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Another Characterization of Alephs: Decompositions of Hyperspace

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Abstract A theorem of Sierpiński of 1919 characterized the cardinality of the continuum by means of lines in two orthogonal directions in the plane: CH if and only if there is a subset S of the plane such that every horizontal cross-section of S is countable and every vertical cross-section of S is co-countable. A theorem of Sikorski of 1951 characterizes the cardinality of an arbitrary set by means of hyperplanes in orthogonal directions in finite powers of that set. A theorem of Davies of 1962 characterizes the cardinality of the continuum by means of lines in nonorthogonal directions in the plane, which, by another theorem of Davies of 1962, may be generalized to finite-dimensional Euclidean space. The main results of this paper unify these analogous theorems of Sikorski and Davies by characterizing the cardinality of an arbitrary set by means of hyperplanes in nonorthogonal directions in that set.

1 Introduction

In 1919, Sierpiński [6] proved the following theorem.

Definition 1.1 Let $S \subseteq \mathbf{R}^2$ and A be a set. Then

1. A is a *vertical cross-section* of S if and only if $\exists r \in \mathbf{R}$ such that $A = \{(x, y) \in S : x = r\}$, and
2. A is a *horizontal cross-section* of S if and only if $\exists r \in \mathbf{R}$ such that $A = \{(x, y) \in S : y = r\}$.

Theorem 1.2 (Sierpiński's Theorem) *CH if and only if* $\exists S \subseteq \mathbf{R}^2$ *such that*

1. *every horizontal cross-section of* S *is countable, and*
2. *every vertical cross-section of* S *is co-countable, that is, every vertical cross-section of* $\mathbf{R}^2 \setminus S$ *is countable.*

Sierpiński's Theorem 1.2 has subsequently undergone a fascinatingly elaborate development. In 1951 one line of investigation issuing from Sierpiński's Theorem 1.2 culminated in the following theorem of Sikorski [7].

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Notation 1.3 Let X be a set and κ be a cardinal. Then

1. $[X]^\kappa = \{Y \subseteq X : |Y| = \kappa\}$,
2. $[X]^{<\kappa} = \{Y \subseteq X : |Y| < \kappa\}$.

Definition 1.4 Let X be a set; $n, m \in \omega$; $\Lambda \in [n]^m$; and Y be a set. Then Y is a Λ-*set* in ${}^nX$ if and only if $\exists \langle a_i \rangle_{i \in \Lambda} \in {}^\Lambda X$ such that $Y = \{x \in {}^nX : \forall i \in \Lambda, x_i = a_i\}$.

Terminology 1.5 *Let X be a set and C be a collection of sets. Then C* covers *X if and only if $\bigcup C = X$.*

Theorem 1.6 (Sikorski's Theorem) *Let X be a nonempty set; $n \in \omega$ such that $n \geq 2$; m be such that $0 < m < n$; and α be an ordinal. Then $|X| < \aleph_{\alpha+m}$ if and only if there is a sequence $\langle S_\Lambda \rangle_{\Lambda \in [n]^m}$ of sets covering ${}^nX$ such that $\forall \Lambda \in [n]^m$, every Λ-set Y in ${}^nX$ intersects S_Λ in fewer than $\aleph_\alpha$ points.*

Sierpiński's Theorem 1.2 is the corollary of Sikorski's Theorem 1.6 obtained by setting $X = \mathbf{R}$, $n = 2$, $m = 1$, and $\alpha = 1$.

In 1962, another line of investigation issuing from Sierpiński's Theorem 1.2 reached a culmination when Davies proved, in effect, the following theorem [3, 4].

Theorem 1.7 (Davies' Theorem) *Let n be a nonzero element of ω. Then the following are equivalent.*

1. $2^{\aleph_0} \leq \aleph_n$.
2. *$\forall k \in \omega$, if $k \geq 2$, then for every sequence $\langle L_i \rangle_{i \in n+2}$ of lines in $\mathbf{R}^k$, no two of which are parallel, there is a sequence $\langle S_i \rangle_{i \in n+2}$ of sets covering $\mathbf{R}^k$ such that $\forall i \in n+2$, every line in $\mathbf{R}^k$ parallel to L_i intersects S_i in finitely many points.*
3. *There is a sequence $\langle L_i \rangle_{i \in n+2}$ of lines in the plane and a sequence $\langle S_i \rangle_{i \in n+2}$ of sets covering $\mathbf{R}^2$ such that $\forall i \in n+2$, every line in $\mathbf{R}^2$ parallel to L_i intersects S_i in finitely many points.*

In 1985, in [5], Freiling formed natural intuitions about randomly selecting real numbers into an attractive philosophical argument for a number of set-theoretic principles, one of the weaker of which, $\mathrm{A}_{\aleph_0}$, he showed to be equivalent to the negation of the continuum hypothesis. It turns out that Freiling's principles are related to Sierpiński's Theorem 1.2 and that many of them can be derived from Sikorski's Theorem 1.6.[1]

The main result of this paper is a generalization of Sierpiński's Theorem 1.2 that embraces both Sikorski's Theorem 1.6 and Davies' Theorem 1.7.[2] The proof of the main result of this paper is based on the proof of Davies in [3].

The following three sections of this paper establish the main result. The fourth section applies the main result to Sikorski's Theorem 1.6 and to Davies' Theorem 1.7, establishes a very general result for finite-dimensional vector spaces, and then takes a look at some of Freiling's principles. The concluding section of this paper presents a succinct recapitulation of Freiling's philosophical argument for $\mathrm{A}_{\aleph_0}$ and indicates how it might lead one to believe that $2^{\aleph_0}$ is weakly inaccessible.

2 From decompositions to cardinalities This section shows how to use certain coverings of a set to say something about its cardinality.

Definition 2.1 $\mathcal{H}$ is an $\mathcal{S}$-indexed *hyperspace* on X if and only if $\mathcal{H}$ is an $\mathcal{S}$-indexed sequence $\langle \mathcal{H}_S \rangle_{S \in \mathcal{S}}$ of partitions of X.

A hyperspace may be thought of as a list of hyperplanes in a space, a hyperplane being identified with the set of hyperplanes parallel to it.

Convention 2.2 *Let $\langle \mathcal{H}_S \rangle_{S \in \mathcal{S}}$ be an $\mathcal{S}$-indexed hyperspace on X and $S \in \mathcal{S}$. We confound $\mathcal{H}_S$ with its associated equivalence relation. Thus, in particular,*

1. *$\forall x \in X$, $[x]_{\mathcal{H}_S}$ stands for the unique $H \in \mathcal{H}_S$ such that $x \in H$, and*
2. *$\forall x, y \in X$, $x \; \mathcal{H}_S \; y$ means that $[x]_{\mathcal{H}_S} = [y]_{\mathcal{H}_S}$.*

Definition 2.3 Let $\mathcal{H}$ be an $\mathcal{S}$-indexed hyperspace on X. Then H is an *S-hyperplane* of $\mathcal{H}$ if and only if

1. $S \in \mathcal{S}$, and
2. $H \in \mathcal{H}_S$.

Definition 2.4 Let $\mathcal{H}$ be an $\mathcal{S}$-indexed hyperspace on X; $\mathcal{S}', \mathcal{S}'' \subseteq \mathcal{S}$; and $A \subseteq X$. Then

1. t is an *$\mathcal{S}''$-invariant $\mathcal{S}'$-translation* of A in $\mathcal{H}$ if and only if
   (a) $t : A \overset{1\text{-}1}{\to} X$,
   (b) $\forall x \in A \; \forall S' \in \mathcal{S}'$, $x \; \mathcal{H}_{S'} \; t(x)$, and
   (c) $\forall x, y \in A \; \forall S'' \in \mathcal{S}''$: $x \; \mathcal{H}_{S''} \; y$ if and only if $t(x) \; \mathcal{H}_{S''} \; t(y)$.
2. T is an $\mathcal{S}''$-invariant *$\mathcal{S}'$-translate* of A in $\mathcal{H}$ if and only if there is an $\mathcal{S}''$-invariant $\mathcal{S}'$-translation t of A in $\mathcal{H}$ such that $T = \mathcal{R}(t)$.

Definition 2.5 Let $\mathcal{H}$ be an $\mathcal{S}$-indexed hyperspace on X; $\mathcal{S}', \mathcal{S}'' \subseteq \mathcal{S}$; and κ be a cardinal. Then $\mathcal{S}'$ *κ-translates* over $\mathcal{S}''$ in $\mathcal{H}$ if and only if $\forall A \in [X]^{<\kappa}$, there is an $\mathcal{S}''$-invariant $\mathcal{S}'$-translate T of A in $\mathcal{H}$ such that $T \cap A = \varnothing$.

