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Uncompactness of *Stit* Logics Containing Generalized *Refref* Conditionals

MING XU

Abstract In this paper we prove the uncompactness of every *stit* logic that contains a generalized *refref* conditional and is a sublogic of the *stit* logic with *refref* equivalence, a syntactical condition of uncompactness that covers infinitely many *stit* logics. This result is established through the uncompactness of every *stit* logic whose semantic structures contain no chain of busy choice sequences with cardinality n, where n is any natural number > 0. The basic idea in the proof is to apply the notion of companions to *stit* sentences in finding busy choice sequences in structures, and to make use of a relation between chains of busy choice sequences and generalized *refref* conditionals in connecting the two conditions of uncompactness mentioned above.

1 Introduction Modal logic of agency has a long tradition which has been represented by many philosophers and logicians in this century.[1] Following this tradition, several theories of agency have been proposed by von Kutschera, Horty, Belnap, and Perloff in a series of articles such as [26], [13], [16], [5], [6], and [7]. These theories are now often referred to as "*stit* theories." They start with "*stit* sentences" such as $[\alpha\ stit\colon\ A]$ (read "α sees to it that A," where α is an agent term and A is any sentence) whose semantic interpretation, based on the branching time theory proposed by Prior [21] and Thomason [24], is roughly that A is guaranteed true due to a choice made by α. If, in this context, the moment at which $[\alpha\ stit\colon\ A]$ is evaluated is the same moment at which α makes the choice, the result theory is called the *deliberative stit*, or *dstit*. If the moment at which $[\alpha\ stit\colon\ A]$ is evaluated is properly later than the moment at which α makes the choice, the result theory is called the *achievement stit*, or *astit*. Two close relatives of these theories are sometimes called *bstit* and *cstit*, where b refers to Brown [9] and c to Chellas ([10] and [11]).[2] Conceptual or technical discussions on *astit*, *bstit*, *cstit*, or *dstit* (including combinations of *stit* with other branches of philosophical logic) can be found, in addition to those mentioned above, in Bartha [1], Belnap and Bartha [4], Horty ([14] and [15]), von Kutschera [27], and

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Xu [31] and [34] (*dstit*), and Belnap ([2] and [3]) and Perloff ([19] and [20]) (*astit*). The present paper focuses on *astit* theory. From now on, we use *stit* for *astit*.

The main purpose of this paper is to prove that for each *stit* logic L,

(1) L is uncompact if L contains a "generalized *refref* conditional" and is a sublogic of the *stit* logic with "*refref* equivalence" (presented in Xu [29]);
(2) L is uncompact if the class of all L-structures includes all finite structures and, for some $n \geqslant 0$, includes no structure containing a chain C of "busy choice sequences" with $|C| > n$, where $|C|$ is the cardinality of C.

Some explanations of the terms used in (1) and (2) seem necessary. "Generalized *refref* conditionals" are those sentences that indicate, say, *doing* implies (or is implied by) *refraining from refraining*, *refraining* implies (or is implied by) *refraining from refraining from refraining*, and so on, each of which amounts to a postulate concerning a certain relation between modes of actions/inactions. A "busy choice sequence" is like a "super task" that indicates a situation in which an agent has infinitely many choice points within a finite time. This notion of busy choice sequences, introduced in Belnap [2] and discussed in Belnap and Perloff [6] and Xu ([29], [30], and [33]), is central to a study of distinct modes of actions/inactions in *stit* theory. For instance, that there is no busy choice sequence is equivalent to that *doing* implies (or is implied by, or is equivalent to) *refraining from refraining from doing*, and hence there are only eight distinct modes of actions/inactions when there is no busy choice sequence.[3] Our study of compactness of the *stit* theories amounts to a study that answers the following questions (and incidentally, the answers are all negative).

(a) When doing is taken to imply (or to be implied by, or to be equivalent to) refraining from refraining, does a sentence following from a set of premises follow from finitely many of them?
(b) When we postulate that there is no busy choice sequence, does a sentence following from a set of premises follow from finitely many of them?
(c) When refraining is taken to imply (or to be implied by, or to be equivalent to) refraining from refraining from refraining, does a sentence following from a set of premises follow from finitely many of them?
(d) When we postulate that there is no chain C of busy choice sequences with $|C| > 1$, does a sentence following from a set of premises follow from finitely many of them?
(e) ...

