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Natural Deduction Based upon Strict Implication for Normal Modal Logics

CLAUDIO CERRATO

Abstract We present systems of Natural Deduction based on Strict Implication for the main normal modal logics between **K** and **S5**. In this work we consider Strict Implication as the main modal operator, and establish a natural correspondence between Strict Implication and strict subproofs.

1 Introduction In this work we present systems of Natural Deduction based on Strict Implication for normal modal logics. Natural Deduction shows clearly the relationship between deducibility and implication (especially in the Fitch variant, see [5], that allows subproofs), and Strict Implication has a central role in modal logics (see Lewis and Langford [8], and Corsi [4]), so that it seems quite natural to construct systems of Natural Deduction for modal logics by considering Strict Implication as the main modal operator.

Systems of Natural Deduction for modal logics were developed in Prawitz [9] only by adding peculiar rules for the introduction and the elimination of modal operators; but that approach really worked well only for **S4** and **S5** (see Bull and Segeberg [2]). In [5] it was suggested that we also modify the structure of deductions by allowing a modal kind of proof, the strict subordinate proof. In spite of their name, these proofs were related to necessity. But necessity is a unary operator while proofs usually involve both hypotheses and conclusions, so in [5] strict subproofs were forced (really as a technical trick) to never have any hypothesis. However, [5] only developed a system of Natural Deduction for (an Intuitionistic version of) **S4**.

In Fitting [6] Natural Deduction was merely treated as a variant of semantic tableaux. Two different kinds of systems were presented, the A-style and the I-style systems, depending on the interpretation given to strict subproofs. One is led to adopt the former when one "thinks of strict subproofs as an argument about a generic alternate world" (so that strict subproofs are related to necessity), while one is led to adopt

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the latter when one "thinks of strict subproofs as an argument about a particular alternate world" (and so strict subproofs are related to possibility). In any case, all the rules are only suitable modifications of the semantic tableaux rules. So, those systems work well because semantic tableaux work well, but lose the flavor of Natural Deduction.

In this work we consider Strict Implication as the main modal operator, and establish a natural correspondence between binary Strict Implication and strict subproofs (here having both hypotheses and conclusions). Strict Implication represents the direct link of deducibility between the hypothesis and the conclusion of a strict subproof. That link is strict because it cannot be influenced (or it can be influenced only in a ruled way) by formulas external to the subproof. We develop systems of Natural Deduction for the fifteen main normal modal logics between **K** and **S5**.

Finally, Hilbert style systems based both on Strict Implication and on Strict Negation were presented in [4], where the equivalence of those systems with many usual normal modal logics was also proved. The correspondence between our modal rules of Natural Deduction and the modal axioms presented in [4] is, in most cases, quite natural and immediate.

2 Natural Deduction for Strict Implication We use the variant of the Natural Deduction presented in [5], which allows us to construct within proofs further subordinate proofs, named *subproofs*. As in [5], we graphically represent a proof as a vertical sequence of items (formulas or subproofs), and explode a subproof into a vertical sequence of items parallel to the parent proof.

A subproof is to be conceived as an *item* of the parent proof and can have a consequence (by applying some specific rule) in its parent proof. The introduction of a subproof is an effect of a rule, namely of the *introduction of hypothesis*, and the first item of a subproof is called an *hypothesis*. Finally, the vertical sequence of items in a subproof stresses the deducibility relationship between the hypothesis and the other items.

The use of subordinate proofs is unessential for the Propositional Calculus (essentially because we can reiterate every formula into any subproof), but offers help when treating modal logics. In fact, modal behaviors are usually characterized by a peculiar kind of subproofs, the *strict* ones (see [5] and [6]), that we adopt and relate to Strict Implication (instead of to necessity, as usual).

Our language is $\mathbf{L}_{\Rightarrow} = \{P, \wedge, \vee, \neg, \rightarrow, \Rightarrow\}$, where $\Rightarrow$ denotes Strict Implication. We define the other operators as follows:

equivalence	$A \leftrightarrow B$	$\underset{\text{def}}{=}$	$(A \rightarrow B) \wedge (B \rightarrow A)$
strict equivalence	$A \Leftrightarrow B$	$\underset{\text{def}}{=}$	$(A \Rightarrow B) \wedge (B \Rightarrow A)$
necessity	$\Box A$	$\underset{\text{def}}{=}$	$\neg A \Rightarrow A$
possibility	$\Diamond A$	$\underset{\text{def}}{=}$	$\neg\Box\neg A$
strict negation	$\sim A$	$\underset{\text{def}}{=}$	$\Box\neg A$

First we show which rules we assume for the Propositional Calculus (PC-rules) (as to notation, we call the usual subproofs and the usual hypotheses *material*):

Introduction of (Material) Hypothesis (hyp):

$$\begin{array}{lll} \vdots & & \\ \mid \underline{A} & & \text{hyp} \\ \mid \;\vdots & & \end{array}$$

Repetition (rep):

$$\begin{array}{ll} A & \\ \vdots & \\ A & \text{rep} \end{array}$$

Reiteration (reit):

$$\begin{array}{ll} A & \\ \mid \underline{B} & \\ \mid \;\vdots & \\ \mid \;A & \text{reit} \end{array}$$

Introduction of Material Implication ($\rightarrow$ I):

$$\begin{array}{ll} \mid \underline{\qquad A} & \\ \mid \qquad \vdots & \\ \mid \qquad B & \\ A \rightarrow B & \rightarrow\text{I} \end{array}$$

Elimination of Material Implication ($\rightarrow$ E):

$$\begin{array}{cl} A & \\ A \rightarrow B & \\ B & \rightarrow\text{E} \end{array}$$

Introduction of Conjunction ($\wedge$ I):

$$\begin{array}{cl} A & \\ B & \\ A \wedge B & \wedge\text{I} \end{array}$$

Elimination of Conjunction ($\wedge$ E):

$$\begin{array}{clccl} A \wedge B & & \qquad & A \wedge B & \\ A & \wedge\text{E} & \qquad & B & \wedge\text{E} \end{array}$$

Introduction of Disjunctionn ($\vee$ I)

$$\begin{array}{clccl} A & & \qquad & B & \\ A \vee B & \vee\text{I} & \qquad & A \vee B & \vee\text{I} \end{array}$$

Elimination of Disjunction ($\vee$E):

$$\begin{array}{l|l l}
A \vee B & & \\
 & A & \\
 & \vdots & \\
 & C & \\
 & B & \\
 & \vdots & \\
 & C & \\
C & & \vee\mathrm{E}
\end{array}$$

Introduction of Negation ($\neg$I):

$$\begin{array}{l|l l}
 & A & \\
 & \vdots & \\
 & \neg A & \\
\neg A & & \neg\mathrm{I}
\end{array}$$

Elimination of Double Negation ($\neg\neg$ E):

$$\begin{array}{l l}
\vdots & \\
\neg\neg A & \\
A & \neg\neg\mathrm{E}
\end{array}$$

Contraposition (contrap):

$$\begin{array}{l|l l}
\vdots & & \\
B & & \\
 & A & \\
 & \vdots & \\
 & \neg B & \\
\neg A & & \text{contrap}
\end{array}$$

These are the same rules used in [5], with the exception of those for negation (since the negation in [5] is not the classical one) which are a suitable modification of the rules for negation presented (in a different context) in Anderson and Belnap [1]. So first, we must convince ourselves that such rules are PC-rules:

Lemma 2.1 *The above system of Natural Deduction is a system for PC.*

Proof: We compare our system with the system for PC presented in Sundholm [10], that differs from ours only for the negation rules (besides our use of subproofs). So, we need only prove that our rules for negation are equivalent to the following ones (see [10]):

Elimination of Double Negation ([10]$\neg$E$_C$):

$$\begin{array}{l l}
\vdots & \\
\neg\neg A & \\
A & [10]\neg\mathrm{E}_C
\end{array}$$

Elimination of Negation ([10]¬E):

$$\begin{array}{ll} \vdots & \\ A & \\ \neg A & \\ B & [10]\neg E \end{array}$$

Introduction of Negation ([10]¬ I):

$$\begin{array}{lll} \vdots & & \\ & \mid\underline{\quad} A & \\ & \mid \quad \vdots & \\ & \mid \quad B & \\ & \mid\underline{\quad} A & \\ & \mid \quad \vdots & \\ & \mid \quad \neg B & \\ \neg A & & [10]\neg\mathrm{I} \end{array}$$