Proposition 2.6 *Let $\mathcal{H}$ be an $\mathcal{S}$-indexed hyperspace on X; $\mathcal{S}', \mathcal{S}'' \subseteq \mathcal{S}$; κ be an infinite cardinal; and $A \in [X]^{<\kappa}$. Suppose that $\mathcal{S}'$ κ-translates over $\mathcal{S}''$ in $\mathcal{H}$. Then there is a sequence $\langle A_\alpha \rangle_{\alpha \in \kappa}$ of pairwise disjoint $\mathcal{S}''$-invariant $\mathcal{S}'$-translates of A in $\mathcal{H}$.*

Proof: By induction on $\alpha < \kappa$. Suppose that $\alpha < \kappa$ and that $\forall \beta < \alpha$, A_β has been defined. Let $B = A \cup \bigcup_{\beta < \alpha} A_\beta$. Then $|B| = |A| (1 + |\alpha|) < \kappa$. Since $\mathcal{S}'$ κ-translates over $\mathcal{S}''$ in $\mathcal{H}$, there is an $\mathcal{S}''$-invariant $\mathcal{S}'$-translation t of B in $\mathcal{H}$ such that $B \cap t[B] = \varnothing$. Let $A_\alpha = t[A]$. □

Definition 2.7 E is an $\mathcal{S}$-indexed *decomposition* of X if and only if E is an $\mathcal{S}$-indexed sequence $\langle E_S \rangle_{S \in \mathcal{S}}$ such that $X = \bigcup_{S \in \mathcal{S}} E_S$.

Definition 2.8 Let $\mathcal{H}$ be an $\mathcal{S}$-indexed hyperspace $\langle \mathcal{H}_S \rangle_{S \in \mathcal{S}}$ on X, $Y \subseteq X$, $\mathcal{S}' \subseteq \mathcal{S}$, E be an $\mathcal{S}'$-indexed decomposition $\langle E_S \rangle_{S \in \mathcal{S}'}$ of Y, and σ be a cardinal. Then E is *σ-fine* in $\mathcal{H}$ if and only if $\forall S \in \mathcal{S}' \; \forall H \in \mathcal{H}_S$, $|H \cap E_S| < \sigma$.

Notation 2.9 Let σ be a cardinal. Then

1. $\sigma^{+0} = \sigma$, and
2. $\forall n \in \omega$, $\sigma^{+(n+1)} = \left(\sigma^{+n}\right)^{+}$.

Notation 2.10 Let $\langle X_i \rangle_{i \in I}$ be a sequence of sets. Then $\prod_{i \in I} X_i$ is the set of all functions f such that

1. $\mathcal{D}(f) = I$, and
2. $\forall i \in I,\ f(i) \in X_i$.

Theorem 2.11 *Let X be a nonempty set, $\mathcal{H}$ be an $\mathcal{S}$-indexed hyperspace on X, $n \in \omega$, and κ and σ be infinite cardinals. Suppose that $\langle \mathcal{S}_i \rangle_{i<n+1}$ is a sequence of subsets of $\mathcal{S}$ such that*

1. $\bigcup_{i<n+1} \mathcal{S}_i = \mathcal{S}$, and
2. $\forall i < n+1$,
   (a) $\mathcal{S}_i$ κ-translates over $\bigcup_{j<i} \mathcal{S}_j$ in $\mathcal{H}$, and
   (b) $|\mathcal{S}_i| < cf\left(\sigma^{+i}\right)$.

Then, if there is a σ-fine $\mathcal{S}$-indexed decomposition of X in $\mathcal{H}$, it follows that $\kappa < \sigma^{+n}$.

Proof: Assume, on the contrary, that $\langle E_S \rangle_{S \in \mathcal{S}}$ is a σ-fine $\mathcal{S}$-indexed decomposition of X in $\mathcal{H}$, but that $\kappa \geq \sigma^{+n}$. Note that since X is nonempty, $\mathcal{S}$ is nonempty, too. We construct a sequence of sequences $\left\langle \langle x_{\vec{\alpha}} \rangle_{\vec{\alpha} \in \prod_{j<i} \sigma^{+j}} \right\rangle_{i<n+2}$ such that

1. $x_{\langle\rangle} \in X$,
2. $\forall i < n+1\ \forall \beta < \sigma^{+i}$, there is a $\bigcup_{j<i} \mathcal{S}_j$-invariant $\mathcal{S}_i$-translation t of
$$\left\{ x_{\vec{\alpha}} : \vec{\alpha} \in \prod_{j<i} \sigma^{+j} \right\}$$
such that
$$\forall \vec{\alpha} \in \prod_{j<i} \sigma^{+j},\ x_{\vec{\alpha} \frown \langle \beta \rangle} = t\left(x_{\vec{\alpha}}\right), \text{ and}$$
3. $\forall i < n+2$, the $x_{\vec{\alpha}}$ of $\langle x_{\vec{\alpha}} \rangle_{\vec{\alpha} \in \prod_{j<i} \sigma^{+j}}$ are pairwise distinct.

Let $x_{\langle\rangle}$ be any element of X. Let $i < n+1$ and suppose that the $x_{\vec{\alpha}}$ for $\vec{\alpha} \in \prod_{j<i} \sigma^{+j}$ have been defined. Let
$$A = \left\{ x_{\vec{\alpha}} : \vec{\alpha} \in \prod_{j<i} \sigma^{+j} \right\}.$$
Then $|A| < \sigma^{+i} \leq \sigma^{+n} \leq \kappa$. Since $\mathcal{S}_i$ κ-translates over $\bigcup_{j<i} \mathcal{S}_j$ in $\mathcal{H}$, there is a sequence $\langle t_\beta \rangle_{\beta < \sigma^{+i}}$ of $\bigcup_{j<i} \mathcal{S}_j$-invariant $\mathcal{S}_i$-translations of A in $\mathcal{H}$ such that the $\mathcal{R}\left(t_\beta\right)$ are pairwise disjoint. $\forall \beta < \sigma^{+i}\ \forall \vec{\alpha} \in \prod_{j<i} \sigma^{+j}$, let $x_{\vec{\alpha} \frown \langle \beta \rangle} = t_\beta\left(x_{\vec{\alpha}}\right)$. Next we construct a sequence
$$\langle \gamma_i \rangle_{i<n+1} \in \prod_{j<n+1} \sigma^{+j}$$
such that
$$\forall i < n+1\ \forall S \in \mathcal{S}_{n-i} \forall \vec{\alpha} \in \prod_{j<n-i} \sigma^{+j},\ x_{\vec{\alpha} \cup \langle \gamma_j \rangle_{j \in (n+1) \setminus (n-i)}} \notin E_S,$$

by induction on i.

Suppose that $i < n+1$ and that $\forall j \in (n+1) \setminus (n-i+1)$, γ_j has been constructed. Let $\vec{\alpha} \in \prod_{j<n-i} \sigma^{+j}$. Let $S \in \mathcal{S}_{n-i}$. Now,

$$\forall \beta < \sigma^{+(n-i)}, x_{\vec{\alpha}} \, \mathcal{H}_S \, x_{\vec{\alpha} \frown \langle \beta \rangle},$$

so

$$\forall \beta, \beta' < \sigma^{+(n-i)}, x_{\vec{\alpha} \frown \langle \beta \rangle} \, \mathcal{H}_S \, x_{\vec{\alpha} \frown \langle \beta' \rangle},$$

whence

$$x_{\vec{\alpha} \frown \langle \beta \rangle \cup \langle \gamma_j \rangle_{j \in (n+1) \setminus (n-i+1)}} \, \mathcal{H}_S \, x_{\vec{\alpha} \frown \langle \beta' \rangle \cup \langle \gamma_j \rangle_{j \in (n+1) \setminus (n-i+1)}}.$$

Let

$$A_{\vec{\alpha}} = \left\{ x_{\vec{\alpha} \frown \langle \beta \rangle \cup \langle \gamma_j \rangle_{j \in (n+1) \setminus (n-i+1)}} : \beta < \sigma^{+(n-i)} \right\}.$$

Let $H_{\vec{\alpha}}$ be the S-hyperplane

$$\left[x_{\vec{\alpha} \frown \langle 0 \rangle \cup \langle \gamma_j \rangle_{j \in (n+1) \setminus (n-i+1)}} \right]_{\mathcal{H}_S}$$

in $\mathcal{H}$. Then $A_{\vec{\alpha}} \subseteq H_{\vec{\alpha}}$. By hypothesis,

$$\left| A_{\vec{\alpha}} \cap E_S \right| \leq \left| H_{\vec{\alpha}} \cap E_S \right| < \sigma.$$

Let $A = \bigcup_{\vec{\alpha} \in \prod_{j<n-i} \sigma^{+j}} A_{\vec{\alpha}}$.

Subclaim 2.12 $|A \cap E_S| < \sigma^{+(n-i)}$.

Subproof: To see this, first suppose that $i < n$. Then

$$\sigma \leq \left| \prod_{j<n-i} \sigma^{+j} \right| < \sigma^{+(n-i)},$$

so

$$|A \cap E_S| \leq \left| \prod_{j<n-i} \sigma^{+j} \right| \cdot \sigma < \sigma^{+(n-i)}.$$

Next suppose that $i = n$. Then

$$|A \cap E_S| = \left| A_{\langle \rangle} \cap E_S \right| < \sigma = \sigma^{+(n-i)}.$$

$\square$

Now, by assumption $|\mathcal{S}_{n-i}| < \mathrm{cf}\left(\sigma^{+(n-i)}\right)$. Thus,

$$\left| A \cap \bigcup_{S \in \mathcal{S}_{n-i}} E_S \right| < \sigma^{+(n-i)}.$$

Now, $\forall \beta \in \sigma^{+(n-i)}$, let

$$A_\beta = \left\{ x_{\vec{\alpha} \frown \langle \beta \rangle \cup \langle \gamma_j \rangle_{j \in (n+1) \setminus (n-i+1)}} : \vec{\alpha} \in \prod_{j<n-i} \sigma^{+j} \right\}.$$

By construction, A is the disjoint union of the A_β. Accordingly, there is a $\beta < \sigma^{+(n-i)}$ such that

$$\left| A_\beta \cap \bigcup_{S \in \mathcal{S}_{n-i}} E_S \right| = \varnothing.$$

Let γ_{n-i} be any such β. Accordingly, $\forall S \in \mathcal{S}$, $x_{\bar{\gamma}} \notin E_S$. But $\langle E_S \rangle_{S \in \mathcal{S}}$ is a decomposition of X, so this is a contradiction. $\square$

3 *From cardinalities to decompositions* This section shows how to use the cardinality of a set to say something about its decompositions, giving a general converse to Theorem 2.11.