Note that there are infinitely many *stit* logics satisfying the antecedent of (1) as well as that of (2). Note also that to establish (1), (2) alone is not enough. We need to show that

(3) each *stit* logic satisfying the antecedent of (1) satisfies that of (2).

Note, finally, that (1) provides a sufficient *syntactical* condition, whereas (2) provides a sufficient *semantic* condition, of uncompactness. We establish (1) through (2), and establish (2) through the use of "companions to *stit* sentences" applied in Xu ([29], [32], [34], and [35]), though the reader's familiarity with that notion is not presupposed. The notion of companions to *stit* sentences is a basic technical notion introduced to obtain syntactical characterizations of *astit* and *dstit* logics. In this paper, for the first time, we apply this notion in the context of chains of busy choice sequences to obtain the uncompactness results.

Uncompactness results in nonclassical logic are familiar phenomena. In modal logic, for example, KM, KW, and KW.3 are known to be uncompact (see Wang [28] and Hughes and Cresswell [18][4] and [17]); and in tense logic, the U,S-tense logic over integer time is an example of uncompact logics (see Reynolds [22]). There are, nevertheless, many modal logics and tense logics that are known to be compact, much more than those that are known to be uncompact. In contrast to this, only two *astit* logics are known to be compact—one is the minimal logic $\mathsf{L}_{\min}$, with a single agent, characterized by the class of all semantic structures, and the other is the largest consistent *stit* logic $\mathsf{L}_{\max}$, with a single agent, characterized by the class of all structures containing no choice points (see [35] for details).[5] This suggests that in the area of *astit*, unlike the area of modal logic or tense logic, or even that of *dstit* (see [31]), compactness might not be a common phenomenon.

Section 2 presents basic *stit* syntactical as well as semantic notions and defines other preliminary notions such as generalized *refref* conditionals and busy choice sequences. Section 3 proves (3), and Section 4 establishes both (1) and (2) above. And finally, Section 5 presents some remarks on our uncompactness results.

2 Preliminaries Although our results concerning "generalized *refref* conditionals" and uncompactness hold for *stit* theories with multiple agents, the formal language used in this paper contains only a single non-truth-functional operator $[\alpha\ stit\!:\ \ ]$ in addition to denumerably many propositional variables and the truth-functional operators $\sim$ and $\wedge$. Formulas are defined as usual, except that $[\alpha stit\!:\ A]$ is a formula whenever A is. We will use A, B, C, and so on, to range over formulas, and use Φ, Ψ, and so on, to range over sets of formulas. Ordinary truth-functional operators such as $\vee$, $\rightarrow$, and $\longleftrightarrow$, and propositional constants $\top$ and $\perp$ are introduced as abbreviations. We will use $[\alpha]A$ as an abbreviation of $[\alpha\ stit\!:\ A]$, and use A^{α} as an abbreviation of $A \wedge {\sim}[\alpha]A$.

Let A be any formula. *A-refraining formulas* (with respect to α) is defined as follows:

(i) $[\alpha]A$ is an A-refraining formula (with respect to α);
(ii) if B is an A-refraining formula (with respect to α), so is $[\alpha]{\sim}B$.

For each A-refraining formula B, the *A-refraining degree of B*, written $\mathsf{Rdg}^{A}(B)$, is defined in a parallel way:

(i) $\mathsf{Rdg}^{A}([\alpha]A) = 0$;
(ii) if B is an A-refraining formula and $\mathsf{Rdg}^{A}(B) = n$, $\mathsf{Rdg}^{A}([\alpha]{\sim}B) = n + 1$.