First, the rules of elimination of double negation, ¬¬ E and [10]¬E$_C$, are the same in both systems; furthermore, by contrap and ¬ I we can prove [10]¬ E and [10]¬ I:

(1)	A		hyp of [10]¬E
(2)	$\neg A$		hyp of [10]¬E
(3)	$\mid\underline{\quad}$	$\neg B$	hyp
(4)	$\mid$	$\neg A$	reit, 2
(5)	$\neg\neg B$		contrap, 1, 3, 4
(6)	B		¬¬E, conclusion of [10]¬E

(1)	$\mid\underline{\quad}$	A			hyp of [10]¬I
	$\mid$	$\vdots$			
(2)	$\mid$	B			hyp of [10]¬I
(3)	$A \to B$				→ I, 1, 2
(4)	$\mid\underline{\quad}$	A			hyp of [10]¬I
	$\mid$	$\vdots$			
(5)	$\mid$	$\neg B$			hyp of[10]¬I
(6)	$\mid$	$A \to B$			reit, 3
(7)	$\mid$	$\mid\underline{\quad}$	A		hyp
(8)	$\mid$	$\mid$	$A \to B$		reit, 6
(9)	$\mid$	$\mid$	B		→ E, 7, 8
(10)	$\mid$	$\mid$	$\mid\underline{\quad}$	$\neg B$	hyp
(11)	$\mid$	$\mid$	$\mid$	$\neg B$	rep, 10
(12)	$\mid$	$\mid$	$\neg\neg B$		contrap, 9, 10, 11
(13)	$\mid$	$\neg A$			contrap, 5, 7, 12
(14)	$\neg A$				¬I, 4, 13, conclusion of [10]¬I.

Vice versa, by [10]¬E and [10]¬I we can prove ¬I and contrap:

(1)	$\quad \vert\underline{\quad A}$	hyp of ¬I
(2)	$\quad \vert\quad \neg A$	hyp of ¬I
(3)	$\quad \vert\underline{\quad A}$	hyp
(4)	$\quad \vert\quad A$	rep, 3
(5)	$\neg A$	[10]¬I , 1, 2, 3, 4, conclusion of ¬I

(1)	A	hyp of contrap
(2)	$\quad \vert\underline{\quad B}$	hyp of contrap
(3)	$\quad \vert\quad \neg A$	hyp of contrap
(4)	$\quad \vert\quad A$	reit, 1
(5)	$\quad \vert\quad \neg B$	[10]¬E , 3, 4
(6)	$\quad \vert\underline{\quad B}$	hyp
(7)	$\quad \vert\quad B$	rep, 6
(8)	$\neg B$	[10]¬I , 2, 5, 6, 7, conclusion of contrap.

So the lemma is proved.

Now, to treat modalities, we increase the expressive power of systems of Natural Deduction by introducing strict subproofs and the corresponding rules related to Strict Implication.

Strict subproofs are like usual proofs, and the behavior of propositional connectives inside strict subproofs is the same as inside usual proofs. The only novelty is that the communication between a strict subproof and its parent proof (conceived as a metaproof) is subject to restrictions, and in this sense the link of deducibility between hypothesis and conclusion is strict. That link is represented in the parent proof as a Strict Implication (⇒I rule); vice versa a Strict Implication can be imported into a strict subproof as an internal representation of a deducibility link, i.e., as a usual material implication (⇒E rule). Furthermore, allowing free communication between a strict subproof and its parent proof leads to the collapse of modalities, while imposing restrictions on that communication leads to the characterization of different modal systems.

Formally, we allow the use of strict subproofs with the corresponding rule of *introduction of strict hypothesis*, and of two related rules for the introduction and the elimination of Strict Implications:

Introduction of Strict Hypothesis (shyp):

$$\begin{array}{ll} \vdots & \\ \vert\!= A & \quad \text{shyp} \\ \vert\ \ \vdots & \end{array}$$

Introduction of Strict Implication (⇒ I):

$$\begin{array}{ll} \vert\underline{\underline{\qquad A}} & \\ \vert\qquad \vdots & \\ \vert\qquad B & \\ A \Rightarrow B & \qquad \Rightarrow \mathrm{I} \end{array}$$

Elimination of Strict Implication (⇒ E):

$$
\begin{array}{lll}
A \Rightarrow B & & \\
\quad \| \quad C & & \\
\quad | \quad \vdots & & \\
\quad | \quad A \rightarrow B & & \Rightarrow \mathrm{E}
\end{array}
$$

We denote strict a subproof by putting a double horizontal line (that recalls the symbol "⇒") just where a strict hypothesis is introduced. The communication between a strict subproof and its parent proof is controlled by specific rules, like ⇒I and ⇒E, so that the usual (material) rules affecting parent proofs (i.e., reit, →I, ¬I, contrap) do not work for strict subproofs.

For example, we do not allow this kind of generic reiteration:

$$
\begin{array}{lll}
A & & \\
\quad \| \; B & & \\
\quad | \; \vdots & & \\
\quad | \; A & & \text{reit;}
\end{array}
$$

otherwise, we shall have:

$$
\begin{array}{llll}
| \quad A & & & \text{hyp} \\
| \quad \quad \| \; \neg A & & & \text{shyp} \\
| \quad \quad | \; A & & & \text{improper reiteration} \\
| \quad \neg A \Rightarrow A & & (= \Box A) & \Rightarrow\mathrm{I} \\
A \rightarrow \Box A & & & \rightarrow\mathrm{I}
\end{array}
$$

and so, $A \rightarrow \Box A$ could be provable: that leads to the collapse of modalities (when we think of "→" as the material implication) or to consider a different kind of implication (see [1]), that is out of our purposes. Since our main modal operator is Strict Implication instead of necessity, necessity needs expressing in terms of Strict Implication; we could choose between:

$$
\begin{array}{lll}
\Box A & = & \neg A \Rightarrow A \text{ and} \\
\Box(A \rightarrow B) & = & A \Rightarrow B;
\end{array}
$$

since we prefer the former as a definition, we must prove the latter as a lemma:

Lemma 2.2 *$A \Rightarrow B$ and $\Box(A \rightarrow B)$ are equivalent, i.e., we can derive each of them from the other.*

Proof: For sake of simplicity we allow to use a tautology inside the proof; that is sound (as proved in Lemmas 3.3 and 3.4), and permits to omit the tedious subordinate proof of the tautology:

$$\begin{array}{lll}
(1) & A \Rightarrow B & \\
(2) & \quad \big| \overline{\overline{\qquad\qquad}}\ \neg(A \to B) & \text{shyp} \\
(3) & \quad \big| \qquad\qquad A \to B & \Rightarrow\text{E}, 1 \\
(4) & \neg(A \to B) \Rightarrow (A \to B) & \Rightarrow\text{I}, 2, 3 \\
(5) & \Box(A \to B) & \Box_{def}, 4.
\end{array}$$

$$\begin{array}{lll}
(1) & \Box(A \to B) & \\
(2) & \neg(A \to B) \Rightarrow (A \to B) & \Box_{def}, 1 \\
(3) & \quad \big| \overline{\overline{\qquad\qquad}}\ A & \text{shyp} \\
(4) & \quad \big| \qquad \neg(A \to B) \to (A \to B) & \Rightarrow\text{E}, 2 \\
(5) & \quad \big| \qquad A \to ((\neg(A \to B) \to (A \to B)) \to B) & \text{tautology} \\
(6) & \quad \big| \qquad (\neg(A \to B) \to (A \to B)) \to B & \to\text{E}, 3, 5 \\
(7) & \quad \big| \qquad B & \to\text{E}, 4, 6 \\
(8) & A \Rightarrow B & \Rightarrow\text{I}, 3, 7.
\end{array}$$

3 The system of Natural Deduction for K We prove the system of Natural Deduction we have introduced is sound and complete for the minimal normal modal logic **K** (see [3]), i.e., there exists a natural deduction **K**-proof of a formula A (as to notation, “A is **K**-provable”) iff there exists a Hilbert style **K**-proof of it (as to notation, “A is a **K**-theorem”).