Definition 3.1 Let $\mathcal{H}$ be an $\mathcal{S}$-indexed hyperspace on X, $n \in \omega$, and τ be a cardinal. Then $\mathcal{H}$ is *τ-fine* to *depth n* if and only if $\forall x \in X$, for every linear ordering $\preceq$ of $\left\{[x]_{\mathcal{H}_S} : S \in \mathcal{S}\right\}$, there is a subset $\mathcal{H}'$ of $\left\{[x]_{\mathcal{H}_S} : S \in \mathcal{S}\right\}$ of size n such that $\forall H' \in \mathcal{H}'$, $\left|\bigcap \left\{H \preceq H' : \neg\exists H'' \in \mathcal{H}' \text{ such that } H \prec H'' \prec H'\right\}\right| \leq \tau$.

Theorem 3.2 *Let $\mathcal{H}$ be an $\mathcal{S}$-indexed hyperspace on X, $n \in \omega$, and τ be a cardinal. Suppose that $\mathcal{S}$ is nonempty and finite and that $\mathcal{H}$ is τ-fine to depth n. Let σ be an infinite cardinal greater than τ. Then if $|X| < \sigma^{+n}$, it follows that there is an $\mathcal{S}$-indexed decomposition of X that is σ-fine in $\mathcal{H}$.*

Proof: Note first that the result is trivial if $n = 0$. Assume, therefore, that $n > 0$. Let us say that H is a *hyperplane* just in case $H \in \bigcup_{S \in \mathcal{S}} \mathcal{H}_S$. Let N be a set of hyperplanes. Let us say that N is a *network* if and only if $\forall \mathcal{S}' \subseteq \mathcal{S} \forall \langle H_S \rangle_{S \in \mathcal{S}'} \in \prod_{S \in \mathcal{S}'} \left(\mathcal{H}_S \cap N\right)$, if $\left|\bigcap_{S \in \mathcal{S}'} H_S\right| \leq \tau$, then $\forall x \in \bigcap_{S \in \mathcal{S}'} H_S \forall S \in \mathcal{S}$, $[x]_{\mathcal{H}_S} \in N$. Clearly, an intersection of networks is a network.

Sublemma 3.3 *Let M be a set of hyperplanes and N be the smallest network such that $N \supseteq M$. Then $|N| \leq \max\{|M|, \sigma\}$.*

Subproof: Define $\langle N_i \rangle_{i \in \omega}$ by induction on i as follows. Let $N_0 = M$. Suppose that N_i has been defined. Then let

$$N_{i+1} = N_i \cup \left\{ H : \exists \mathcal{S}' \subseteq \mathcal{S} \exists \langle H_S \rangle_{S \in \mathcal{S}'} \in \prod_{S \in \mathcal{S}'} \left(\mathcal{H}_S \cap N_i\right) \exists x \in \bigcap_{S \in \mathcal{S}'} H_S \exists S \in \mathcal{S} \right.$$

such that

$$\left. \left| \bigcap_{S \in \mathcal{S}'} H_S \right| \leq \tau \;\&\; H = [x]_{\mathcal{H}_S} \right\}$$

Clearly, $N = \bigcup_{i \in \omega} N_i$. Moreover, $|N_0| \leq \max\{|M|, \sigma\}$, and $\forall i \in \omega$, if $|N_i| \leq \max\{|M|, \sigma\}$, then $\left|N_{i+1}\right| \leq \max\{|M|, \sigma\} \cdot \tau = \max\{|M|, \sigma\}$. Thus, $|N| \leq \omega \cdot \max\{|M|, \sigma\} = \max\{|M|, \sigma\}$. $\square$

Sublemma 3.4 *Let $k \in \omega$ and N be a network such that $|N| \leq \sigma^{+k}$. Then there is a well-ordering $\prec$ of N such that*

($*_k$) *$\forall H \in N$, there are fewer than σ finite sets $\mathcal{H}'$ of hyperplanes such that*

(1_k) *H is the $\prec$-greatest element of $\mathcal{H}'$,*

(2_k) *$\bigcap \mathcal{H}' \neq \varnothing$, and*

(3_k) *$\exists \mathcal{H}'' \in [\mathcal{H}']^{k+1}$ such that $\forall H'' \in \mathcal{H}''$, $\left|\bigcap \left\{H' \in \mathcal{H}' : H' \preceq H'' \;\&\; \neg\exists H''' \in \mathcal{H}'' \text{ such that } H' \prec H''' \prec H''\right\}\right| \leq \tau$.*

Subproof: By induction on k. First assume that $k = 0$. Then $|N| \leq \sigma$. Let $\prec$ be any well-ordering of N. Then below any H there are, of course, fewer than σ finite sets of whatever kind. Next, assume that the sublemma is true for k. We seek to show it true for $k + 1$. To this end, assume that N is a network and that $|N| \leq \sigma^{+(k+1)}$. If $|N| < \sigma^{+(k+1)}$, then the result is trivial by the induction hypothesis, so assume that $|N| = \sigma^{+(k+1)}$. Let $\langle H_\alpha \rangle_{\alpha < \sigma^{+(k+1)}}$ enumerate N. $\forall \alpha \in \sigma^{+(k+1)} \setminus \sigma^{+k}$, let N_α be the smallest network containing $\left\{H_\beta : \beta < \alpha\right\}$. Then, by the preceding sublemma, $\forall \alpha \in \sigma^{+(k+1)} \setminus \sigma^{+k}$, $|N_\alpha| = \sigma^{+k}$. $\forall \alpha \in \sigma^{+(k+1)} \setminus \sigma^{+k}$, let $\prec_\alpha$ be a well-ordering of N_α as guaranteed by the induction hypothesis. Now, $\forall H \in N$, let $\alpha(H)$ be the least α such that $H \in N_\alpha$. Finally, define the well-ordering $\prec$ by $H_1 \prec H_2$ if and only if either

$$\alpha(H_1) < \alpha(H_2)$$

or

$$\alpha(H_1) = \alpha(H_2) \;\&\; H_1 \prec_{\alpha(H_1)} H_2.$$

We need to show $(*_{k+1})$. To this end, let $H \in N$ and suppose that $\mathcal{H}'$ is a finite set of hyperplanes that satisfies (1_{k+1}) – (3_{k+1}). Let $\mathcal{H}''$ be as in (3_{k+1}), so that in particular, $\mathcal{H}''$ is of size $k+2$, and let H'' be the $\prec$-least element of $\mathcal{H}''$.

We show that $\alpha(H'') = \alpha(H)$. For suppose otherwise. Then $\alpha(H'') < \alpha(H)$, and moreover, by (3_{k+1}), $\left|\bigcap\left\{H' \in \mathcal{H}' : H' \preceq H''\right\}\right| \leq \tau$. But then, by (2_{k+1}), $H \in N_{\alpha(H'')}$, whence $\alpha(H) \leq \alpha(H'')$, which contradicts our supposition. Accordingly, $\left\{H' \in \mathcal{H}' : H' \succeq H''\right\}$ satisfies (1_k) – (3_k) with respect to $\prec_{\alpha(H)}$. Now there are by construction of $\prec_{\alpha(H)}$, fewer than σ finite sets $\mathcal{H}^*$ of hyperplanes satisfying (1_k) – (3_k) with respect to $\prec_{\alpha(H)}$. Let $\mathcal{H}^*$ be such a set. $\forall x \in \bigcap H^*$, there are only finitely many supersets $\mathcal{H}^{**}$ of $\mathcal{H}^*$ satisfying (2_k) with respect to $\prec_{\alpha(H)}$ and such that $x \in \bigcap \mathcal{H}^{**}$. Since, by (3_k), $\left|\bigcap \mathcal{H}^*\right| \leq \tau < \sigma$, there are fewer than σ finite sets $\mathcal{H}'$ satisfying (1_{k+1}) – (3_{k+1}) with respect to $\prec_{\alpha(H)}$. □

Now we continue with the proof of the theorem. We have assumed that $n > 0$. Thus, $|X| \leq \sigma^{+(n-1)}$, whence $\left|\bigcup_{S \in \mathcal{S}} \mathcal{H}_S\right| \leq \sigma^{+(n-1)}$. Apply the preceding sublemma with $k = n - 1$, getting a well-ordering of $\prec$ of the network N of all hyperplanes satisfying $(*_{n-1})$. In particular, in (3_{n-1}), $k + 1 = (n-1) + 1 = n$. Now, $\forall H \in N$, let $E_H = \{x \in X : H \text{ is the } \prec\text{-greatest element of } N \text{ including } x\}$. Then $E_H \subseteq H$, and since $\prec$ satisfies $(*_{n-1})$ and $\mathcal{H}$ is τ-fine to depth n, $|E_H| < \sigma$. Now, $\forall S \in \mathcal{S}$, let $E_S = \bigcup_{H \in \mathcal{H}_S} E_H$. Then $\forall H \in \mathcal{H}_S$, $H \cap E_S = E_H$, so $|H \cap E_S| < \sigma$. But the E_H and hence, since $\mathcal{S}$ is nonempty, the E_S decompose X. Thus, $\langle E_S \rangle_{S \in \mathcal{S}}$ is an $\mathcal{S}$-indexed decomposition of X that is σ-fine in $\mathcal{H}$. □

4 ***Cardinalities and decompositions*** This section presents the main theorem of this paper and proves it by combining the results of the previous two sections. Before proceeding, it is convenient to introduce a special family of hyperspaces.