Let us fix a propositional variable q. The *q-refref equivalence* (*refref equivalence* for short) is the formula $[\alpha]q \longleftrightarrow [\alpha]{\sim}[\alpha]{\sim}[\alpha]q$. A ($q$-)*refref conditional* is either $[\alpha]q \rightarrow [\alpha]{\sim}[\alpha]{\sim}[\alpha]q$ or $[\alpha]{\sim}[\alpha]{\sim}[\alpha]q \rightarrow [\alpha]q$. A *generalized* ($q$-)*refref conditional* is any formula of the form $A \rightarrow [\alpha]{\sim}[\alpha]{\sim}A$ or $[\alpha]{\sim}[\alpha]{\sim}A \rightarrow A$, where A is a q-refraining formula. In this context, if $\mathsf{Rdg}^{q}(A) = n$, we call $A \rightarrow [\alpha]{\sim}[\alpha]{\sim}A$ and $[\alpha]{\sim}[\alpha]{\sim}A \rightarrow A$ generalized *refref* conditionals *with degree n*. Clearly, a *refref* conditional is a generalized *refref* conditional with degree 0.

Based on branching time theory proposed by Prior and Thomason (see [21], [24], and [25]), a *semantic structure for astit* (briefly, a *structure*) is any quintuple $\mathfrak{F} = \langle T, \leqslant, \mathit{Instant}, \mathit{Agent}, \mathit{Choice}\rangle$ satisfying the following postulates: $\langle T, \leqslant\rangle$ is a tree-like frame, that is, T is a nonempty set, whose elements w, m, and so on, are called *moments*, and $\leqslant$ is a partial order on T subject to *historical connection*, $\forall m\forall m'\exists w(w \leqslant m \wedge w \leqslant m')$, and *no downward branching*, $\forall m\forall w\forall w'(w \leqslant m \wedge w' \leqslant m \rightarrow w \leqslant w' \vee w' \leqslant w)$. We use $w < m$ for $w \leqslant m$ and $w \neq m$. A maximal chain h of moments in T (or a branch of the tree) is called a *history*, representing a possible course of history. We use H, H' and so on, to range over sets of histories, and for each $w \in T$, let $H_{(w)} = \{h\colon w \in h\}$. Two histories h and h' are *undivided at* w, written $h \equiv_w h'$, if and only if $\exists w'(w < w' \wedge w' \in h \cap h')$. *Instant*, whose elements i, i', and so on, are called *instants*, is a partition of T satisfying *unique intersection*, $\forall i\forall h\exists! m(\{m\} = i \cap h)$, and *order preservation*, $\forall i\forall i'\forall h\forall h'(m_{(i,h)} \leqslant m_{(i',h)} \longleftrightarrow m_{(i,h')} \leqslant m_{(i',h')})$ where $\{m_{(i,h)}\} = i \cap h$, and so on. We define $i_{(m)}$ to be the instant to which m belongs, and $w < i$ if and only if $\exists m(m \in i \wedge w < m)$. Provided $w < i$, we use $i|_{>w}$ for $\{m\colon m \in i \wedge w < m\}$. *Agent* is a nonempty set whose elements a, b, and so on, are called *agents*. *Choice* is a function on $\mathit{Agent} \times T$ such that for each $a \in \mathit{Agent}$ and each $w \in T$, $\mathit{Choice}(a, w)$ is a partition of $H_{(w)}$. Elements of $\mathit{Choice}(a, w)$ are called *possible choices for a at* w. h and h' are *choice equivalent for a at* w, written $h \equiv^a_w h'$, if and only if $\exists H(H \in \mathit{Choice}(a, w) \wedge h, h' \in H)$. Provided $m, m' \in i|_{>w}$, m and m' are *choice equivalent for a at* w, written $m \equiv^a_w m'$, if and only if $\exists h\exists h'(h \equiv^a_w h' \wedge \{m\} = i \cap h \wedge \{m'\} = i \cap h')$. The function *Choice* is subject to the conditions *no choice between undivided histories*, $\forall h\forall h'\forall a\forall w(h \equiv_w h' \rightarrow h \equiv^a_w h')$, and *independence of agents*: for each $w \in T$, and for each function s_w on *Agent* such that $s_w(a) \in \mathit{Choice}(a, w)$ for all $a \in \mathit{Agent}$, $\bigcap_{a\in \mathit{Agent}} s_w(a) \neq \varnothing$. An agent a has *vacuous choice at* w if and only if $\mathit{Choice}(a, w) = \{H_{(w)}\}$.

