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On theorems of Gödel and Kreisel: Completeness and Markov's Principle

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Abstract In 1957, Gödel proved that completeness for intuitionistic predicate logic **HPL** implies forms of Markov's Principle, **MP**. The result first appeared, with Kreisel's refinements and elaborations, in Kreisel [14]. Featuring large in the Gödel-Kreisel proofs are applications of the axiom of dependent choice, **DC**. Also in play is a form of Herbrand's Theorem, one allowing a reduction of **HPL** derivations for negated prenex formulae to derivations of negations of conjunctions of suitable instances. First, we here show how to deduce Gödel's results by alternative means, ones arguably more elementary than those of Kreisel [14]. We avoid **DC** and Herbrand's Theorem by marshalling simple facts about negative translations and Markov's Rule. Second, the theorems of Gödel and Kreisel are commonly interpreted as demonstrating the unprovability of completeness for **HPL**, if means of proof are confined within strictly intuitionistic metamathematics. In the closing section, we assay some doubts about such interpretations.

1 Markov's Principle and the Gödel-Kreisel Theorems In intuitionistic and constructivistic circles, '**MP**' or 'Markov's Principle' stands for a variety of attempts to shoehorn an idea into one or another formalism. The idea, in brief, is that extensions of semidecidable predicates are fixed under double negation. Here follow three attempts to formalize this idea.

1. **Definition:** *Markov's Principle* **MP** is the schema

$$\neg\neg\exists n M(n) \rightarrow \exists n M(n)$$

   for primitive recursive numerical predicates $M(n)$.

2. **Definition:** *Parametrized Markov's Principle* $\mathbf{MP}(\alpha)$ is the schema

$$\forall\alpha\neg\neg\exists n M(\alpha, n) \rightarrow \forall\alpha\exists n M(\alpha, n)$$

   for predicates $M(\alpha, n)$ primitive recursive in α, where α is a binary-valued function of natural numbers.

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3. **Definition:** *Weak Parametrized Markov's Principle* **WMP**(α) is

$$\forall\alpha\neg\neg\exists n M(\alpha, n) \rightarrow \neg\neg\forall\alpha\exists n M(\alpha, n)$$

for $M(\alpha, n)$ primitive recursive in α, where α is once again a binary-valued function.

The second and third renditions make room for the prospect that M be primitive recursive not only with respect to such basis functions as successor and identity but also with respect to an arbitrary intuitionistic function (perhaps a choice sequence) α.

One should be aware that there is far more variety to formulations of the idea behind **MP** than is captured in these three schemata. A survey of variant renditions of **MP** and of their behaviors with respect to Heyting's arithmetic appears in Smorynski [18]. For a study of equivalents of **MP** within a language for intuitionistic set theory—or within second-order Heyting arithmetic—see McCarty [17]. Suffice it to remark here that none of the above formulations of Markov's Principle is derivable in (appropriate natural extensions of) the formal arithmetic **HA**. Either Kreisel [13] or Troelstra [19] is a useful reference on that fact.

2 Strengths of completeness On moving from classical into intuitionistic mathematics, we find that even (deceptively) rudimentary notions such as inequality and finiteness splinter into a range of distinguishable subnotions. The situation is little different in intuitionistic metamathematics: we must there distinguish between variant forms of completeness for **HPL**, even when it comes to completeness for individual formulae.

1. **Definition:** **HPL** is *strongly complete* whenever $\models \varphi$ implies that $\vdash \varphi$, for all formulae φ.
2. **Definition:** **HPL** is *weakly complete* whenever $\models \varphi$ implies that $\neg\neg \vdash \varphi$, for all formulae φ.

Should we assume **MP**, these two types of completeness coincide extensionally. In the absence of **MP**, they are sharply separable: Kreisel proved weak completeness for the negative fragment of **HPL** constructively in [12]. It follows from the third of the Gödel-Kreisel theorems listed below that no strong completeness result can be had for the same fragment without **MP**.

We can now present what Gödel and Kreisel proved and the latter published. There are three discriminable results:

1. **Theorem 1:** Strong completeness for **HPL** implies **MP**α.
2. **Theorem 2:** Weak completeness for **HPL** implies **WMP**α.
3. **Theorem 3:** Strong completeness for the negative fragment of **HPL** implies **MP**.