Definition 4.1 Let X and n be sets and let $S \subseteq n$. Then H is an S-*hyperplane* in ${}^nX$ if and only if $\exists f : n \setminus S \to X$ such that $H = \{x \in {}^nX : \forall i \in n \setminus S, x_i = f(i)\}$.

Definition 4.2 Let X be a nonempty set and m and n be cardinals such that $m \leq n$. Then $\mathrm{S}_m^n(X)$, that is, the *Sikorski hyperspace* of m-dimensional hyperplanes in ${}^nX$, is the sequence $\langle \mathcal{H}_S \rangle_{S \in [n]^m}$ such that $\forall S \in [n]^m$, $\mathcal{H}_S$ is the set of S-hyperplanes in ${}^nX$.

Clearly, $\mathrm{S}_m^n(X)$ is an $[n]^m$-indexed hyperspace on ${}^nX$. The main theorem of this paper is as follows.

Theorem 4.3 (Main Theorem) *Let X be an infinite set, σ be an infinite cardinal, and $n \in \omega$. Then the following are equivalent.*

1. $|X| < \sigma^{+n}$.
2. *For every $\tau < \sigma$ and for every $\mathcal{S}$-indexed hyperspace $\mathcal{H}$ on X such that $\mathcal{S}$ is nonempty and finite, if $\mathcal{H}$ is τ-fine to depth n, then there is an $\mathcal{S}$-indexed decomposition of X that is σ-fine in $\mathcal{H}$.*
3. *There is an $\mathcal{S}$-indexed hyperspace $\mathcal{H}$ on X for which there are an $\mathcal{S}$-indexed decomposition of X that is σ-fine in $\mathcal{H}$ and a sequence $\langle \mathcal{S}_i \rangle_{i<n+1}$ of subsets of $\mathcal{S}$ such that*
   (a) $\bigcup_{i<n+1} \mathcal{S}_i = \mathcal{S}$, *and*
   (b) $\forall i < n+1$,
      (i) *$\mathcal{S}_i$ $|X|$-translates over $\bigcup_{j<i} \mathcal{S}_j$ in $\mathcal{H}$, and*
      (ii) $|\mathcal{S}_i| < cf(\sigma^{+i})$.

Proof: That (1) implies (2) is a trivial consequence of the Theorem 3.2. That (2) implies (3) may be seen as follows. Since X is infinite, assume without loss of generality that $X = {}^{n+1}Y$. Let $\mathcal{H} = \mathrm{S}_1^{n+1}(Y)$ and $\mathcal{S} = [n+1]^1$. Then $\mathcal{H}$ is an $\mathcal{S}$-indexed hyperspace on X, $\mathcal{S}$ is finite, and $\mathcal{H}$ is 1-fine to depth n. Next, $\forall i < n+1$, let $\mathcal{S}_i = \{\{i\}\}$. Now, let $i < n+1$ and $A \subseteq X$ such that $|A| < |Y|$. Then let $B = \{b \in Y : \exists x \in A \text{ such that } x_i = b\}$ and $f : B \xrightarrow{1-1} Y \setminus B$. Define t to be the function from A into X such that $\forall x \in X$, $t(x) =$ the element y of X such that $y_i = f(x_i)$ and such that $\forall j < n$, if $j \neq i$, then $y_j = x_j$. Then t is a $\bigcup_{j<i} \mathcal{S}_j$-invariant $\mathcal{S}_i$ translation of A in $\mathcal{H}$. Accordingly, $\forall i \in n+1$, $\mathcal{S}_i$ $|Y|$-translates over $\bigcup_{j<i} \mathcal{S}_j$ in $\mathcal{H}$. Therefore, by (2), there is an $\mathcal{S}$-indexed decomposition of X that is σ-fine in $\mathcal{H}$. Since $|X| = |Y|$, we are done. That (3) implies (1) is a trivial consequence of the Theorem 2.11. □

This theorem may be expressed in measure-theoretic terms by means of the second theorem following, that is, Theorem 4.7.

Definition 4.4 Let $\mathcal{H}$ be a set of subsets of X, σ be a cardinal, and $A \subseteq X$. Then

1. A is *σ-null over $\mathcal{H}$* if and only if $\forall H \in \mathcal{H}$, $|A \cap H| < \sigma$, and
2. A is *σ-full over $\mathcal{H}$* if and only if $X \setminus A$ is σ-null over $\mathcal{H}$.

Definition 4.5 Let $\mathcal{H}$ be an $\mathcal{S}$-indexed hyperspace on X, σ be a cardinal, and $A \subseteq X$. Then

1. A is *σ-null in $\mathcal{H}$* if and only if $\exists S \in \mathcal{S}$ such that A is σ-null over $\mathcal{H}_S$, and
2. A is *σ-full in $\mathcal{H}$* if and only if $\exists S \in \mathcal{S}$ such that A is σ-full over $\mathcal{H}_S$.

Theorem 4.6 *Let X be a set, κ be an infinite cardinal, $\mathcal{A}$ be a κ-algebra on X, τ be an infinite cardinal no greater than κ, and μ be a τ-additive measure on $\mathcal{A}$. Suppose that $\mathcal{M} \subseteq \mathcal{P}(X)$ such that*

1. *$\forall \mathcal{M}' \in [\mathcal{M}]^{<\kappa}$, $\bigcup \mathcal{M}' \in \mathcal{M}$, and*
2. *$\forall A \in \mathcal{A}$, if $\exists M \in \mathcal{M}$ such that $A \subseteq M$, then $\mu(A) = 0$.*

Then there is a unique measure ν on the κ-algebra generated by $\mathcal{A} \cup \mathcal{M}$ that extends μ and is such that $\forall M \in \mathcal{M}$, $\nu(M) = 0$. Moreover, this measure ν is τ-additive.

Proof: Let $\mathcal{N} = \{N : \exists M \in \mathcal{M} \text{ such that } N \subseteq M\}$. Then $\mathcal{N}$ satisfies (1) and (2) above, too; that is,

1. $\forall \mathcal{N}' \in [\mathcal{N}]^{<\kappa}$, $\bigcup \mathcal{N}' \in \mathcal{N}$, and
2. $\forall A \in \mathcal{A}$, if $\exists N \in \mathcal{M}$ such that $A \subseteq N$, then $\mu(A) = 0$.

We begin by showing that there is an extension $\bar{\mu}$ of μ to the κ-algebra generated by $\mathcal{A} \cup \mathcal{N}$ such that $\forall N \in \mathcal{N}$, $\bar{\mu}(N) = 0$.

Let $\mathcal{B} = \{A \triangle N : A \in \mathcal{A} \,\&\, N \in \mathcal{N}\}$. We show that $\mathcal{B}$ is a κ-algebra. To this end, suppose first that $B \in \mathcal{B}$. Then there are $A \in \mathcal{A}$ and $N \in \mathcal{N}$ such that $B = A \triangle N$. Moreover, $X \setminus B = (X \setminus A) \triangle N$, so $X \setminus B \in \mathcal{B}$. Accordingly, $\mathcal{B}$ is closed under complements.

Suppose next that $|I| < \kappa$ and that $\forall i \in I$, $B_i \in \mathcal{B}$. Then $\forall i \in I$, there are $A_i \in \mathcal{A}$ and $N_i \in \mathcal{N}$ such that $B_i = A_i \triangle N_i$. Let $C = \left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcup_{i\in I} B_i\right)$, and $D = \left(\bigcup_{i\in I} B_i\right) \setminus \left(\bigcup_{i\in I} A_i\right)$. Then $\bigcup_{i\in I} B_i = \left(\bigcup_{i\in I} A_i\right) \triangle (C \cup D)$.

Now, $C, D \subseteq \bigcup_{i\in I} N_i$, so, by (1) (for $\mathcal{N}$) and the fact that every subset of an element of $\mathcal{N}$ is an element of $\mathcal{N}$, $C \cup D \in \mathcal{N}$, whence $\bigcup_{i\in I} B_i \in \mathcal{B}$. Accordingly, $\mathcal{B}$ is closed under unions of fewer than κ sets, so $\mathcal{B}$ is indeed a κ-algebra.

