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Counting Functions

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Abstract Counting functions are shown to be closed under composition.

The proof by Pelletier and Martin [1] of Post's Functional Completeness Theorem contains a very complex argument that shows in effect that counting functions are closed under composition. The purpose of this note is to give a simple proof of this result.

A function f^n is an *n-ary truth function* iff the domain of f^n consists of the set of n-tuples of truth values (t and f) and f^n assigns t or f to each member of the domain. Let R_n be an n-tuple of truth values. $O_{n,i}$, for $1 \leq i \leq n$, is an *operator* iff, for every $R_n, O_{n,i}R_n$ is an n-tuple of truth values that differs from R_n in and only in the ith place. A function f^n is *counting* iff f^n is an n-ary truth function and for every operator $O_{n,i}$ either $f^n(R_n) = f^n(O_{n,i}R_n)$ for every R_n or $f^n(R_n) \neq f^n(O_{n,i}R_n)$ for every R_n. (Suppose $f^2 = \{\langle\langle tt\rangle, f\rangle, \langle\langle tf\rangle, t\rangle, \langle\langle ft\rangle, f\rangle, \langle\langle ff\rangle, t\rangle\}$. f^2 is counting since $f^2(R_2) = f^2(O_{2,1}R_2)$ for every R_2 and $f^2(R_2) \neq f^2(O_{2,2}R_2)$ for every R_2.)

Theorem 1 *If function f^n is counting and $\langle O_{n,i_1}, \ldots, O_{n,i_m}\rangle$ is a sequence of operators then either $f^n(R_n) = f^n(O_{n,i_1} \ldots O_{n,i_m}R_n)$ for every R_n or $f^n(R_n) \neq f^n(O_{n,i_1} \ldots O_{n,i_m}R_n)$ for every R_n.*

Proof: Assume the antecedent. We use induction on the length m of the sequence of operators. Basis step: $m = 1$. Immediate. Induction step: $m > 1$. By the induction hypothesis either $f^n(R_n) = f^n(O_{n,i_2} \ldots O_{n,i_{m+1}}R_n)$ for every R_n or $f^n(R_n) \neq f^n(O_{n,i_2} \ldots O_{n,i_{m+1}}R_n)$ for every R_n. So either $f^n(O_{n,i_1}R_n) = f^n(O_{n,i_1}O_{n,i_2} \ldots O_{n,i_{m+1}}R_n)$ for every R_n or $f^n(O_{n,i_1}R_n) \neq f^n(O_{n,i_1}O_{n,i_2} \ldots O_{n,i_{m+1}}R_n)$ for every R_n. Since either $f^n(R_n) = f^n(O_{n,i_1}R_n)$ for every R_n or $f^n(R_n) \neq f^n(O_{n,i_1}R_n)$ for every R_n, either $f^n(R_n) = f^n(O_{n,i_1} \ldots O_{n,i_{m+1}}R_n)$ for every R_n or $f^n(R_n) \neq f^n(O_{n,i_1} \ldots O_{n,i_{m+1}}R_n)$ for every R_n.

Theorem 2 *If functions $g^m, h_1^n, \ldots h_m^n$ are counting and $f^n = g^m(h_1^n, \ldots h_m^n)$ then f^n is counting.*

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Proof: Assume the antecedent. Then f^n is an n-ary truth function. Suppose $O_{n,i}$ is an operator. Let $y \in X$ iff $h_y^n(R_n) \neq h_y^n(O_{n,i}R_n)$ for every R_n. If X is empty then $g^m(h_1^n(R_n), \ldots h_m^n(R_n)) = g^m(h_1^n(O_{n,i}R_n), \ldots h_m^n(O_{n,i}R_n))$ for every R_n. So $f^n(R_n) = f^n(O_{n,i}R_n)$ for every R_n. So f^n is counting. If X is non-empty then $X = \{k_1, \ldots k_r\}$ (for $r \leq m$). Then $g^m(O_{m,k_1} \ldots O_{m,k_r}(h_1^n(R_n), \ldots h_m^n(R_n)) = g^m(h_1^n(O_{n,i}R_n), \ldots h_m^n(O_{n,i}R_n))$ for every R_n. By Theorem 1, $g^m(h_1^n(R_n), \ldots h_m^n(R_n)) = g^m(O_{m,k_1} \ldots O_{m,k_r}(h_1^n(R_n), \ldots h_m^n(R_n)))$ for every R_n or $g^m(h_1^n(R_n), \ldots h_m^n(R_n)) \neq g^m(O_{m,k_1} \ldots O_{m,k_r}(h_1^n(R_n), \ldots h_m^n(R_n)))$ for every R_n. So $f^n(R_n) = f^n(O_{n,i}R_n)$ for every R_n or $f^n(R_n) \neq f^n(O_{n,i}R_n)$ for every R_n. So f^n is counting.

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REFERENCE

[1] Pelletier, F. and N. Martin, "Post's Functional Completeness Theorem," *Notre Dame Journal of Formal Logic*, vol. 31 (1990), pp. 462–475.

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