The language used for our systems of Natural Deduction is not the same as for usual Hilbert style systems: in fact, we adopt $\mathbf{L}_{\Rightarrow}$, which has $\Rightarrow$ (instead of $\Box$) as primitive operator, while Hilbert style systems use $\mathbf{L}_{\Box} = \{P, \wedge, \vee, \neg, \to, \Box\}$, where $A \Rightarrow B \overset{=}{_{\text{def}}} \Box(A \to B)$, $\Diamond = \neg\Box\neg$ and $\sim = \Box\neg$, and the equivalence between $\Box A$ and $\neg A \Rightarrow A$ is provable. Thus we introduce two functions, $^{\Box}$ and $^{\Rightarrow}$, that transform formulas of $\mathbf{L}_{\Rightarrow}$ into formulas of $\mathbf{L}_{\Box}$, and vice versa (for simplicity of notation, we avoid stressing the languages by $_{\Rightarrow}$ and $_{\Box}$, when clear from the context):

$$\begin{array}{lcll}
P^{\Box} & = & P & \text{when P is an atomic formula} \\
(\neg A)^{\Box} & = & \neg A^{\Box} & \\
(A \wedge B)^{\Box} & = & A^{\Box} \wedge B^{\Box} & \\
(A \vee B)^{\Box} & = & A^{\Box} \vee B^{\Box} & \\
(A \to B)^{\Box} & = & A^{\Box} \to B^{\Box} & \\
(A \Rightarrow B)^{\Box} & = & \Box(A \to B)^{\Box} &
\end{array}$$

$$\begin{array}{lcll}
P^{\Rightarrow} & = & P & \text{when P is an atomic formula} \\
(\neg A)^{\Rightarrow} & = & \neg A^{\Rightarrow} & \\
(A \wedge B)^{\Rightarrow} & = & A^{\Rightarrow} \wedge B^{\Rightarrow} & \\
(A \vee B)^{\Rightarrow} & = & A^{\Rightarrow} \vee B^{\Rightarrow} & \\
(A \to B)^{\Rightarrow} & = & A^{\Rightarrow} \to B^{\Rightarrow} & \\
(\Box A)^{\Rightarrow} & = & (\neg A)^{\Rightarrow} \Rightarrow A^{\Rightarrow}. &
\end{array}$$

The last two conditions mean that $(A \Rightarrow_{\Rightarrow} B)^{\Box} = A^{\Box} \Rightarrow_{\Box} B^{\Box}$ and that $(\Box_{\Box} A)^{\Rightarrow} = \Box_{\Rightarrow} A^{\Rightarrow}$, i.e., that the $^{\Box}$-translation of a Strict Implication is a Strict Implication and that the $^{\Rightarrow}$-translation of a necessity is a necessity. However, $^{\Box}$ and $^{\Rightarrow}$ are not one the inverse of the other (since, e.g., $(\Box A)^{\Box\Rightarrow} = (\neg A \Rightarrow A)^{\Box\Rightarrow} = (\Box(\neg A \to A))^{\Rightarrow} = \neg(\neg A \to A) \Rightarrow (\neg A \to A) \neq \Box A$); anyway $A^{\Box\Rightarrow}$ is equivalent to A, and $A^{\Rightarrow\Box}$ is equivalent to A, too, and those equivalences are Natural Deduction and Hilbert style provable, respectively.

So, when proving Soundness and Completeness we really show that "if a $\mathbf{L}_{\Rightarrow}$-formula A is **K**-provable then the corresponding $\mathbf{L}_{\Box}$-formula $A^{\Box}$ is a **K**-theorem" and "if a $\mathbf{L}_{\Box}$-formula A is a **K**-theorem then the corresponding $\mathbf{L}_{\Rightarrow}$-formula $A^{\Rightarrow}$ is **K**-provable." We recall what the system **K** is in the Hilbert style (see [3]):

(1) it is a modal system, i.e., a system closed under the rule of inference RPL: $\dfrac{A_1, A_2, \ldots, A_n}{A}$ where A is a tautological consequence of $A_1, A_2, \ldots, A_n$;
(2) it contains the axiom schema K: $\Box(A \to B) \to (\Box A \to \Box B)$;
(3) it is closed under the rule of necessitation RN: $\dfrac{A}{\Box A}$.

We do not consider the axiom schema Df$\Diamond$: $\Diamond A \leftrightarrow \neg\Box\neg A$ (see [3]), since our language does not contain an explicit possibility symbol; furthermore, Condition 1 implies that **K** contains the usual Propositional Calculus, and it is equivalent to require **K**

(1a) to contain all the tautologies,
(1b) to be closed under the Modus Ponens.

To prove Soundness of our system of Natural Deduction we need a technical lemma:

Lemma 3.1 *The following* $\mathbf{L}_{\Box}$*-sentences are* **K***-theorems:*

(a) $\Box(A \to A)$
(b) $\Box(A \to B) \to \Box(C \to (A \to B))$
(c) $\Box(A \to (B \to C)) \to (\Box(A \to B) \to \Box(A \to C))$
(d) $\Box(A \to B) \to (\Box(A \to C) \to \Box(A \to (B \wedge C))$
(e) $\Box(\neg\neg A \to A)$
(f) $\Box(A \to B) \to (\Box(B \to C) \to \Box(A \to C))$
(g1) $\Box((A \wedge B) \to A)$
(g2) $\Box((A \wedge B) \to B)$
(h1) $\Box(A \to (A \vee B))$
(h2) $\Box(B \to (A \vee B))$
(i) $\Box A \to \Box(B \to A)$
(j) $\Box(A \to B) \to (\Box\neg B \to \Box\neg A)$
(k) $\Box(A \to B) \to (\neg\Box B \to \neg\Box A)$.

Proof: Immediate.

Note that we can prove the above sentences only using tautologies, axiom K, Modus Ponens and applying Necessitation to tautologies; thus, recalling that $A \Rightarrow B$ and $\Box(A \to B)$ are also **S1**-equivalent, all of these proofs are still valid for the Lewis' system **S1** (see [8] and [7]) when directly reading $A \Rightarrow B$ instead of $\Box(A \to B)$. That could be useful when trying to develop systems of Natural Deduction based on Strict Implication for the Lewis' systems completely into $\mathbf{L}_{\Rightarrow}$.

Theorem 3.2 (Soundness) *If a* $\mathbf{L}_{\Rightarrow}$*-formula is* **K***-provable then its* $^{\Box}$*-translation is a* **K***-theorem.*

Proof: We adapt to our case the quasi-proofs of [5] and [1]. A quasi-proof is a proof whose items can also be formulas of kind $\Box^n M$ (where M is a theorem of the modal system **K**—possibly a tautology — $\Box^0 M = M$ and $\Box^{n+1} M = \Box(\Box^n M)$). We gradually transform a Natural Deduction proof into a quasi-proof where the only used rules are $\to$E (that is MP), repetition (that is sound, but not necessary in a Hilbert style proof), and $\Rightarrow_{def}$ (that is sound, too). So, replacing the formulas $\Box^n M$ by their corresponding proofs in the Hilbert style, we obtain the required Hilbert style proof.

As in [1], we start from the innermost subproofs, i.e., proofs without any subordinate proof (really they are quasi-proofs without formulas of kind $\Box^n M$) and transform their parent proofs into innermost quasi-proofs. Step by step we transform all of the proof into a quasi-proof where the only used rules are $\to$E, rep, and $\Rightarrow_{def}$.

We start from a Natural Deduction in $\mathbf{L}_\Rightarrow$ and arrive to a Hilbert style proof in $\mathbf{L}_\Box$, so that we must transform the original $\mathbf{L}_\Rightarrow$-formulas into the corresponding $\mathbf{L}_\Box$-formulas: we can do it either at the first step, for all the formulas together, or step by step, just for the formulas of the transforming parent proof. In any case, we handle only formulas of $\mathbf{L}_\Box$: the translated formulas have the same connectives used in the original ones, while the new formulas contain the necessity operator, $\Box$. We transform "$\Rightarrow$" into "$\Box$", and vice versa, by using $\Rightarrow_{def}$.

We distinguish two cases:

1. the hypothesis of an innermost quasi-proof is material;
2. the hypothesis of an innermost quasi-proof is strict.