The following fact, which is a consequence of *no choice between undivided histories*, is established as §1.2 in Xu [30].

Fact 2.1 *Let $\mathfrak{F}$ be any structure in which $w' < w < m \in i$ and $m' \in i|_{>w}$. Then for every agent a, $m' \equiv^a_{w'} m$. And in particular, if $w' \leqslant w$ and $m' \equiv^a_w m$, then $m' \equiv^a_{w'} m$.*

A model $\mathfrak{M}$ on $\mathfrak{F}$ is a pair $\langle \mathfrak{F}, V\rangle$, where $\mathfrak{F} = \langle T, \leqslant, \mathit{Instant}, \mathit{Agent}, \mathit{Choice}\rangle$ is a structure, and V is a valuation assigning to each agent term α an agent $V(\alpha) = \bar{\alpha} \in \mathit{Agent}$,[6] and to each propositional variable a subset of $\{\langle m, h\rangle\colon m \in h\}$. That a formula A is *true in* $\mathfrak{M}$ *at* a moment/history pair $\langle m, h\rangle$ with $m \in h$, written $\mathfrak{M} \models_{m/h} A$, is defined recursively as follows, where $i = i_{(m)}$ and q is any propositional variable:

$\mathfrak{M} \models_{m/h} q$	iff	$\langle m, h\rangle \in V(q)$;
$\mathfrak{M} \models_{m/h} {\sim}A$	iff	$\mathfrak{M} \not\models_{m/h} A$ (not $\mathfrak{M} \models_{m/h} A$);
$\mathfrak{M} \models_{m/h} A \wedge B$	iff	$\mathfrak{M} \models_{m/h} A$ and $\mathfrak{M} \models_{m/h} B$;
$\mathfrak{M} \models_{m/h} [\alpha]A$	iff	there is a $w < m$ and an $m'' \in i|_{>w}$ such that

(i) $\forall m'\forall h'(m' \equiv^{\bar{\alpha}}_w m \wedge m' \in h' \rightarrow \mathfrak{M} \models_{m'/h'} A)$,
(ii) $\exists h''(m'' \in h'' \wedge \mathfrak{M} \not\models_{m''/h''} A)$.

With reference to the clause defining $\mathfrak{M} \models_{m/h} [\alpha]A$ above, we call (i) *the positive condition*, (ii) *the negative condition*, w a *witness to* $[\alpha]A$ *at* m, and m'' a *counter*. A is *settled true at* m in $\mathfrak{M}$, written $\mathfrak{M} \models_m A$, if and only if $\mathfrak{M} \models_{m/h} A$ for all h in $\mathfrak{M}$

with $m \in h$. For each set Φ of formulas, and for each model $\mathfrak{M}$, $\mathfrak{M} \models_{m/h} \Phi$ if and only if $\mathfrak{M} \models_{m/h} A$ for all $A \in \Phi$, and $\mathfrak{M} \models_m \Phi$ if and only if $\mathfrak{M} \models_{m/h} \Phi$ for every h in $\mathfrak{M}$ with $m \in h$. A (or Φ) *has a model* $\mathfrak{M}$ if $\mathfrak{M} \models_{m/h} A$ ($\mathfrak{M} \models_{m/h} \Phi$) for some m, h in $\mathfrak{M}$ with $m \in h$. $\mathfrak{F}$ is a *structure of* A, or simply an A-*structure* (a *structure of* Φ, or a Φ-*structure*), written $\mathfrak{F} \models A$ ($\mathfrak{F} \models \Phi$), if and only if $\mathfrak{M} \models_m A$ ($\mathfrak{M} \models_m \Phi$) for every model $\mathfrak{M}$ on $\mathfrak{F}$ and every m in $\mathfrak{M}$. If $\mathfrak{F} \models A$, we also say that A is *valid in* $\mathfrak{F}$.

The following are easily provable consequences of our semantic definitions and are useful for our upcoming discussions.