Now, suppose $B \in \mathcal{B}$; $A_1, A_2 \in \mathcal{A}$; $N_1, N_2 \in \mathcal{N}$; and $A_1 \triangle N_1 = B = A_2 \triangle N_2$. Then $A_1 \triangle A_2 = N_1 \triangle N_2 \in \mathcal{N}$, so, by (2) (for $\mathcal{N}$), $\mu(A_1 \triangle A_2) = 0$, whence $\mu(A_1) = \mu(A_2)$. Accordingly, define $\bar{\mu}$ to be the unique function on $\mathcal{B}$ such that $\forall A \in \mathcal{A}\ \forall N \in \mathcal{N}$,

$$\bar{\mu}(A \triangle N) = \mu(A).$$

Clearly, $\bar{\mu}$ is an extension of μ. We show that $\bar{\mu}$ is τ-additive. To this end, suppose that $|I| < \tau$ and that $\langle B_i \rangle_{i\in I}$ is a sequence of pairwise disjoint sets drawn from $\mathcal{B}$. Then $\forall i \in I$, there are $A_i \in \mathcal{A}$ and $N_i \in \mathcal{N}$ such that $B_i = A_i \triangle N_i$. As above, let $C = \left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcup_{i\in I} B_i\right)$ and $D = \left(\bigcup_{i\in I} B_i\right) \setminus \left(\bigcup_{i\in I} A_i\right)$, so that $C \cup D \in \mathcal{N}$ and $\bigcup_{i\in I} B_i = \left(\bigcup_{i\in I} A_i\right) \triangle (C \cup D)$. Then $\bar{\mu}\left(\bigcup_{i\in I} B_i\right) = \mu\left(\bigcup_{i\in I} A_i\right)$. Now, let $E = \bigcup_{i,j\in I, i\neq j} \left(A_i \cap A_j\right)$. Suppose $i, j \in I$ and $i \neq j$. Then, since B_i and B_j are disjoint,

$$A_i \cap A_j \subseteq N_i \cup N_j \in \mathcal{N},$$

whence

$$\mu\left(A_i \cap A_j\right) = 0.$$

Accordingly, $\mu(E) = 0$, whence

$$\mu\left(\bigcup_{i\in I} A_i\right) = \mu\left(\left(\bigcup_{i\in I} A_i\right) \setminus E\right)$$

$$= \mu\left(\bigcup_{i\in I}(A_i \setminus E)\right) = \sum_{i\in I}\mu(A_i \setminus E)$$
$$= \sum_{i\in I}\mu(A_i) = \sum_{i\in I}\bar{\mu}(B_i).$$

Thus, $\bar{\mu}\left(\bigcup_{i\in I} B_i\right) = \sum_{i\in I}\bar{\mu}(B_i)$. Accordingly, $\bar{\mu}$ is τ-additive.

Now let ν be $\bar{\mu}$ restricted to the κ-algebra $\mathcal{C}$ generated by $\mathcal{A}\cup\mathcal{M}$. Then ν is, as desired, a measure on $\mathcal{C}$ that extends μ and is 0 on $\mathcal{M}$. To see that ν is unique, suppose that ν' is also such a measure. Let $C \in \mathcal{C}$. Then $C \in \mathcal{B}$, so there are $A \in \mathcal{A}$ and $N \in \mathcal{N}$ such that $C = A \triangle N$, and there is an $M \in \mathcal{M}$ such that $N \subseteq M$. Now, $\nu'(A) = \mu(A)$ and $\nu'(M) = 0$. Hence, $\nu'(A) = \nu'(A \setminus M) \leq \nu'(C) \leq \nu'(A \cup M) = \nu'(A)$. Thus, $\nu'(C) = \nu'(A) = \mu(A) = \nu(C)$. Accordingly, ν is unique, and, by construction, it is τ-additive. □

Theorem 4.7 *Let $\mathcal{H}$ be an $\mathcal{S}$-indexed hyperspace on X and σ be an infinite cardinal. Let κ be an infinite cardinal, $\mathcal{A}$ be a κ-algebra on X, τ be an infinite cardinal no greater than κ, and μ be a τ-additive measure on $\mathcal{A}$. Let $\mathcal{M} = \left\{\bigcup_{S\in\mathcal{S}} A_S : \forall S \in \mathcal{S},\ A_S \text{ is } \sigma\text{-null over } \mathcal{H}_S\right\}$. Suppose that κ is no greater than $cf(\sigma)$ and that $\forall A \in \mathcal{M}$, if $\mu(A)$ is defined, then $\mu(A) = 0$. Then the following are equivalent.*

1. *$\forall A \in \mathcal{A}$, if $\mu(A) \neq 0$, then there is no $\mathcal{S}$-indexed decomposition of A that is σ-fine in $\mathcal{H}$.*
2. *There is a τ-additive measure ν defined on the κ-algebra generated by $\mathcal{A}\cup\mathcal{M}$ that extends μ and is such that $\forall M \in \mathcal{M}$, $\nu(M) = 0$.*
3. *There is a finitely additive measure ν on X that extends μ and is such that $\forall M \in \mathcal{M}$, $\nu(M) = 0$.*

Proof: We first show that (1) implies (2). To this end, assume (1). Note that $\mathcal{M}$ is closed under unions of fewer than $\mathrm{cf}(\sigma)$ sets. Let $A \in \mathcal{A}$ and suppose that there is an $M \in \mathcal{M}$ such that $A \subseteq M$. We seek to show that $\mu(A) = 0$. Assume otherwise. Since $M \in \mathcal{M}$, there is a sequence $\langle M_S\rangle_{S\in\mathcal{S}}$ such that $\forall S \in \mathcal{S}$, M_S is σ-null over $\mathcal{H}_S$ and such that $M = \bigcup_{S\in\mathcal{S}} M_S$. But then, $\langle A \cap M_S\rangle_{S\in\mathcal{S}}$ is an $\mathcal{S}$-indexed decomposition of A that is σ-fine in $\mathcal{H}$, which contradicts (1). Accordingly, $\mu(A)$ does equal zero. By the immediately preceding theorem, it follows that (2) is true.

That (3) follows from (2) is trivial. To see that (1) follows from (3), let ν be as in (3). Let $A \in \mathcal{A}$ such that $\mu(A) \neq 0$. Suppose $\langle E_S\rangle_{S\in\mathcal{S}}$ is a σ-fine $\mathcal{S}$-indexed decomposition of A in $\mathcal{H}$. Then $\mu(A) = \nu\left(\bigcup_{S\in\mathcal{S}} E_S\right) = 0$, which is a contradiction. Hence (1) holds. □

5 Applications In this section, we apply Main Theorem 4.3 to Sikorski's Theorem 1.6, Davies' Theorem 1.7, finite-dimensional vector spaces, and some of Freiling's principles. We begin by restating Sikorski's Theorem 1.6 and proving it as an immediate consequence of Main Theorem 4.3.

Proposition 5.1 *Let n be an element of ω no less than 2, X be a set of size n, m be such that $0 < m < n$, and $\preceq$ be any linear ordering of $[X]^m$. Then there is a* *subset $\mathcal{S}$ of $[X]^m$ of size $n-m$ such that $\forall S \in \mathcal{S}$, $\bigcap\left\{S' \preceq S : \neg\exists S'' \in \mathcal{S}\text{ such that } S' \prec S'' \prec S\right\} = \varnothing$.*

Proof: By induction on n. If $n = 2$, then the result is trivial. Accordingly, assume that the result is true for n; we seek to show that it is true for $n+1$. To this end, let X be a set of size $n+1$ and $\preceq$ be any linear ordering of $[X]^m$. If $m = n$, then the result is again trivial, so assume that $m < n$. $\forall S \in [X]^m$, let $[S]_\prec$ stand for $\left\{S' : S' \prec S\right\}$. Now let S be the $\preceq$-first element of $[X]^m$ such that $S \cap \bigcap [S]_\prec = \varnothing$, let $\mathcal{S} = [X]^m \setminus [S]_\prec$, and finally, let $x \in \bigcap [S]_\prec$. Then $[X \setminus \{x\}]^m \subseteq \mathcal{S}$. By the induction hypothesis, there is a subset $\mathcal{S}'$ of $[X \setminus \{x\}]^m$ of size $n-m$ such that $\forall S' \in \mathcal{S}', \bigcap\left\{S'' \in [X \setminus \{x\}]^m : S'' \preceq S' \,\&\, \neg\exists S''' \in \mathcal{S}' \text{ such that } S'' \prec S''' \prec S'\right\} = \varnothing$. Then $\{S\} \cup \mathcal{S}'$ is as desired. $\square$

Theorem 5.2 (Sikorski's Theorem) *Let X be a set, n be a nonzero element of ω, m be such that $0 < m \le n$, and σ be an infinite cardinal. Then $|X| < \sigma^{+(n-m)}$ if and only if $\exists \langle E_S\rangle_{S\in[n]^m}$ such that*

1. *${}^nX = \bigcup_{S\in[n]^m} E_S$, and*
2. *$\forall S \in [n]^m$, every S-hyperplane in ${}^nX$ intersects E_S in fewer than σ points.*

Proof: If X is finite or if $n = m$, then the theorem is trivial, so assume that X is infinite and that $m < n$. Let $\mathcal{H}$ be $\mathrm{S}^n_m(X)$. Let $\mathcal{S} \subseteq [n]^m$ such that $\bigcap \mathcal{S} = \varnothing$. Then $\forall \langle H_S\rangle_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \mathcal{H}_S$, $\left|\bigcap_{S\in\mathcal{S}} H_S\right| \le 1$. Accordingly, by the preceding proposition, $\mathcal{H}$ is 1-fine to depth $n-m$.