Case 1: Let

$$\begin{array}{|l} \underline{A_0} \\ \quad\vdots \\ A_n \end{array}$$

be an innermost quasi-proof with a material hypothesis. We transform the parent proof into a quasi-proof by substituting the "subproof item" with a sequence of items; namely, for each A_i we add a sequence of items, the last one being $A_0 \to A_i$:

(i) if A_i is A_0
then we add $A_0 \to A_0$ tautology;

(ii) if A_i is obtained repeating some previous A_j (and so $A_i = A_j$)
then we have already added $A_0 \to A_i$ $(= A_0 \to A_j)$
so we add now $A_0 \to A_i$ rep;

(iii) if A_i is obtained reiterating an A_i placed in the parent proof

then we have already	(1) A_i	
so we add now	(2) $A_i \to (A_0 \to A_i)$	tautology
	(3) $A_0 \to A_i$	$\to$E, 1, 2;

(iv) if A_i is obtained from some previous $A_k = A_j \to A_i$ and A_j by $\to$E rule

then we have already added	(1) $A_0 \to (A_j \to A_i)$	$(= A_0 \to A_k)$
and	(2) $A_0 \to A_j$	
so we add now	(3) $(A_0 \to (A_j \to A_i)) \to$ $((A_0 \to A_j) \to (A_0 \to A_i))$	tautology
	(4) $(A_0 \to A_j) \to (A_0 \to A_i)$	$\to$E, 1, 3
	(5) $A_0 \to A_i$	$\to$E, 2, 4;

we can also use that proof for the rule $\wedge$I by using the tautology $(A_0 \to B) \to ((A_0 \to C) \to (A_0 \to (B \wedge C)))$;

(v) if A_i is obtained applying the rule $\neg\neg$E to a previous $A_j = \neg\neg A_i$

then we have already added	(1) $A_0 \to \neg\neg A_i$	
so we add now	(2) $(A_0 \to \neg\neg A_i) \to (A_0 \to A_i)$	tautology
	(3) $A_0 \to A_i$	$\to$E, 1, 2;

we can also use that proof for the rules $\wedge$E and $\vee$I by using the tautologies $(A_0 \to (B \wedge C)) \to (A_0 \to B)$, $(A_0 \to (B \wedge C)) \to (A_0 \to C)$, $(A_0 \to B) \to (A_0 \to (B \vee C))$, $(A_0 \to C) \to (A_0 \to (B \vee C))$;

(vi) if A_i is $\Box^n M (n \geq 0)$ (this case is not allowed for the initial innermost subproofs)

then we add	(1) $\Box^n M$	quasi-proof def.
	(2) $\Box^n M \to (A_0 \to \Box^n M)$	tautology
	(3) $A_0 \to \Box^n M$	$\to$E, 1, 2;

(vii) since the hypothesis of the innermost quasi-proof is material, none of its items can be obtained by $\Rightarrow$E; furthermore, since the quasi-proof is an innermost one, none of its items can be consequence of $\to$I, $\vee$E, $\neg$I, $\Rightarrow$I or contrap.

We need some final steps to complete the transformation of the parent proof into a quasi-proof: in fact, in the parent proof there is a formula, C, that was the consequence of the subproof we have just transformed. So now, we must justify that formula in some other way. Namely, just before C, we add a sequence of items that now justify it:

(viii) if C was obtained applying the $\to$I rule (and so $C = A_0 \to A_n$)

then we have just added	(1) $A_0 \to A_n$	
justifying	(2) $A_0 \to A_n$	rep;

(ix) if C was obtained applying the $\vee$E rule (so $C = A_n = B_m$, i.e., it is the last item of two already transformed innermost quasi-proofs having A_0 and B_0 as first item, respectively)

then we have already	(1) $A_0 \vee B_0$	
we have already added	(2) $B_0 \to C$	$(= B_0 \to B_m)$
we have just added	(3) $A_0 \to C$	$(= A_0 \to A_n)$
so we add now	(4) $(A_0 \vee B_0) \to ((B_0 \to C)$ $\to ((A_0 \to C) \to C))$	tautology
justifying	(5) C	$\to$E, 1, 2, 3, 4;

(x) if C was obtained applying the $\neg$I rule (and so $C = \neg A_0 = A_n$)

then we have just added	(1) $A_0 \to \neg A_0$	$(= A_0 \to A_n)$
so we add now	(2) $(A_0 \to \neg A_0) \to \neg A_0$	tautology
justifying	(3) $\neg A_0$	$\to$E, 1, 2;

(xi) if C was obtained by contraposition (and so $C = \neg A_0$ and $A_n = \neg B$, for some previous B in the parent proof)

then we have already	(1) B	
we have just added	(2) $A_0 \to \neg B$	$(= A_0 \to A_n)$
so we add now	(3) $B \to ((A_0 \to \neg B) \to \neg A_0)$	tautology
and	(4) $(A_0 \to \neg B) \to \neg A_0$	$\to$E, 1, 3
justifying	(5) $\neg A_0$	$\to$E, 2, 4;

(xii) finally, since the hypothesis of the innermost quasi-proof was material, C could not be obtained applying the $\Rightarrow$I rule.

Case 2: Let

$$\begin{array}{|l} \models A_0 \\ \quad\vdots \\ A_n \end{array}$$

be an innermost quasi-proof with a strict hypothesis. As in the previous case we transform the parent proof into a quasi-proof by substituting the "subproof item" with a sequence of items; namely, for each A_i we add a sequence of items, the last one being $\Box(A_0 \to A_i)$:

(i) if A_i is A_0

then we add $\Box(A_0 \to A_0)$ Lemma 3.1a;

(ii) if A_i is obtained repeating some previous A_j (and so $A_i = A_j$)

then we have already added	$\Box(A_0 \to A_i)$	$(= \Box(A_0 \to A_j))$
so we add now	$\Box(A_0 \to A_i)$	rep;

(iii) if A_i is obtained applying the rule $\Rightarrow$E to a formula $A \Rightarrow B$ placed in the parent proof (and so $A_i = A \to B$)

then we have already	(1) $A \Rightarrow B$	
so we add now	(2) $\Box(A \to B)$	$\Rightarrow_{def}$, 1
	(3) $\Box(A \to B) \to$ $\Box(A_0 \to (A \to B))$	Lemma 3.1b
	(4) $\Box(A_0 \to (A \to B))$ $(= \Box(A_0 \to A_i))$;	$\to$E, 2, 3

(iv) if A_i is obtained from some previous $A_k = A_j \to A_i$ and A_j by $\to$E rule

then we have already added	(1) $\Box(A_0 \to (A_j \to A_i))$	$(= \Box(A_0 \to A_k))$
and	(2) $\Box(A_0 \to A_j)$	
so we add now	(3) $\Box(A_0 \to (A_j \to A_i))$ $\to (\Box(A_0 \to A_j)$ $\to \Box(A_0 \to A_i))$	Lemma 3.1c
	(4) $\Box(A_0 \to A_j)$ $\to \Box(A_0 \to A_i)$	$\to$E, 1, 3
	(5) $\Box(A_0 \to A_i)$	$\to$E, 2, 4;

(we can also use that proof for the rule $\wedge$I by using Lemma 3.1d);

(v) if A_i is obtained applying the rule $\neg\neg$E. to a previous $A_j = \neg\neg A_i$

then we have already added	(1) $\Box(A_0 \to \neg\neg A_i)$	$(= \Box(A_0 \to A_j))$
so we add now	(2) $\Box(\neg\neg A_i \to A_i)$	Lemma 3.1e
	(3) $\Box(A_0 \to \neg\neg A_i)$ $\to (\Box(\neg\neg A_i \to A_i)$ $\to \Box(A_0 \to A_i))$	Lemma 3.1f
	(4) $\Box(\neg\neg A_i \to A_i) \to$ $\Box(A_0 \to A_i)$	$\to$E, 2, 3
	(5) $\Box(A_0 \to A_i)$	$\to$E, 1, 4;

(we can also use that proof for the rules $\wedge$E and $\vee$I by using Lemmas 3.1g1, 3.1g2, 3.1h1, 3.1h2);

(vi) if A_i is $\Box^n M (n \geq 0)$ (this case is not allowed for the initial innermost subproofs)

then we add	(1) $\Box^{n+1} M$	quasi-proof def.
	(2) $\Box^{n+1} M \to \Box(A_0 \to \Box^n M)$	Lemma 3.1i
	(3) $\Box(A_0 \to \Box^n M)$	$\to$E, 1, 2;

(vii) since the hypothesis of the innermost quasi-proof is strict, none of its items could be obtained by reiteration; furthermore, since the quasi-proof is an innermost one, none of its items could be a consequence of $\to$I, $\vee$E, $\neg$I, $\Rightarrow$I or contrap.

