.

$$(f)(A)(a)\,[f\colon wxA \rightarrow A \;\&\; a \in A \;.\!\supset\!.\; (\exists^1 h)(h\colon w \rightarrow A \;\&\; h(0) = a \;\&$$
$$(n)(n \in w \;.\!\supset\!.\; h(n+) = f(\langle n, h(n)\rangle)))]\,.$$

(The superscripted "1" is for uniqueness.) But Burali-Forti tries to get along without the + (successor) function and also takes certain features of addition as axiomatic. It remains highly questionable that the "alternative method" of Peano's 1901 *Formulaire* owes itself to Burali-Forti or that Burali-Forti's method should be described as having been followed in its general idea by the "Meanwhile . . ." passage of Russell's paper (p. 157). The author's link between Burali-Forti's method, §32 of the 1901 *Formulaire*, and Russell, is quite tenuous.

Another part of the support for Rodriguez-Consuegra's conclusions is flawed. This is the book's discussion of the Principle of Abstraction. In *Principles*, Russell objected to definition by abstraction because, beginning from a class of like entitites, abstraction cannot hope to find a unique property common to all and only members of the class which underwrites the likeness of its members. There are many such properties which can be said to generate the likeness. Some classes are like under the equivalence relation 'similarity'. Abstraction would then take the cardinal number of a class *C* to be that property—actually, the class which is its extension—that all and only classes similar to *C* have in common. For instance, let C be $\{a\}$. The classes $\{a\}$, $\{b\}$, $\{\{a\}\}$, etc., are similar. So we are supposed to abstract away the property they have in common, take its extension, and regard this as the cardinal number 1. Russell objects, however, that there are other properties that all and only classes similar to $\{a\}$ have in common. Thus abstraction cannot yield "the" cardinal number 1.

Russell, however, distills out what is useful about abstraction, namely the Principle of Abstraction:

> Every equivalence relation R (that is exemplified) can be viewed as the relative product of some function S and its converse $\breve{S}$.

The relative product of a function S and its converse is the function $S/\breve{S}$. We have that $\{a\}$ bears $S/\breve{S}$ to $\{b\}$ if and only if there is some x such that $\{a\}$ has S to x and x has S to $\{b\}$. Taking R as 'similarity' as we did above, and taking a particular S, we get an $\underset{\sim}{x}$ which will render one notion of "the cardinal number 1". Finding this $\underset{\sim}{x}$ does not, of course, solve the problem of uniqueness. For a function S', we get an $\underset{\sim}{x}$ which renders a different notion of "the cardinal number 1".

So why is the Principle of Abstraction so important in finding nominal definitions? In the author's view, it simply isn't! But this is wrong. The principle is important to Russell because it can be used to demonstrate that the range of any S such that:

$$(x)(y)(x \operatorname{sim} y \;.\!\equiv.\; x(S/\breve{S})y)$$

will have all the formal properties of the cardinal numbers. Indeed, Russell proves in "Sur la logique des relations" that the range $\breve{\sigma}$ of any such choice of S, when 'similarity' is restricted to finite classes, forms a progression satisfying the Peano Postulates (§4 *2). Hence, although definition by abstraction is worthless, Russell's Principle of Abstraction reveals that we can find our nominal definition if we simply choose a particular S. Is there a philosophical ground for one choice over another? Russell offers no explicit answer in "Sur la logique des relations". (Frege's logical analysis of number does give strong argument which Russell would later embrace.) In his proof of the Principle of Abstraction at *6.2, however, he reveals a prejudice for a particular choice of S, namely:

$$Sx = \{z : z \operatorname{sim} x\},$$

for $\underset{\sim}{x}$ in the domain of the relation 'similarity'. The range of this S is just the Frege cardinal numbers. And when the domain of 'similarity' is restricted to finite classes, we arrive at Frege's construction of natural numbers as finite cardinals.

Unfortunately, this is entirely lost in the book. Blind to the real nature of the Principle of Abstraction, the author focuses on Russell's claim that the principle is "presupposed" in definition by abstraction. This leads him to hold that Russell's demonstration that Arithmetic holds for the range of any choice of an appropriate S is just an attempt to get along with definition by abstraction. The author therefore finds the appearance in the work of both Russell's criticism of definition by abstraction and the Principle of Abstraction "somewhat obscure"—unless one can think of those criticisms (with the missing "Meanwhile . . ." passage) as a late addition (p. 191). In the earlier 1900 manuscript, Russell worked with the Principle of Abstraction, again saying it is "presupposed" in definition by abstraction. The author directs us to the following conditional definition (p. 160),

$$\text{sim} = S/\breve{S} \;.\!\supset\!.\; Nc = \breve{\sigma}. \quad \text{Df},$$

and the margin note: "This Df won't do. There may be many such relations as *S*. *Nc* must be indefinable." Rodriguez-Consuegra believes that the Principle of Abstraction could not provide a nominal definition and therefore must have played other philosophical roles in Russell's philosophy.

To be sure, Russell's margin note indicates that sometime in writing the early work, he knew that the principle could not give a nominal definition this way. But it is surely a small step for Russell to see that although there are many such relations *S*, each will suffice for a nominal definition. The step, explicit in *Principles*, is quite distinct from any of Burali-Forti's ideas. It is certainly reasonable to think that Russell had come to see it earlier. Thus, the absence of the "Meanwhile . . ." passage is surely not enough to suggest that Russell knew of no nominal definition of "cardinal number" in "Sur la logique des relations" (French Draft) or even at sometime nearer to completion of the early 1900 manuscript. That Russell did not have a string "$--- =_{\text{Df}} ---$" forming a nominal definition of the cardinal number of a class is no telling evidence against Russell's codiscovery. Russell's method of the Principle of Abstraction itself attests to his independence from Burali-Forti and Peano in coming to a nominal definition. The Principle of Abstraction is not "useless" (as is definition by abstraction) and Russell never came to any such conclusion in *Principia*, in "secret" or otherwise. (Witness *72.66 and Russell's following comments, p. 442.)

These two weaknesses—that of (A) and (B)—are an unfortunate drawback to the book. Nonetheless, they do not destroy its originality or importance in calling attention to the significant achievements of the Italian school. Readers should eagerly await the companion volume which carries the story of Russell's development to *Principia*.

Gregory Landini
Department of Philosophy
University of Iowa
Iowa City, Iowa 52242