Now, $\forall i \in n-m+1$, let $\mathcal{S}_i = [m+i]^m \setminus [m+(i-1)]^m$. Then $\bigcup_{i<n-m+1} \mathcal{S}_i = [n]^m$ and $\forall i < n-m+1$, $\mathcal{S}_i$ $|X|$-translates over $\bigcup_{j<i} \mathcal{S}_j$ in $\mathcal{H}$. Sikorski's Theorem is now an easy consequence of Main Theorem 4.3. $\square$

Next we restate Davies' Theorem 1.7 and show that it, too, is an immediate consequence of Main Theorem 4.3. Before proceeding, it is convenient to introduce another special family of hyperspaces.

Definition 5.3 Let V be a vector space and π be a subspace of V. Then H is a *π-hyperplane in V* if and only if $\exists v \in V$ such that $H = v + \pi$, that is, such that $H = \{v + w : w \in \pi\}$.

Definition 5.4 Let V be a vector space and Π be a set of subspaces of V. Then H is a *Π-hyperplane of V* if and only if $\exists \pi \in \Pi$ such that H is a π-hyperplane in V.

Definition 5.5 Let V be a vector space and Π be a set of subspaces of V. Then $\mathrm{B}_\Pi(V)$, that is, the *Bagemihl hyperspace*[3]of Π hyperplanes of V, is the sequence $\left\langle \mathcal{H}_\pi\right\rangle_{\pi\in\Pi}$ such that $\forall \pi \in \Pi$, $\mathcal{H}_\pi$ is the set of π-hyperplanes in V.

Clearly, $\mathrm{B}_\Pi(V)$ is a Π-indexed hyperspace on V.

Theorem 5.6 (Davies' Theorem) *Let $n \in \omega$, σ be an infinite cardinal, k be a nonzero element of ω, and $\langle L_i\rangle_{i\in n+1}$ be a sequence of lines in $\mathbf{R}^k$, no two of which are parallel. Then $2^{\aleph_0} < \sigma^{+n}$ if and only if $\exists \langle S_i\rangle_{i\in n+1}$ such that*

1. *$\mathbf{R}^k = \bigcup_{i\in n+1} S_i$, and*
2. *$\forall i \in n+1$, every line in $\mathbf{R}^k$ parallel to L_i intersects S_i in fewer than σ points.*

Proof: If $n = 0$, then the result is trivial, so suppose that $n > 0$, in which case $k > 1$. $\forall i \in n+1$, let π_i be a nonzero vector in $\mathbf{R}^k$ parallel to L_i; let $\Pi = \{\pi : i < n+1\}$; and finally, let $\mathcal{H} = \mathrm{B}_\Pi\left(\mathbf{R}^k\right)$. Then $\mathcal{H}$ is a Π-indexed hyperspace on $\mathbf{R}^k$ which is 1-fine to depth n. Now, $\forall i \in n+1$, let $\mathcal{S}_i = \{\pi_i\}$. Then $\bigcup_{i<n+1} \mathcal{S}_i = \Pi$ and $\forall i < n+1$, $\mathcal{S}_i$ $2^{\aleph_0}$-translates over $\bigcup_{j<i} \mathcal{S}_j$ in $\mathcal{H}$. Davies' Theorem is now an easy consequence of Main Theorem 4.3. □

Next we apply Main Theorem 4.3 to vector spaces in general.

Definition 5.7 Let V be a nontrivial vector space, Π be a set of nontrivial subspaces of V, and $n \in \omega$. Then Π is *n-good* if and only if for every linear ordering $\preceq$ of Π, there is a subset $\mathcal{S}$ of Π of size n such that $\forall \pi \in \mathcal{S}$, $\bigcap \left\{\pi' \preceq \pi : \neg\exists \pi'' \in \mathcal{S} \text{ such that } \pi' \prec \pi'' \prec \pi\right\}$ is 0-dimensional.

Lemma 5.8 *Let V be a nontrivial vector space over an infinite field F, Π be a set of nontrivial subspaces of V, and $n \in \omega$. Suppose that Π is n-good, but not $(n+1)$-good. Then $\exists \langle \Pi_i \rangle_{i<n+1}$ such that*

1. $\Pi = \bigcup_{i<n+1} \Pi_i$ *and*
2. $\forall A \in [V]^{<|F|}\ \forall i < n+1\ \exists v \in \bigcap \Pi_i$ *such that* $A \cap (v + A) = \varnothing$.

Proof: Since Π is not $(n+1)$-good, there is a linear ordering $\preceq$ of Π for which there is no subset $\mathcal{S}$ of Π of size $n+1$ such that $\forall \pi \in \mathcal{S}$, $\bigcap \left\{\pi' \preceq \pi : \neg\exists \pi'' \in \mathcal{S} \text{ such that } \pi' \prec \pi'' \prec \pi\right\}$ is 0-dimensional. On the other hand, since Π is n-good, there is a subset $\mathcal{S}$ of Π of size n such that $\forall \pi \in \mathcal{S}$, $\bigcap \left\{\pi' \preceq \pi : \neg\exists \pi'' \in \mathcal{S} \text{ such that } \pi' \prec \pi'' \prec \pi\right\}$ is 0-dimensional.

Let $\langle \pi_i \rangle_{i \in n}$ enumerate S in $\prec$ order. If $n = 0$, let $\Pi_0 = \Pi$. If $n > 0$: let $\Pi_0 = \{\pi : \pi \prec \pi_0\}$; $\forall i$ such that $0 < i < n$, let $\Pi_i = \{\pi : \pi_{i-1} \preceq \pi \prec \pi_i\}$; and let $\Pi_n = \{\pi : \pi_{n-1} \preceq \pi\}$. By choice of $\prec$, we may assume that $\forall i < n$, π_i is the $\prec$-first element of Π not in $\left\{\pi_j : j < i\right\}$ such that $\bigcap \left\{\pi' \preceq \pi_i : \neg\exists \pi'' \in \mathcal{S} \text{ such that } \pi' \prec \pi'' \prec \pi_i\right\}$ is 0-dimensional, in which case $\forall i < n$, $\bigcap \Pi_i$ is nontrivial. Moreover, by choice of $\prec$, $\bigcap \Pi_n$ is nontrivial.

Now, $\forall i < n+1$, let v_i be a nonzero element of $\bigcap \Pi_i$. Let $A \in [V]^{<|F|}$ and $i < n+1$. Then let $W = \{w_1 - w_2 : w_1, w_2 \in A\}$. Since $|F|$ is infinite, $|W| < |F|$. Accordingly, there is a scalar multiple v of v_i such that $v \notin W$. Then A and $v + A$ are disjoint. □

Theorem 5.9 *Let F be a field, V be a vector space over F, $n \in \omega$, and σ be an infinite cardinal. Suppose that the dimension of V is at least 2 and that $|V| = |F|$ (which is the case, for example, when F is infinite and V is finite dimensional). Then the following are equivalent.*

1. $|V| < \sigma^{+n}$.
2. *For every nontrivial finite set Π of nontrivial subspaces of V that is n-good, there is a sequence $\langle E_\pi \rangle_{\pi \in \Pi}$ such that*
   (a) $V = \bigcup_{\pi \in \Pi} E_\pi$ *and*
   (b) $\forall \pi \in \Pi\ \forall v \in V$, $|(v + \pi) \cap E_\pi| < \sigma$.

3. *There is a nontrivial finite set* Π *of nontrivial subspaces of* V *that is not* $(n+1)$*-good for which there is a sequence* $\langle E_\pi \rangle_{\pi \in \Pi}$ *such that*
   (a) $V = \bigcup_{\pi \in \Pi} E_\pi$ *and*
   (b) $\forall \pi \in \Pi\ \forall v \in V,\ |(v+\pi) \cap E_\pi| < \sigma$.

Proof: Note first that since it is assumed that $|V| = |F|$ and that the dimension of V is at least 2, F is automatically assumed to be infinite. We show first that (1) implies (2). To this end, assume (1) and suppose that Π is a nontrivial finite set of nontrivial subspaces of V that is n-good. Let $\mathcal{H} = \mathrm{B}_\Pi(V)$. We show that $\mathcal{H}$ is 1-fine to depth n. To this end, let $v \in V$ and $\forall \pi \in \Pi$, let H_π be the unique element of $\mathcal{H}_\pi$ such that $v \in H_\pi$, that is, let $H_\pi = v + \pi$.

Let $\preceq$ be any linear ordering of $\{H_\pi : \pi \in \Pi\}$. Since the H_π are distinct, this induces a linear ordering $\prec'$ on Π. Since Π is n-good, there is a subset $\mathcal{S}$ of Π of size n such that $\forall \pi \in \mathcal{S}, \bigcap \left\{\pi' \preceq' \pi : \neg\exists \pi'' \in \mathcal{S} \text{ such that } \pi' \prec' \pi'' \prec' \pi\right\}$ is 0-dimensional. But then $\forall \pi \in \mathcal{S}, \bigcap \left\{H' \preceq H_\pi : \neg\exists \pi'' \in \mathcal{S} \text{ such that } H' \prec H_{\pi''} \prec H_\pi\right\} = \{v\}$. Accordingly, $\mathcal{H}$ is 1-fine to depth n, whence by Main Theorem 4.3, (2) holds.