As in Case 1, to complete the transformation of the parent proof into a quasi-proof, we must justify in the parent proof the consequence, C, of the subproof we have transformed; so, just before C, we add a sequence of items that now justify it:

(viii) if C was obtained applying the $\Rightarrow$I rule (and so $C = A_0 \Rightarrow A_n$)

then we have just added	(1) $\Box(A_0 \to A_n)$	
justifying	(2) $A_0 \Rightarrow A_n$	$\Rightarrow_{def}$, 1;

(ix) finally, since the hypothesis of the innermost quasi-proof was strict, C could not be obtained by applying the rules $\to$I, $\vee$E, $\neg$I and contrap.

So, starting from the innermost subproofs and transforming step by step the innermost quasi-proofs in such a way, we transform the Natural Deduction proof into a quasi-proof that uses only the rules $\to$E (i.e., MP), repetition and $\Rightarrow_{def}$ (that are sound). So, this quasi-proof is really a Hilbert style proof that contains unjustified formulas $\Box^n M$. Then, we substitute every formula $\Box^n M$ with the sequence M, $\Box^1 M$, ..., $\Box^{n-1} M$, $\Box^n M$ where each item (except the first one, M) is obtained applying the rule RN to the previous one: so, we have a Hilbert style proof whose only unjustified formulas are **K**-theorems (i.e., M); finally, we substitute all the **K**-theorems with their corresponding Hilbert style **K**-proofs, obtaining a full Hilbert style proof.

This finishes the proof of Theorem 3.2.

Before proving Completeness we need some other technical Lemmas (3.3–3.6) concerned with formulas of $\mathbf{L}_{\Rightarrow}$ (as to notation, hereinafter we simply call "proof" a Natural Deduction proof):

Lemma 3.3 *If A is a tautology then it is* **K**-*provable.*

Proof: Every tautology is PC-provable (by Lemma 2.1) and every PC-proof is a **K**-proof (without strict subproofs).

Lemma 3.4 *If A is* **K***-provable then it can be put as an item at every place inside a proof (but not replacing a hypothesis or a consequence of a subproof).*

Proof: If A is **K**-provable then there exists a **K**-proof P of it, that is a sequence of items (justified by rules) whose first member is a subproof and whose last one is A:

$$\text{P:}\quad \begin{array}{l|ll} & \underline{\ }D & \text{hyp (or shyp)} \\ & \ \vdots & \\ A. & & \end{array}$$

Given a **K**-proof Q, we can add P at every place in any subproof Q′ (with the two exceptions specified in the statement) giving rise to a **K**-proof again. In fact, the new items have no side effect on the old items of Q, and all of the proof P becomes a subproof completely justified by the same rules. Formally, we distinguish two cases:

Case 1: we want to put A before all the items of Q, so

$$\text{the proof Q}\quad \begin{array}{l|l} & \underline{\ }F \\ & \ \vdots \\ G & \end{array} \quad \text{becomes} \quad \begin{array}{l|l} & \underline{\ }D \\ & \ \vdots \\ A & \\ & \underline{\ }F \\ & \ \vdots \\ G & \end{array}$$

Case 2: we want to put A after a formula B of Q, so

$$\text{the proof Q}\quad \begin{array}{c} \vdots \\ B \\ \vdots \end{array} \quad \text{becomes} \quad \begin{array}{l|ll} \vdots & & \\ B & & \\ & \underline{\ }D & \text{beginning of P} \\ & \ \vdots & \\ A & & \text{end of P} \\ B & & \text{rep (to reconnect to Q)} \\ \vdots & & \end{array}$$

and in both cases we obtain a **K**-proof again.

The reason for the two (really unessential) exceptions in the statement is that the role of hypothesis (conclusion) depends on its position in the proof (first item, for hypothesis; just before the subproof, for conclusion). So, when shifting, a hypothesis (conclusion) is no longer justified; moreover, the subproof P cannot assume that role, since a hypothesis (conclusion) is required to be a formula.

Lemma 3.5 *If A is* **K***-provable then $\Box A$ is* **K***-provable, too.*

Proof: Let us consider the following proof of $\Box$A:

$$\begin{array}{l|lll} & \underline{\underline{\qquad\quad}}\neg A & & \text{shyp} \\ & \qquad\quad A & & \text{Lemma 3.4 (}A\text{ is }\mathbf{K}\text{-provable by the hypothesis)} \\ \neg A \Rightarrow A & & (\underset{\text{def}}{=} \Box A) & \Rightarrow\text{I.} \end{array}$$

Lemma 3.6 *The axiom schema* **K**, $\Box(A \to B) \to (\Box A \to \Box B)$, *is* **K***-provable.*

Proof:

(1)	$\square(A \to B)$	hyp
(2)	$A \Rightarrow B$	Lemma 2.2
(3)	$\neg A \Rightarrow A \quad =_{\mathrm{def}} \square A$	hyp
(4)	$A \Rightarrow B$	reit, 2
(5)	$\neg B$	shyp
(6)	$A \to B$	$\Rightarrow$E, 4
(7)	$\neg A \to A$	$\Rightarrow$E, 3
(8)	$(\neg A \to A) \to A$	tautology, Lemmas 3.4 and 3.5
(9)	A	$\to$E, 7, 8
(10)	B	$\to$E, 6, 9
(11)	$\neg B \Rightarrow B \quad =_{\mathrm{def}} \square B$	$\Rightarrow$I, 5, 10
(12)	$\square A \to \square B$	$\to$ I, 3, 11
(13)	$\square(A \to B) \to (\square A \to \square B)$	$\to$ I, 1, 12.

Now we can prove the Completeness Theorem for **K**:

Theorem 3.7 (Completeness) *If an* $\mathbf{L}_{\square}$*-formula is a* **K***-theorem, then its* $^{\Rightarrow}$*-translation is* **K***-provable.*

Proof: We prove the thesis by induction on the complexity of Hilbert style proofs:

(1a) the $^{\Rightarrow}$-translation of a tautology is still a tautology in $\mathbf{L}_{\Rightarrow}$, so that is **K**-provable, by Lemma 3.4;

(1b) the system of Natural Deduction is closed under the $^{\Rightarrow}$-translation of Modus Ponens: in fact, if both A and $A \to B$ are **K**-theorems, then by the inductive hypothesis, both $A^{\Rightarrow}$ and $(A \to B)^{\Rightarrow} = A^{\Rightarrow} \to B^{\Rightarrow}$ are **K**-provable. So the following proof:

$A^{\Rightarrow} \to B^{\Rightarrow}$	Lemma 3.5
$A^{\Rightarrow}$	Lemma 3.5
$B^{\Rightarrow}$	$\to$E

is a **K**-proof of $B^{\Rightarrow}$, i.e., the thesis;

(2) the $^{\Rightarrow}$-translation of the axiom schema K (i.e., $\square_{\Rightarrow}(A^{\Rightarrow} \to B^{\Rightarrow}) \to (\square_{\Rightarrow} A^{\Rightarrow} \to \square_{\Rightarrow} B^{\Rightarrow})$) is **K**-provable, by Lemma 3.7;

(3) the system of Natural Deduction is closed under the $^{\Rightarrow}$-translation of the rule of Necessitation, RN, (i.e., $\dfrac{A^{\Rightarrow}}{\square_{\Rightarrow} A^{\Rightarrow})}$), by Lemma 3.6.