The relevant notion of *negative formula* is familiar and is defined in Troelstra and van Dalen [20]. We refer throughout to the Gödel-Kreisel theorems as '**Theorem 1**,' '**Theorem 2**' and '**Theorem 3**,' respectively.

3 Completeness and Markov's Rule What our proof does exploit, in place of Herbrand's Theorem, is the elementary fact that numerous intuitionistic systems, **HA** among them, are closed under the *proof rule* **MR**. This is so even though these systems generally do not derive **MP**.

Definition A formal system S is *closed under Markov's Rule* **MR** provided that, for any primitive recursive $M(x)$, whenever $S \vdash \neg\neg\exists x M(x)$, $S \vdash \exists x M(x)$.

A particularly elegant proof that **HA** and its relatives are closed under **MR** comes from noting that these systems admit the Friedman-Dragalin translation. For details on the translation and relevant proofs, one can consult either Friedman [4] or the comprehensive Troelstra and van Dalen [20]. From these sources it is plain that proofs of closure under **MR** are formulable within primitive recursive arithmetic. Hence, it would seem that our proofs can be carried out within any extension of primitive recursive arithmetic in which suitable notions of satisfaction and validity for formulae and of completeness are formalizable.

4 Proving Theorem 3 Although Kreisel began his exposition of the theorems in [14] by proving Theorem 1 first and then deducing the others as corollaries, since Theorem 3 has such an easy proof—on our way of doing things—we prefer to begin there.

Theorem 3 *Strong completeness for the negative fragment of* **HPL** *implies* **MP**.

Proof: Let $M(n)$ be a primitive recursive predicate and let **Q** be a natural finite set of arithmetic axioms and recursion equations sufficient for the following.

For any n, $\mathbf{N} \models M(n)$ if and only if $\mathbf{Q} \vdash M(n)$.

N is the standard model of arithmetic. We may assume that the axioms of **Q** are drawn exclusively from the negative fragment of the language of **HA**—extended, perhaps, by new symbols for the primitive recursive functions that enter into the definition of M. Since **Q** will be provably sound with respect to **N**,

$\mathbf{N} \models \exists x M(x)$ iff $\mathbf{Q} \vdash \exists x M(x)$.

The axioms of **Q** are all equations or, at worst, implications among and negations of equations. Hence, **Q** will deduce the Friedman-Dragalin translation of any of its axioms and **Q** is, therefore, closed under **MR**.

Now, assume that $\mathbf{N} \models \neg\neg\exists n M(n)$ and that completeness holds for negative formulae in the language of **Q**, among them the **Q** axioms. Since $\mathbf{N} \models \neg\neg\exists n M(n)$, we know from the last line displayed that $\neg\neg(\mathbf{Q} \vdash \exists x M(x))$. Let $\Im$ be any model for **Q**. It follows from the soundness of **HPL** that $\Im \models \neg\neg\exists n M(n)$. Since $\Im$ is an arbitrary model of **Q**,

$\mathbf{Q} \models \neg\neg\exists n M(n)$.

Given that **HPL** is complete for negative formulae, it follows that $\mathbf{Q} \vdash \neg\neg\exists n M(n)$. Since **Q** is known to be closed under **MR**, we have that

$\mathbf{Q} \vdash \exists n M(n)$.

Finally, the soundness theorem reënters to yield

$\mathbf{N} \models \exists n M(n)$.

Hence, **MP** follows from completeness for negative formulae in the language of **HA**—extended, perhaps, with symbols for primitive recursive functions.

5 On pure predicate logic One could, with suitable circumspection and without invoking **DC**, reformulate the foregoing to apply to the negative fragment of pure predicate logic. For that, we need not assume that **HPL** is complete for the negative fragment of the language of **Q**. We require only that **HPL** be complete for a simple language with a finite number of predicate signs—and without function signs or numerals.