We show next that (2) implies (3). Assume (2). Since the dimension of V is at least 2, there is a sequence $\langle L_i \rangle_{i \in n+1}$ of pairwise distinct lines in V through the origin. Let $\Pi = \{L_i : i \in n+1\}$. Then Π is n-good, but not $(n+1)$-good. By (2), there is a sequence $\langle E_\pi \rangle_{\pi \in \Pi}$ such that

1. $V = \bigcup_{\pi \in \Pi} E_\pi$ and
2. $\forall \pi \in \Pi\ \forall v \in V,\ |(v+\pi) \cap E_\pi| < \sigma$.

Thus (3) holds. By the preceding lemma and Main Theorem 4.3, (3) implies (1). □

This is certainly not the most general theorem that can be proved about vector spaces but it is, in the present context, probably the most interesting one. A question that needs to be addressed is, Which Π are n-good? Here we can make several observations.

First, suppose that $0 < m \leq n \in \omega$ and that $\langle v_i \rangle_{i<n}$ is a sequence of n linear independent vectors drawn from V. $\forall S \in [n]^m$, let π_S be the m-dimensional subspace of V spanned by $\{v_i : i \in S\}$. Then $\forall \mathcal{S} \subseteq [n]^m$, $\bigcap_{S \in \mathcal{S}} \pi_S$ is trivial if and only if $\bigcap \mathcal{S} = \varnothing$. Let $\Pi = \{\pi_S : S \in [n]^m\}$. By Proposition 5.1, Π is $(n-m)$-good. It is also easy to see that Π is not $(n-m+1)$-good—let $\prec$ be any ordering of Π such that $\forall S, S' \in [n]^m$, if $\max S < \max S'$, then $\pi_S \prec \pi_{S'}$. Since for every infinite cardinality, there is a field of that cardinality, Sikorski's Theorem 5.2 follows from the preceding theorem.

Next, let Π be a set of $(n+1)$ lines through the origin. Then, as noted in the proof of the preceding theorem, Π is n-good, but not $(n+1)$-good. Thus, a generalized version of Davies' Theorem 5.6 also follows from the preceding theorem.

To see something new—something not involving hyperplanes all of the same dimension—let $V = \mathbf{R}^3$ and Π consist of the x, y-plane and the z-axis. Then Π is 1-good, but not 2-good. Hence, for example, CH if and only if there are sets E_z and E_{xy} such that

1. $\mathbf{R}^3 = E_z \cup E_{xy}$,
2. every line parallel to the z-axis intersects E_z in countably many points, and
3. every plane parallel to the x, y-plane intersects E_{xy} in countably many points.

To see something else new—something like Davies' Theorem 5.6, but for nontrivial hyperplanes—let $V = \mathbf{R}^3$; v_0, v_1, and v_2 be nonzero vectors in the x-, y-, and z-directions; and v_3 be a nonzero vector not in any of the coordinate planes, for example, the vector $\langle 1, 1, 1\rangle$. Define Π to be the set of six planes generated by pairs of these vectors. Then it is easy to verify that Π is 2-good, but not 3-good. Hence, for example, CH if and only if there is a decomposition $\langle E_\pi\rangle_{\pi\in\Pi}$ of $\mathbf{R}^3$ such that $\forall\pi \in \Pi$, every plane parallel to π intersects E_π in only finitely many points.

In principle, we can compute of a given a set of hyperplanes just how "good" it is and see thereby just what kinds of Sierpiński-type theorems we can get for it. It is particularly interesting to re-express these results for finite-dimensional Euclidean space in terms of Lebesgue measure.

Notation 5.10

1. λ is the Lebesgue measure on $\mathbf{R}$.
2. $\forall k \in \omega$, λ^k is the Lebesgue measure on $\mathbf{R}^k$.

Proposition 5.11 *Let σ be an infinite cardinal—such as $\aleph_0$ or $\aleph_1$—such that $\forall A \in [\mathbf{R}]^{<\sigma}$, $\lambda(A) = 0$; $\kappa = \min\{\aleph_1, cf(\sigma)\}$; and $n \in \omega$. Let $k \in \omega$ such that $k \geq 2$. Then the following are equivalent.*

1. $2^{\aleph_0} \geq \sigma^{+n}$.
2. *For every nontrivial finite set Π of nontrivial subspaces of $\mathbf{R}^k$, if Π is n-good, then there is a κ-additive measure ν such that*
   (a) $\nu \supseteq \lambda^k$, *and*
   (b) $\forall A \subseteq \mathbf{R}^k$, *if $\exists\pi \in \Pi$ such that every hyperplane in $\mathbf{R}^k$ parallel to π intersects A in fewer than σ points, then $\nu(A) = 0$.*
3. *There is a nontrivial finite set Π of nontrivial subspaces of $\mathbf{R}^k$ that is not $(n+1)$-good for which there is a a finitely-additive measure ν such that*
   (a) $\nu \supseteq \lambda^k$, *and*
   (b) $\forall A \subseteq \mathbf{R}^k$, *if $\exists\pi \in \Pi$ such that every hyperplane in $\mathbf{R}^k$ parallel to π intersects A in fewer than σ points, then $\nu(A) = 0$.*

Proof: The result is immediate by Main Theorem 4.3, its measure-theoretic reformulation Theorem 4.7, and the Fubini Theorem for Lebesgue measure. □

Corollary 5.12 *The following are equivalent.*

1. *¬CH.*
2. *There is a countably-additive measure ν that extends the Lebesgue measure for the plane and that gives measure 0 to every subset of the plane all of whose vertical or all of whose horizontal cross-sections are countable.*
3. *There is a finitely-additive measure ν that extends the Lebesgue measure for the plane and that gives measure 0 to every subset of the plane all of whose vertical or all of whose horizontal cross-sections are countable.*

Let us conclude this section by turning to Freiling's principles. In [5] Freiling proved the following theorems related to Sikorski's Theorem 1.6.

Definition 5.13 $\mathrm{A}_{<2^{\aleph_0}}$ if and only if $\forall f : 2^{\aleph_0} \longrightarrow \left[2^{\aleph_0}\right]^{<2^{\aleph_0}}$ $\exists x_1, x_2 \in 2^{\aleph_0}$ such that both $x_2 \notin f(x_1)$ & $x_1 \notin f(x_2)$.

Proposition 5.14 $\neg A_{<2^{\aleph_0}}$.

Definition 5.15 Let n be a nonzero element of ω. Then $\mathrm{A}^n_{\aleph_0}$ if and only if $\forall f : \left[2^{\aleph_0}\right]^{n-1} \longrightarrow \left[2^{\aleph_0}\right]^{<\aleph_1}$ $\exists X \in \left[2^{\aleph_0}\right]^n$ such that $\forall x \in X$, $x \notin f(X \setminus \{x\})$.

Proposition 5.16 *Let n be an element of ω not less than* 2*. Then $A^n_{\aleph_0}$ if and only if* $2^{\aleph_0} \geq \aleph_n$.

Freiling [5] remarks, without proof, that the following proposition is true.

Definition 5.17 Let n be a nonzero element of ω. Then A^n_{finite} if and only if $\forall f : \left[2^{\aleph_0}\right]^{n-1} \longrightarrow \left[2^{\aleph_0}\right]^{<\aleph_0}$ $\exists X \in \left[2^{\aleph_0}\right]^n$ such that $\forall x \in X$, $x \notin f(X \setminus \{x\})$.

Proposition 5.18 *Let n be an element of ω no less than* 2*. Then A^{n+1}_{finite} if and only if* $2^{\aleph_0} \geq \aleph_n$.

We may generalize Freiling's results as follows.

Definition 5.19 Let X be a set and σ, n, and m be cardinals. Then $\mathrm{F}(X, \sigma, n, m)$ if and only if $\forall f : [X]^n \longrightarrow \left[[X]^{m+1}\right]^{<\sigma}$ $\exists A \in [X]^{n+m+1}$ such that $\forall B \in [A]^{m+1}$, $B \notin f(A \setminus B)$.