4 *Categorical strict subproofs and system T* In the previous section we considered only hypothetical strict subproofs, i.e., those having a nonempty hypothesis. Now, we also allow categorical strict subproofs, that have empty strict hypotheses. Actually, the following Natural Deduction proof

$$\begin{array}{l|ll} & \models & \text{empty shyp} \\ & \vdots & \\ & A & \\ & \models A & \text{shyp} \\ & \vdots & \\ & B & \\ B & & \text{conclusion,} \end{array}$$

should represent the Modus Ponens for strict implication:

$$MP\Rightarrow \qquad \frac{A \qquad A \Rightarrow B}{B}.$$

In such an interpretation, the first occurrence of A (in MP$\Rightarrow$) means that there exists a strict proof of A without any hypothesis (i.e., a categorical strict proof). [4] considered the Ax.R.: $A \wedge (A \Rightarrow B) \Rightarrow B$, which directly represent that situation. In this work, we add these new rules:

Introduction of empty strict hypothesis (eshyp):

$$\begin{array}{l|ll} \vdots & & \\ & \models & \text{eshyp} \\ & \vdots & \end{array}$$

Introduction of empty strict implication (e$\Rightarrow$I):

$$\begin{array}{l|ll} & \models & \\ & \vdots & \\ & A & \\ A & & \text{e}\Rightarrow\text{I} \end{array}$$

Elimination of empty strict implication (e$\Rightarrow$E):

$$\begin{array}{l|ll} A & & \\ & \models & \\ & \vdots & \\ & A & \text{e}\Rightarrow\text{E} \end{array}$$

We remark that the rules e$\Rightarrow$I and e$\Rightarrow$E are the empty-strict version of the rules $\Rightarrow$I and $\Rightarrow$E, respectively (that can be graphically seen just reading "$\Rightarrow$A"—that must not be confused with $\top \Rightarrow A$—instead of A in the parent proof). Note that rule e$\Rightarrow$E allows free reiterations into categorical strict subproofs. Finally, the rule $\Rightarrow$E still works for categorical strict subproofs; now, in the case of an empty strict hypothesis it appears as:

Elimination of Strict Implication ($\Rightarrow$E):

$$\begin{array}{l} A \Rightarrow B \\ \left| \begin{array}{l} \hline\hline \\ \qquad \vdots \\ \qquad A \to B \qquad \Rightarrow\text{E} \end{array}\right. \end{array}$$

We prove the system of Natural Deduction we have introduced is sound and complete for the normal modal logic **T** (= **K**T, see Chellas [3]), i.e., a formula A is **T**-provable iff it is a **T**-theorem. We recall **T** is **K** plus the axiom schema

$$\mathrm{T} : \Box A \to A.$$

Theorem 4.1 (Soundness) *If a* $\mathbf{L}_{\Rightarrow}$*-formula is* **T***-provable then its* $^{\Box}$*-translation is a* **T***-theorem.*

Proof: We use the same proof used for **K** with suitable modifications to also consider categorical strict subproofs. Namely, we add the new case of an innermost quasi-proof with an empty strict hypothesis (note that since we are proving Soundness of the system of Natural Deduction for **T**, we can use **T**-theorems into the quasi-proofs):

Case 3: Let

$$\left| \begin{array}{l} \hline\hline \\ A_1 \\ \vdots \\ A_n \end{array}\right.$$

be an innermost quasi-proof with an empty strict hypothesis. As in the other cases, we transform the parent proof into a quasi-proof by substituting the "subproof item" with a sequence of items; namely, for each A_i we add a sequence of items, the last one being the same A_i. In most cases we can easily readapt the proofs in Step 2 using tautologies instead of the modal theorems of Lemma 3.1:

(i) A_0 does not exist since the hypothesis is empty;

(ii) if A_i is obtained repeating some previous A_j (and so $A_i = A_j$)

then we have already added	A_j	
so we add now	A_j	rep;

(iiia) if A_i is obtained applying the rule $\Rightarrow$E to a formula $A \Rightarrow B$ placed in the parent proof (and so $A_i = A \to B$)

then we have already	(1) $A \Rightarrow B$	
so we add now	(2) $\Box(A \to B)$	$\Rightarrow_{def}$,1
	(3) $\Box(A \to B) \to (A \to B)$	Axiom T
	(4) $A \to B$	$\to$E, 2, 3;

(iiib) if A_i is obtained applying the rule e$\Rightarrow$E to a formula A placed in the parent proof (and so $A_i = A$)

then we have already	(1) A	
so we add now	(2) A	rep;

(iv) if A_i is obtained from some previous $A_k = A_j \to A_i$ and A_j by $\to$E rule

then we have already added	(1) $A_j \to A_i$	$(= A_k)$
and	(2) A_j	
so we add now	(3) A_i	$\to$E, 1, 2;

(that also holds for the rule $\wedge$I, by suitable uses of tautologies),

(v) if A_i is obtained applying the rule $\neg\neg$E. to a previous $A_j = \neg\neg A_i$

then we have already added	(1) $\neg\neg A_i$	$(= A_j)$
so we add now	(2) $\neg\neg A_i \to A_i$	tautology
	(3) A_i	$\to$E, 1, 2;

(that also holds for the rules $\wedge$E and $\vee$I, by suitable uses of tautologies),

(vi) if A_i is $\Box^n M (n \geq 0)$ (this case is not allowed for the initial innermost subproofs)
then we add $\Box^n M$ quasi-proof def;

(vii) since the hypothesis of the innermost quasi-proof is strict, we cannot have any reiteration. Furthermore, since the quasi-proof is an innermost one, none of its items can be obtained by $\to$I, $\vee$E, $\neg$I, $\Rightarrow$I, e$\Rightarrow$I or contrap.

As in Cases 1 and 2 of Theorem 3.2, to complete the transformation of the parent proof into a quasi-proof we must justify in the parent proof the consequence, C, of the subproof we have just transformed. So, just before C, we add a sequence of items that now justify it:

(viii) if C was obtained applying the e$\Rightarrow$I rule (and so $C = A_n$)

then we have just added	(1) A_n	
justifying	(2) A_n	rep, 1;

(ix) finally, since the hypothesis of the innermost quasi-proof was strict, C could not be obtained applying the rules $\to$I, $\vee$E, $\neg$I, contrap, and, since the hypothesis of the innermost quasi-proof was empty, C could not be obtained applying the rule $\Rightarrow$I.

So, proceeding as for **K**, we transform the Natural Deduction proof of a $\mathbf{L}_{\Rightarrow}$-formula into an Hilbert style **T**-proof of its $^{\Box}$-translation. This completes the proof of Theorem 4.1.

Theorem 4.2 (Completeness) *If a* $\mathbf{L}_{\Box}$*-formula is a* **T***-theorem then its* $^{\Rightarrow}$*-translation is* **T***-provable.*

Proof: Since the system of Natural Deduction for **T** is really an extension of the system for **K** (and so all the previous lemmas still hold) we need only prove that the $^{\Rightarrow}$-translation of the axiom schema T is provable:

(1)	$\mid$ ——— $\neg A \Rightarrow A$	$=_{\text{def}} \Box A$	hyp
(2)	$\mid \quad \mid$ ═══		eshyp
(3)	$\mid \quad \mid \quad \neg A \to A$		$\Rightarrow$E, 1
(4)	$\mid \quad \mid \quad (\neg A \to A) \to A$		tautology, Lemmas 3.3 and 3.4
(5)	$\mid \quad \mid \quad A$		$\to$E, 3, 4
(6)	$\mid \quad A$		e$\Rightarrow$I, 2, 5
(7)	$(\neg A \Rightarrow A) \to A$	$=_{\text{def}} \Box A \to A$	$\to$ I, 1, 6.

5 K4 and S4 When we introduced strict subproofs, we showed that the reiteration of generic formulas into strict subproofs cannot be allowed, so as to avoid the collapse of modalities. Now, we allow the reiteration of some peculiar modal formulas, the strict implications:

Reiteration of Strict Implication (⇒reit):

$$\begin{array}{lll}
A \Rightarrow B & & \\
\vert\overline{\overline{\qquad}} & C & \\
\vert & \vdots & \\
\vert & A \Rightarrow B & \Rightarrow\text{reit.}
\end{array}$$

That rule is the immediate translation of the [4] Ax.T.: $(A \Rightarrow B) \Rightarrow (C \Rightarrow (A \Rightarrow B))$, and corresponds to the usual axiom schema

$$4 : \Box A \to \Box\Box A.$$

Adding the axiom schema 4 to **K** and to **T** we obtain **K**4 and **S4**, respectively; adding the rule ⇒reit to the systems of Natural Deduction for **K** and for **T** we obtain systems of Natural Deduction for **K**4 and for **S4**, respectively, as we prove in the following theorems.