First, arrange matters so that **Q** supports Gödel's negative translation: make it so that, if φ^g is the Gödel translation of φ, $\mathbf{Q} \vdash \mathbf{Q}^g$. Second and without loss of generality, we can assume that $\exists n M(n)$ is, in fact, the 'halting statement' $\exists n T(e, e, n)$. We know from standard proofs of the undecidability of pure predicate logic, such as those appearing in Boolos and Jeffrey [1] on pp. 122ff or in Cutland [2] on pp. 109ff, that there are pure predicate formulae **S** and $\exists x H(x)$ which represent rudimentary machine behaviors in predicate logic. **S** describes the initial condition for a computation and the instructions composing machine e's program while $\exists x H(x)$ asserts that the computation of $\{e\}(e)$ eventually halts. For these formulae, it is easy to see that

$$\mathbf{N} \models \exists n M(n) \text{ iff } \mathbf{Q} \vdash \exists n M(n) \text{ iff } \mathbf{S} \vdash \exists x H(x).$$

It is also plain that

$$\mathbf{S} \vdash_c \exists x H(x) \text{ iff } \mathbf{Q} \vdash_c \exists n M(n),$$

where $\vdash_c$ refers to derivability in classical predicate logic. These biconditionals are provable, even intuitionistically, without the use of **DC**.

To prove our result from the completeness of **HPL** for pure redicate logic, instead of completeness for an arithmetical language, it will suffice to show how to use the former completeness property to go from the assumption that

$$\mathbf{Q} \models \neg\neg\exists n M(n).$$

to the conclusion that $\mathbf{Q} \vdash \neg\neg\exists n M(n)$. For completeness played a role solely at this juncture in the argument. To that end, we assume that

$$\mathbf{Q} \models \neg\neg\exists n M(n).$$

It follows that $\mathbf{N} \models \neg\neg\exists n M(n)$. This, in turn, implies that

$$\neg\neg(\mathbf{S} \vdash \exists x H(x)\ ).$$

Thanks to the negative translation—applied now to the language of pure predicate logic—we have that

$$\neg\neg(\mathbf{S}^g \vdash \neg\neg\exists x H(x)^g\ ).$$

As before, we can then show that it follows from the soundness of **HPL** that

$$\mathbf{S}^g \models \neg\neg\exists x H(x)^g.$$

$\mathbf{S}^g$ is (or is equivalent to) a negative formula of the language of pure predicate logic, as is $\neg\neg\exists x H(x)^g$. We now assume that **HPL** is complete for single formulae in the negative fragment of this language. It then follows that

$$\mathbf{S}^g \vdash \neg\neg\exists x H(x)^g.$$

It is a consequence of the negative translation theorem that

$$\mathbf{S} \vdash_c \exists x H(x).$$

From the above, we know that $\mathbf{Q} \vdash_c \exists n M(n)$ and, by the negative translation once again, that

$$\mathbf{Q} \vdash \neg\neg\exists n M(n).$$

Now, we can continue as above but without assuming completeness for the language of **Q** and with the assumption that pure predicate logic is complete for negative formulae in its place.

6 Proving Theorem 1 Similar arguments—with extensions of the axiom system we called **Q**—give parallel proofs for the more general Gödel-Kreisel results, the ones involving parametrized forms $\mathbf{MP}\alpha$ and $\mathbf{WMP}\alpha$.

Theorem 1 Strong completeness for **HPL** implies $\mathbf{MP}\alpha$.

Proof Assume that **HPL** is strongly complete for single formulae and that the antecedent of $\mathbf{MP}\alpha$ holds. Let $M(\alpha, n)$ be a primitive recursive predicate of functions $\alpha : (\mathbf{N} \Rightarrow \mathbf{2})$ and natural numbers n. For the present, fix a function α. As before, let **Q** be a natural finite set of axioms and recursion equations sufficient for the following to hold for all n:

$$\mathbf{N} \models M(\alpha, n) \text{ if and only if } \mathbf{Q}+\mathbf{A}_\alpha \vdash M(\alpha, n).$$

Here, **N** is the standard model and $\mathbf{A}_\alpha$ is the set of all formal equations of the form $\alpha(n) = m$ such that function α does indeed output m on input n. We assume that a function symbol—autonymously known as 'α'—is appended to the usual language for arithmetic. As above, such axiom systems as **Q** or $\mathbf{Q}+\mathbf{A}_\alpha$ are closed under **MR**, since the axioms of $\mathbf{Q}+\mathbf{A}_\alpha$ plainly deduce their own Friedman-Dragalin translations.