Clearly,

$$\mathrm{A}_{<2^{\aleph_0}} \iff \mathrm{F}\left(2^{\aleph_0}, 2^{\aleph_0}, 1, 0\right), \tag{1}$$
$$\mathrm{A}^n_{\aleph_0} \iff \mathrm{F}\left(2^{\aleph_0}, \aleph_1, n-1, 0\right), \text{ and} \tag{2}$$
$$\mathrm{A}^n_{\text{finite}} \iff \mathrm{F}\left(2^{\aleph_0}, \aleph_0, n-1, 0\right). \tag{3}$$

Proposition 5.20 *Let X be an infinite set, σ be an infinite cardinal, and $n, m \in \omega$. Then $|X| \geq \sigma^{+n}$ if and only if $F(X, \sigma, n, m)$.*

Proof: Note first that the proposition is trivial if $n = 0$; hence, assume that $n > 0$. We begin with the left-to-right direction. To this end, assume that $|X| \geq \sigma^{+n}$. Let $\mathcal{H} = \mathrm{S}^{n+m+1}_{m+1}(X)$. By Sikorski's Theorem 5.2, there is no $[n+m+1]^{m+1}$-indexed decomposition of ${}^{n+m+1}X$ that is σ-fine in $\mathcal{H}$. Let

$$D = \left\{x \in {}^{n+m+1}X : \exists i, j \in n+m+1 \text{ such that } i \neq j \;\&\; x_i = x_j\right\}.$$

Now let μ be the smallest probability measure on ${}^{n+m+1}X$ that assigns 0 to D, that is, let μ be the function whose domain is $\left\{\varnothing, D, {}^{n+m+1}X \setminus D, {}^{n+m+1}X\right\}$, that assigns the value 0 to $\varnothing$ and to D, and that assigns the value 1 to ${}^{n+m+1}X \setminus D$ and to ${}^{n+m+1}X$. Since there is no $[n+m+1]^{m+1}$-indexed decomposition of ${}^{n+m+1}X$ that is σ-fine in $\mathcal{H}$, there is no $[n+m+1]^{m+1}$-indexed decomposition of ${}^{n+m+1}X \setminus D$ that is σ-fine in $\mathcal{H}$. Hence, by Theorem 4.7, there is a finitely-additive measure ν on ${}^{n+m+1}X$ that extends μ and that assigns the value 0 to every subset of ${}^{n+m+1}X$ that is σ-null in $\mathcal{H}$.

We now show that $\mathrm{F}(X,\sigma,n,m)$ is true. To this end, suppose that $f:[X]^n\longrightarrow\left[[X]^{m+1}\right]^{<\sigma}$. Now, $\forall S\in[n+m+1]^{m+1}$, let

$$F_S=\left\{x\in{}^{n+m+1}X:\{x_i:i\in S\}\in f\left(\{x_i:i\in(n+m+1)\setminus S\}\right)\right\}.$$

Let

$$F=\bigcup_{S\in[n+m+1]^{m+1}}F_S.$$

Then F is a finite union of sets σ-null in $\mathcal{H}$, so $\nu(F)=0$. Hence,

$$\nu\left({}^{n+m+1}X\setminus(F\cup D)\right)=1,$$

so there is an $x\in{}^{n+m+1}X\setminus(F\cup D)$. Let $A=\{x_i:i<n+m+1\}$. Then A is as desired.

Next we prove the right-to-left direction. To this end, assume that $|X|<\sigma^{+n}$. Let $\mathcal{H}$ be as before. Then, by Sikorski's Theorem 5.2 again, there is an $[n+m+1]^{m+1}$-indexed decomposition $\langle E_S\rangle_{S\in[n+m+1]^{m+1}}$ of ${}^{n+m+1}X$ that is σ-fine in $\mathcal{H}$. Let

$$F=\left\{x\in{}^{n+m+1}X:\exists S\in[n+m+1]^{m+1}\,\exists\pi:n+m+1\xrightarrow[\text{onto}]{1-1}n+m+1\right.$$

such that

$$\left.\pi[n]=(n+m+1)\setminus S\;\&\;\left\langle x_{\pi(i)}\right\rangle_{i<n+m+1}\in E_S\right\}.$$

F is a finite union of sets σ-null over $\mathcal{H}_{(n+m+1)\setminus n}$. Moreover, since E is a decomposition of ${}^{n+m+1}X$,

$$\forall x\in{}^{n+m+1}X\exists\pi:n+m+1\xrightarrow[\text{onto}]{1-1}n+m+1$$

such that

$$\left\langle x_{\pi(i)}\right\rangle_{i<n+m+1}\in F.$$

Let $f:[X]^n\longrightarrow[X]^{m+1}$ whose graph as a subset of ${}^{n+m+1}X$ is $F\setminus D$, where D is defined as above. Then f is a counterexample to $\mathrm{F}(X,\sigma,n,m)$. □

It is possible to prove the preceding proposition in one or more direct ways that avoid using measure-theoretic techniques, but the proof given has the advantage of showing that if $\mathrm{F}(X,\sigma,n,m)$ and $f:[X]^n\longrightarrow\left[[X]^{m+1}\right]^{<\sigma}$, then there is not only one, but *measure one* sets $A\in[X]^{n+m+1}$ such that $\forall B\in[A]^{m+1}$, $B\notin f(A\setminus B)$.

6 *Is $2^{\aleph_0}$ weakly inaccessible?* Freiling's principle $\mathrm{A}_{\aleph_0}$ is the special case $\mathrm{A}^2_{\aleph_0}$ of his principle $\mathrm{A}^n_{\aleph_0}$. The principle $\mathrm{A}_{\aleph_0}$ is then, the same as $\mathrm{F}\left(2^{\aleph_0},\aleph_1,1,0\right)$ which we have just seen, is equivalent to $\neg$CH.

The gist of Freiling's philosophical argument for $\mathrm{A}_{\aleph_0}$ is as follows. Let $\mathbf{I}$ be the closed unit interval $[0,1]$ of real numbers. Let $f:\mathbf{I}\longrightarrow[\mathbf{I}]^{<\aleph_1}$. Pick x and y at random from $\mathbf{I}$. Now, $\forall a\in\mathbf{I}$, $P(y\in f(x)\mid x=a)$, that is, the probability that $y\in f(x)$ given that $x=a$, is 0. Hence—and here is the crux of the argument—$P(y\in f(x))$, that is, the probability that $y \in f(x)$, is 0. By symmetry, $P(x \in f(y)) = 0$, too. Thus,

$$P(y \in f(x) \text{ or } x \in f(y)) \leq P(y \in f(x)) + P(x \in f(y)) = 0,$$

whence

$$P(y \notin f(x) \;\&\; x \notin f(y)) = 1.$$

Accordingly, $\exists x, y \in \mathbf{I}$ such that $y \notin f(x) \;\&\; x \notin f(y)$.

For every nonzero element n of ω, Freiling presents a similarly attractive argument for $\mathrm{F}\left(2^{\aleph_0}, \aleph_1, n, 0\right)$: let $f : [\mathbf{I}]^n \longrightarrow [\mathbf{I}]^{<\aleph_1}$; pick $x_0, x_1, \ldots, x_{n-1}$, and y at random from $\mathbf{I}$; and check to see whether $y \in f(\{x_0, x_1, \ldots, x_{n-1}\})$. $\mathrm{F}\left(2^{\aleph_0}, \aleph_1, n, 0\right)$ is, as we know, equivalent to the proposition that $2^{\aleph_0} \geq \aleph_{n+1}$. Thus, Freiling has presented an attractive argument for believing that $2^{\aleph_0} > \aleph_\omega$.

Sikorski's Theorem 5.2 indicates how little the principles $\mathrm{F}\left(2^{\aleph_0}, \aleph_1, n, 0\right)$ have to do with the reals per se: if one is willing to believe that $2^{\aleph_0} > \aleph_\omega$, then perhaps one might as well believe that for every ordinal α, $2^{\aleph_\alpha} > \aleph_{\alpha+\omega}$.

In any case, if one believes that small subsets of the reals, that is, sets of cardinality less than that of the continuum, should have *measure* 0, and if one believes in Freiling's argument for $\mathrm{F}\left(2^{\aleph_0}, \aleph_1, 1, 0\right)$, then perhaps one ought to believe as well that $\mathrm{F}\left(2^{\aleph_0}, 2^{\aleph_0}, 1, 0\right)$. However, this proposition is simply false. On the other hand, one might simply believe that $\forall \sigma < 2^{\aleph_0}$, $\mathrm{F}\left(2^{\aleph_0}, \sigma^+, 1, 0\right)$. This has the interesting consequence that $2^{\aleph_0} \geq \aleph_{\omega_1}$.

Proposition 6.1 *Let κ be an infinite cardinal. Then κ is a limit cardinal if and only if $\forall \sigma < \kappa$, $F\left(\kappa, \sigma^+, 1, 0\right)$.*

Proof: Trivial. □

Corollary 6.2 *Suppose that $\forall \sigma < 2^{\aleph_0}$, $F\left(2^{\aleph_0}, \sigma^+, 1, 0\right)$. Then $2^{\aleph_0} \geq \aleph_{\omega_1}$.*

Moreover, if one believes that small unions of small sets should be of measure 0, then $2^{\aleph_0}$ should be regular, whence $2^{\aleph_0}$ should be weakly inaccessible.

Corollary 6.3 *Suppose that $\forall \sigma < 2^{\aleph_0}$, $F\left(2^{\aleph_0}, \sigma^+, 1, 0\right)$, and that $2^{\aleph_0}$ is regular. Then $2^{\aleph_0}$ is weakly inaccessible.*

For more on Freiling's philosophical argument against CH, see [9] and [2].[4]

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NOTES

1. For more on the history of Sierpiński's Theorem 1.2, see [8].

2. [4] actually contains a generalization of Davies' Theorem 1.7 in a direction other than that taken here. I plan to generalize the results of this paper in that direction in a future work.

3. In [1], Bagemihl obtained the first Sierpiński-type results concerning three or more lines in the plane.

4. For more on decompositions of hyperspace, cf. also [10].

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