Theorem 5.1 (Soundness) *If a* $\mathbf{L}_{\Rightarrow}$*-formula is* **K**4*-provable (***S4***-provable) its* $^{\Box}$*-translation is a* **K**4*-theorem (***S4***-theorem).*

Proof: For **K**4, we use the same proof used for **K** with suitable modifications to consider the new rule ⇒reit. Namely, we add a subcase for strict subproofs (now we allow the use of **K**4-theorems):

If A_i is obtained by the rule ⇒reit reiterating a formula $A \Rightarrow B$ placed in the parent proof (and so $A_i = A \Rightarrow B$)

then we already have	(1) $A \Rightarrow B$	
so we add now	(2) $\Box(A \to B)$	$\Rightarrow_{def}$,1
	(3) $\Box(A \to B) \to \Box\Box(A \to B)$	Axiom 4
	(4) $\Box\Box(A \to B)$	$\to$E, 2, 3
	(5) $\Box\Box(A \to B) \to$ $\Box(A_0 \to \Box(A \to B))$	Lemma 3.1i
	(6) $\Box(A_0 \to \Box(A \to B)$	$\to$E, 4, 5
	(7) $\Box(A_0 \to (A \Rightarrow B))$ $(= \Box(A_0 \to A_i))$	6, $\Rightarrow_{def}$, Eq

For **S4**, we use the same proof used for **T**, merely adding the above subcase for hypothetical strict subproofs. For categorical strict subproofs nothing changes, since the reiteration of Strict Implication is a particular case of rule e⇒E.

Theorem 5.2 (Completeness) *The* $^{\Rightarrow}$*-translation of the axiom schema 4 is provable from the rule ⇒reit (in a system of Natural Deduction containing the* **K** *one).*

Proof:

$$\begin{array}{llll}
(1) & \vert\underline{\ \neg A \Rightarrow A\ } & \overset{=}{_{\text{def}}}\Box A & \text{hyp} \\
(2) & \vert\quad \vert\overline{\overline{\qquad}}\neg(\neg A \Rightarrow A) & & \text{shyp} \\
(3) & \vert\quad \vert\qquad \neg A \Rightarrow A & & \Rightarrow\text{reit, } 1 \\
(4) & \vert\ \neg(\neg A \Rightarrow A) \Rightarrow (\neg A \Rightarrow A) & \overset{=}{_{\text{def}}}\Box\Box A & \Rightarrow\text{I, } 2, 3 \\
(5) & \Box A \to \Box\Box A & & \to\text{ I, } 1, 4.
\end{array}$$

6 Reiterating negations of Strict Implications into strict subproofs (5-systems) In this section we allow the reiteration of negations of modal sentences (i.e., negations of strict implications) into strict subproofs. So we consider the following rule:

Reiteration of negation of strict implication ($\neg \Rightarrow$reit):

$$\begin{array}{lll} \neg(A \Rightarrow B) & & \\ \quad\Vert & C & \\ & \vdots & \\ & \neg(A \Rightarrow B) & \neg \Rightarrow \text{reit.} \end{array}$$

That rule corresponds to the axiom schema

$$5 : \Diamond A \to \Box\Diamond A.$$

Adding the axiom schema 5 to **K**, **K**4, **S4** we obtain **K**5, **K**45, **S5** (= **K**T5), respectively. Adding the rule $\neg \Rightarrow$reit to the systems of Natural Deduction for **K**, **K**4, **S4** we obtain systems for **K**5, **K**45, **S5**, respectively, as we prove in the following theorems.

Theorem 6.1 (Soundness) *If a* $\mathbf{L}_{\Rightarrow}$*-formula A is* **K**5*-provable (***K**45*-provable,* **S5***-provable) then its* $^{\Box}$*-translation is a* **K**5*-theorem (***K**45*-theorem,* **S5***-theorem).*

Proof: For **K**5 (**K**45, **S5**), we use the same proof used for K (**K**4, **S4**) with suitable modifications to consider the new rule. Namely, we add a subcase for strict subproofs (now we allow the use into quasi-proofs of **K**5-theorems, **K**45-theorems, **S5**-theorems, respectively):

if A_i is obtained by the rule $\neg \Rightarrow$reit reiterating a formula $\neg(A \Rightarrow B)$ placed in the parent proof (and so $A_i = \neg(A \Rightarrow B)$)

then we already have	(1)	$\neg(A \Rightarrow B)$	
so we add now	(2)	$\neg\Box(A \to B)$	$\Rightarrow_{def}$,1
	(3)	$\Diamond\neg(A \to B) \to \Box\Diamond\neg(A \to B)$	Axiom 5
	(4)	$\neg\Box(A \to B) \to \Box\neg\Box(A \to B)$	$\Diamond_{def}$, 3
	(5)	$\Box\neg\Box(A \to B)$	$\to$E, 2, 4
	(6)	$\Box\neg\Box(A \to B) \to$	
		$\Box(A_0 \to \neg\Box(A \to B))$	Lemma 3.1i
	(7)	$\Box(A_0 \to \neg\Box(A \to B))$	$\to$E, 5, 6
	(8)	$\Box(A_0 \to \neg(A \Rightarrow B))$	7, $\Rightarrow_{def}$, Eq
		$(= \Box(A_0 \to A_i))$	

Theorem 6.2 (Completeness) *The* $^{\Rightarrow}$*-translation of the axiom schema 5 is provable from the rule* $\neg \Rightarrow$*reit (in a system of Natural Deduction containing the* **K** *one).*

Proof:

(1)	$\neg(\neg\neg A \Rightarrow \neg A)$	$\overset{=}{\text{def}}\neg\Box\neg A \overset{=}{\text{def}}\Diamond A$	hyp
(2)	$\quad\Vert\ \neg\neg(\neg\neg A \Rightarrow \neg A)$	$\overset{=}{\text{def}}\neg\Diamond A$	shyp
(3)	$\quad\ \ \neg(\neg\neg A \Rightarrow \neg A)$	$\overset{=}{\text{def}}\Diamond A$	$\neg \Rightarrow$reit, 1
(4)	$\neg\Diamond A \Rightarrow \Diamond A$	$\overset{=}{\text{def}}\Box\Diamond A$	$\Rightarrow$I, 2, 3
(5)	$\Diamond A \to \Box\Diamond A$		$\to$I, 1, 4.

7 Necessity of negation as strict negation (B-systems) Now, we consider the Natural Deduction counterpart of the axiom schema

$$B : A \to \Box\Diamond A.$$

That axiom schema was introduced when considering the strict negation "$\Box\neg$" as an Intuitionistic negation: in fact, when rewriting the Intuitionistic axiom about double negation with "$\Box\neg$" we obtain $A \to (\Box\neg)(\Box\neg)A$, that is the axiom schema B; on the contrary, when rewriting the classical converse axiom about double negation with "$\Box\neg$" we obtain $(\Box\neg)(\Box\neg)A \to A$, that leads to the collapse of modalities ([7]). In our systems of Natural Deduction, we consider $\sim \overset{=}{_{\text{def}}} \Box\neg$ as the negation affected when a strict subproof communicates (in both the directions) with its parent proof, while the usual negation still works into the internal of any subproof. We should consider the strict version of the rules $\neg$E, $\neg$I and contrap; but the first one works only inside the subproofs, so that we do not have to change it; as to $\neg$I, its version for strict subproofs appears as

$$\begin{array}{l|l} & \underline{\underline{\quad}}\ A \\ & \quad\ \vdots \\ & \quad \neg A \\ \sim A & \end{array}$$

(in fact, the internal negation remains "$\neg$", while the external one becomes "$\sim$"); that rule is really equivalent to the rule $\Rightarrow$I, so that we do not consider it. Thus, the only new rule for modal negation is a suitable version of contraposition for strict subproofs:

B contraposition (B-contrap):

$$\begin{array}{l|l} A & \\ & \underline{\underline{\quad}}\ B \\ & \quad\ \vdots \\ & \sim A \\ \sim B & \end{array} \qquad \text{B-contrap,}$$

where "$\sim$" has the role of negation from and to the strict subproof.

That rule is quite different from the [4] Ax.S.: $(A \Rightarrow (B \vee \sim (A \Rightarrow B))$, and corresponds to the axiom schema B.

Adding the axiom schema B to the systems **K**, **K**T (=**T**), **K**4 we obtain the systems **K**B, **K**BT (=**B**), **K**B4, respectively; adding the rule B-contrap to the rules for **K** (**K**T, **K**4) we obtain a system of Natural Deduction for **K**B (**K**BT, **K**B4), as we prove in the following theorems.