From the assumption that $\mathbf{N} \models \forall\alpha\neg\neg\exists n M(\alpha, n)$ it follows, for our function α, that

$$\neg\neg(\ \mathbf{Q}+\mathbf{A}_\alpha \vdash \exists x M(\alpha, x) \ ).$$

Just as in the earlier argument, we can show that

$$\mathbf{Q}+\mathbf{A}_\alpha \models \neg\neg\exists x M(\alpha, x).$$

Now, any model for **Q** can be expanded to a model of **Q** plus $\mathbf{A}_\alpha$. Therefore, $\mathbf{Q} \models \neg\neg\exists x M(\alpha, x)$.

From this point on, the argument proceeds as before to the conclusion that $\forall\alpha\exists n M(\alpha, n)$ is true.

Corollary Theorem 2.

Proof: Immediate.

Note We could have exercised here the circumspection necessary to rely solely upon completeness for pure predicate logic. We would reason much as before, but, this time, we busy ourselves with formulae **S** and $\exists x H(x)$ describing the initial and halting behavior of a machine armed with an oracle for α.

7 Who's afraid of Markov's Principle? Ought we to conclude from the theorems of Gödel and Kreisel that a purely intuitionistic proof of the completeness of **HPL** will always lay beyond our grasp? One should answer "Yes" if there are convincing arguments that **MP** is not, strictly speaking, intuitionistically correct. Yet these arguments—at least such as are in popular circulation—are not wholly convincing. It has been argued that **MP** stands in permanent conflict with the *meanings* of the intuitionistic logical signs, while remaining in perfect congress with the classical construal of the same signs. Naturally, appeal to this kind of conflict and congress has force only if we agree (1) that the logical signs of the intuitionist differ in meaning from those of the classical mathematician and (2) that from the meanings of the intuitionistic signs the failure of **MP** is explicable. But it is nowise plain that we ought to agree quite so much. The first set of remarks below concern claim (1) while the second take up (2), concluding with a brief examination of an argument of Professor Dummett's.

8 Is the disagreement a matter of meaning? Doubtless, there are profound mathematical differences separating the intuitionists from their classical cousins. The greater the weight of those differences, however, the less willing one should be to rest them all upon a slender support, that of an as-yet-undeveloped idea for an 'intuitionistic semantics' of the logical signs. It may well be that our best response to the realities of intuitionistic mathematics is not to call upon a presumptive semantical divergence between the sayings of intuitionistic mathematicians and those of their classical colleagues. It appears more appropriate—both historically and philosophically—to allow that intuitionists chose first to differ in their mathematics and, only at some later date, found the leisure to mull over a precise semantics which might capture a portion of that choice. And, when that 'mulling over' took place, it occurred not under the guiding star of meaning and of metaphysics, but of mathematics.

If we look to the meanings of logical signs to explain and underwrite the disagreements between classical and intuitionistic mathematics, there remain problems about the very understanding of what intuitionists are up to. On the assumption of a divergence in meaning, it is difficult to interpret even such familiar historical claims as "Brouwer believed himself to have proved the failure of the law of the excluded third." For, if the intuitionist's assertions differ in meaning from assertions classically expressed, it is likely that what the intuitionist refuses in denying the universal validity of $\varphi \vee \neg\varphi$ is not what the classicist asserts by demanding it. Worse, perhaps even the univocity of 'acceptance' and 'refusal' is questionable. So what, precisely, ought we to take Brouwer as having (quasi)refused? Perhaps this worry was part of the message Kreisel meant to convey in his remark about Hilbert from footnote 3 of Kreisel [15]:

> Considering that the intended meaning of the intuitionistic disjunction is different from that of classical disjunction, the rejection of *tertium non datur* is much more like depriving non-commutative algebra of the rule $ab = ba$ than a boxer of the use of his fists.

Besides, even if talk of meanings were the best way to clarify what remains unclear about intuitionism, it nowise follows that the contested meanings can, even in such seemingly simple cases as arithmetic, be reduced to a disagreement over the meanings of the *logical signs*. Were intuitionism's disagreements with classical mathematics semantical at bottom, it would not follow that every principle of intuitionistic arithmetic was liable to be called up to judgment before the court of pure logic. (It is worth noting in this connection that we do not pass judgments on the adequacy of principles of classical mathematics on the basis of their congruence with the meanings of the classical logical signs. Why ought intuitionistic mathematics to be any different?) Indeed, talk of reducing the intuitionist's disagreement to one over meanings stands at odds with a *basic* idea of intuitionism: that the certification for any piece of mathematics should be neither a bit of logic nor a bit of metaphysics. Yet philosophers' talk of alternative meanings is plainly a matter of both.