Theorem 7.1 (Soundness) *If a* $\mathbf{L}_{\Rightarrow}$*-formula A is* **K***B-provable (***K***BT-provable,* **K***B4-provable) then its* $^{\Box}$*-translation is a* **K***B-theorem (***K***BT-theorem,* **K***B4-theorem).*

Proof: For **K**B (**K**BT, **K**B4), we use the same proof used for **K** (**K**T, **K**4) with suitable modifications to consider the new rule. We recall that the $^{\Box}$-translation of a strict negation is not a strict negation: in fact, $(\sim A)^{\Box} = (\Box\neg A)^{\Box} = \neg\neg A^{\Box} \Rightarrow_{\Box} \neg A^{\Box} = \Box(\neg\neg A^{\Box} \to \neg A^{\Box}) \neq \Box\neg A^{\Box} = \sim A^{\Box}$, but anyway $(\sim A)^{\Box} \leftrightarrow \sim A^{\Box}$, so that we can pass from such a formula to the other by using the "substitution of equivalents," Eq, that is a Hilbert style sound rule in any **K**-system (actually also in **S1**). We add a subcase for strict subproofs (now we allow into quasi-proofs the use of **K**B-theorems,

KBT-theorems, **K**B4-theorems, respectively):

if C was obtained by applying the rule B-contrap (so $C = \neg\neg A_0 \Rightarrow \neg A_0$ and $A_n = \neg\neg A \Rightarrow \neg A$, for some previous A in the parent proof)

then we have already	(1)	A	
we have just added	(2)	$\Box(A_0 \to (\neg\neg A \Rightarrow \neg A))$	$(= \Box(A_0 \to A_n))$
	(3)	$\Box(A_0 \to\ \sim A)$	$\sim_{def}$, Eq, 2
so we add now	(4)	$A \to \Box\Diamond A$	Axiom B
	(5)	$\Box\Diamond A \underset{\text{def}}{=}\ \sim\ \sim A$	$\to$E, 1, 4
	(6)	$\Box(A_0 \to\ \sim A) \to$ $(\sim\ \sim A \to\ \sim A_0)$	Lemma 3.1j, $\sim_{def}$
	(7)	$\sim\ \sim A \to\ \sim A_0$	$\to$E, 3, 6
	(8)	$\sim A_0$	$\to$E, 5, 7
justifying	(9)	$\neg\neg A_0 \Rightarrow \neg A_0$	$\sim_{def}$, $\Rightarrow_{def}$, 8.

For systems containing the axiom schema T we should consider the case in which B-contrap is applied to categorical strict subproofs. Such an extension is unessential, since systems of Natural Deduction with rule B-contrap working only on hypothetical strict subproofs are just proved complete. For categorical strict subproofs, any generic formula X can be a consequence of that rule:

B contraposition (B-contrap):

$$\begin{array}{ll} A\,\big|\!\!= & \\ \;\;\big|\quad\vdots & \\ \;\;\big|\;\sim A & \\ X & \qquad\text{B-contrap} \end{array}$$

in fact, when proving Soundness, we must add only a new subcase for innermost quasi-proofs with empty strict hypothesis, that is:

if X was obtained by applying the rule B-contrap to a categorical strict subproof (and so $A_n = \neg\neg A \Rightarrow \neg A$, for some previous A in the parent proof)

then we have already	(1)	A	
we have just added	(2)	$\neg\neg A \Rightarrow \neg A$	$(= A_n)$
	(3)	$\Box\neg A$	$\sim_{def}$, 2
so we add now	(4)	$\Box\neg A \to \neg A$	Axiom T
	(5)	$\neg A$	$\to$E, 3, 4
	(6)	$A \to (\neg A \to X)$	tautology
	(7)	$\neg A \to X$	$\to$E, 1, 6
justifying	(8)	X	$\to$E, 5, 7.

This finishes the proof of Theorem 7.1.

Theorem 7.2 (Completeness) *The $^{\Rightarrow}$-translation of the axiom schema B is provable from the rule B-contrap (in a system of Natural Deduction containing the* **K** *one).*

Proof:

(1)	A		hyp
(2)	$\sim A$		shyp
(3)	$\sim A$		rep, 2
(4)	$\sim\sim A$	$=_{\text{def}} \Box\Diamond A$	B-contrap
(5)	$A \to \Box\Diamond A$		$\to$ I, 1, 4.

8 The axiom schema D Finally, we complete our work considering the Natural Deduction counterpart of the axiom schema

$$D : \Box A \to \Diamond A.$$

This axiom schema is introduced when considering necessity and possibility as obligation and permission, respectively. So, we add a rule that corresponds to the axiom schema D, but that is really more directly linked to strict negation rather than to Strict Implication. Formally, we introduce another variant of contraposition for strict subproofs:

D contraposition (D-contrap):

$\sim A$
$\quad \neg B$
$\quad \vdots$
$\quad A$
$\neg \sim B$ D-contrap

The rule D-contrap corresponds to the usual axiom schema D ([4] used the Ax.D.: $\sim\ \sim T$). Adding the axiom schema D to the systems **K**, **K**4, **K**5, **K**45 and **K**B we obtain the systems **K**D, **K**D4, **K**D5, **K**D45 and **K**BD, respectively; adding the rule D-contrap to the rules for **K** (**K**4, **K**5, **K**45, **K**B) we obtain a system of Natural Deduction for **K**D (**K**D4, **K**D5, **K**D45, **K**BD), as we prove in the following theorems.

Theorem 8.1 (Soundness) *If a* $\mathbf{L}_{\Rightarrow}$*-formula* A *is* **K***D-provable (***K***D4-provable,* **K***D5-provable,* **K***D45-provable,* **K***BD-provable) then its* $^{\Box}$*-translation is a* **K***D-theorem (***K***D4-theorem,* **K***D5-theorem,* **K***D45-theorem,* **K***BD-theorem).*

Proof: For **K**D (**K**D4, **K**D5, **K**D45, **K**BD), we use the same proof used for **K** (**K**4, **K**5, **K**45, **K**B) with suitable modifications to consider the new rule. Namely, we add a subcase for strict subproofs (now we allow into quasi-proofs the use of **K**D-theorems, **K**D4-theorems, **K**D5-theorems, **K**D45-theorems and **K**BD-theorems, respectively):

if C was obtained by applying the rule D-contrap (so $C = \neg(\neg A_0 \Rightarrow A_0)$)(—with $A_0 = \neg B$—)(and $A_n = A$ for some previous $\neg\neg A \Rightarrow \neg$ A in the parent proof)

then we have already	(1)	$\neg\neg A \Rightarrow \neg A$	
	(2)	$\Box\neg A$	$\Rightarrow_{def}$, 1
we have just added	(3)	$\Box(A_0 \to A)$	$= \Box(A_0 \to A_n)$
so we add now	(4)	$\Box\neg A \to \Diamond\neg A$	Axiom D
	(5)	$\Diamond\neg A \quad = \neg\Box\neg\neg A$	$\to$E, 2, 4
	(6)	$\neg\Box A$	5, tautology, Eq

	(7)	$\Box(A_0 \to A) \to (\neg\Box A \to \neg\Box A_0)$	Lemma 3.1k
	(8)	$\neg\Box A \to \neg\Box A_0$	$\to$E, 3, 7
	(9)	$\neg\Box A_0$	$\to$E, 6, 8
justifying	(10)	$\neg(\neg A_0 \Rightarrow A_0)$	$\Rightarrow_{def}$, Eq, 9.

This finishes the proof.

We remark that the axiom schema D can be easily derived from the axiom schema T (see [3]). So we do not need to consider the rule D-contrap in systems of Natural Deduction for logics that contain the axiom schema T. The version of rule D-contrap for categorical strict subproofs is unessential, as argued in the previous section for the corresponding version of the rule B-contrap.

Theorem 8.2 (Completeness) *The $^{\Rightarrow}$-translation of the axiom schema D is provable from the rule D-contrap (in a system of Natural Deduction containing the* **K** *one).*

Proof:

(1)	$\quad\mid\underline{\quad} \Box A$		hyp
(2)	$\quad\mid\ \Box\neg\neg A$	$= \sim\neg A$	1, **K**-theorem
(3)	$\quad\mid\quad \mid\underline{\underline{\quad\quad}}\neg A$		shyp
(4)	$\quad\mid\quad \mid\quad\ \neg A$		rep, 3
(5)	$\quad\mid\ \neg\Box\neg A$	$= \neg\sim A = \Diamond A$	D-contrap, 2, 3, 4
(6)	$\Box A \to \Diamond A$		$\to$ I, 1, 5

where step 2 condenses the **K**-proof of $\Box\neg\neg A$ from $\Box A$ (see Lemmas 3.3–3.6).

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via di Bravetta 340
I–00164, Roma
Italy