9 Is MP consistent with Heyting's interpretation? One who insists that reflection upon the intuitionistic construals of the logical operators plainly proves that **MP** is constructively unacceptable has some explaining to do about mathematical expression and about the power of those construals. It is well known that **MP** is *Dialectica*-interpretable and, hence, that it is consistent with such theories as $\mathbf{HA}^\omega$—intuitionistic arithmetic in all finite types—plus the Axiom of Choice. **MP** is also demonstrably consistent with the intuitionistic set theory **IZF** plus **DC** and Brouwer's Theorem. Therefore, if there is some aspect of the mere *explanation* of the meanings of the logical operators, as represented, say, in Heyting's interpretation, which is at odds with **MP** then, whatever that aspect is, it would exceed, in its mathematical powers, the reach of all of $\mathbf{HA}^\omega$ and all of intuitionistic set theory. On the other hand, **MP** is known to be inconsistent with theories of lawless sequences and with Brouwer's ideas on the creative subject. Are we then to claim, in refusing **MP** on the basis of meanings alone, that some part of the meanings of the connectives and quantifiers requires the acceptance of such relative arcana as lawless sequences?

Lastly, it is worth pointing out that Dummett's argument against the intuitionistic character of **MP** on the basis of its presumptive incongruence with the meanings of intuitionistic statements (in Dummett [3]) does not seem wholly successful. Dummett considered **MP** in the form

$$\neg\neg\exists n.P(n) \rightarrow \exists n.P(n)$$

where *P(n)* is decidable. He wrote on pages 246 and 247 of [3],

> [T]he intuitionistic statement that $\neg\neg\exists n.P(n)$, or that $\neg\forall n\neg P(n)$, does not express the classical proposition that there exists an n such that $P(n)$, or that it will not happen that we check each n in turn, and find, in every case, that $\neg P(n)$. The intuitionistic statement merely expresses that we shall never be able to prove that $\forall n\neg P(n)$; i.e. that, for however large a number m we may have verified that $\forall n \leq m\neg P(m)$, the possibility will remain open that we may find an $n > m$ for which $P(n)$; and, from this proposition, $\exists n.P(n)$ does not follow, even in its classical sense.

One can interpret this talk of mathematical possibility in terms of double negation. Should we do so, Dummett's last assertion is tantamount to the claim that $\exists n.P(n)$ does not follow (intuitionistically) logically from its double negation, even in the presence of the assumption that *P(n)* is decidable. This is uncontestable, since **MP** is no principle of intuitionistic *logic*. What remains highly contestable is the implicit claim—that seems to be required if we are to conclude from the above that **MP** is constructively inappropriate—that, if **MP** be true at all, it must be true as a matter of logic (given, perhaps, some semantical reflections).

One might agree with Dummett over the status of **MP** in logic (plus, perhaps, semantics) and continue to maintain that **MP** is a mathematically correct statement governing the behavior of the constructions which guarantee the intuitionistic truth of statements such as "$P(n)$ is decidable." Talk of "mathematical possibilities" can equally well be brought to our aid in describing this behavior. If we allow that the antecedent of **MP** asserts the possibility that, for decidable *P(n)*, there exists an *m* such that *P(m)*, then it seems reasonable to insist that this possibility be revealed to me in reality by working the constructive recipe that, for each *n*, decides the truth of *P(n)*. In these terms, **MP** might be construed as telling me that, by working a recipe in virtue of which *P(n)* is decidable to attempt to enumerate the truths $\neg P(0)$, $\neg P(1)$, . . ., I will eventually discover *P(m)* for some *m*. There seems no intuitionistic prohibition against allowing that this is *one sort of mathematical* fact in which the presumed "possibility" that $\exists n.P(n)$ may consist. Professor Dummett has here said little to convince one that this reading of **MP** remains inherently nonintuitionistic.

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