Fact 2.2 *Let $\mathfrak{M} = \langle T, \leqslant, Instant, Agent, Choice, V\rangle$ be any model. Then the following hold:*

1. *if $\mathfrak{M} \models_{m/h} [\alpha]A$, $\mathfrak{M} \models_m [\alpha]A$;*
2. *if $\mathfrak{M} \models_m [\alpha]A$, and if w is a witness to $[\alpha]A$ at m, then w is the unique witness to $[\alpha]A$ at m;*[7]
3. *if $\mathfrak{M} \models_m [\alpha]A$ with witness w, $Choice(\bar{\alpha}, w) \neq \{H_{(w)}\}$.*

It has been shown in [29] and [32] that each of the following A1 – A8 is valid in every structure, where A, B, C are any formulas.

A1	$\sim[\alpha]\top$
A2	$[\alpha]A \to A$
A3	$[\alpha]A \to [\alpha][\alpha]A$
A4	$[\alpha]A \wedge [\alpha]B \to [\alpha](A \wedge B)$
A5	$[\alpha]([\alpha]A \wedge B) \to [\alpha](A \wedge B)$
A6	$[\alpha](A \wedge B) \wedge \sim[\alpha]B \to [\alpha](A \wedge B^{\alpha})$
A7	$[\alpha](\sim[\alpha](A \wedge B) \wedge B^{\alpha}) \longleftrightarrow [\alpha](\sim[\alpha]A \wedge B^{\alpha})$
A8	$[\alpha]A \longleftrightarrow [\alpha](A \wedge B^{\alpha}) \vee [\alpha](A \wedge \sim[\alpha](A \wedge B^{\alpha}))$

A *stit logic with a single agent* (*stit logic* for short) is a set L of formulas that contains all instances of truth-functional tautologies and all instances of A1 – A8, and is closed under substitution, modus ponens, and RE (i.e., $A \longleftrightarrow B \in \mathsf{L}$ only if $[\alpha]A \longleftrightarrow [\alpha]B \in \mathsf{L}$). For each *stit* logic L, let us use $\mathfrak{C}(\mathsf{L})$ for the class of all L-structures, that is, $\mathfrak{C}(\mathsf{L}) = \{\mathfrak{F}\colon \mathfrak{F} \models \mathsf{L}\}$. For the smallest *stit* logic $\mathsf{L}_{\min} = \bigcap\{\mathsf{L}\colon \mathsf{L}$ is a *stit* logic$\}$, $\mathfrak{C}(\mathsf{L}_{\min})$ is the class of all structures (see [35]).[8] The smallest *stit* logic L_{refref} containing *refref* equivalence is referred to as the *stit* logic with *refref* equivalence, and is shown in [29] and [30] to be characterized by the class of all structures containing no "busy choice sequences" (to be defined below) and $\mathfrak{C}(\mathsf{L}_{refref}) = \{\mathfrak{F}\colon \mathfrak{F}$ contains no busy choice sequences$\}$.[9] The largest consistent *stit* logic $\mathsf{L}_{\max} = \bigcup\{\mathsf{L}\colon \mathsf{L}$ is a consistent *stit* logic$\}$ is characterized by the class of all frames containing no choice points, that is, frames containing no moments at which an agent has nonvacuous choice (see [35]).

Let $\mathfrak{C}$ be any class of structures. A formula A *follows from* a set Φ of formulas with respect to $\mathfrak{C}$, written $\Phi \models_{\mathfrak{C}} A$, if and only if for every $\mathfrak{F} \in \mathfrak{C}$, every $\mathfrak{M}$ on $\mathfrak{F}$, and every m, h in $\mathfrak{M}$ with $m \in h$, $\mathfrak{M} \models_{m/h} \Phi$ only if $\mathfrak{M} \models_{m/h} A$. For any $\mathfrak{C}$, $\models_{\mathfrak{C}}$ is *compact* if and only if whenever $\Phi \models_{\mathfrak{C}} A$, $\Psi \models_{\mathfrak{C}} A$ for some finite subset Ψ of Φ, or equivalently, for every set Φ of formulas, Φ has a model $\mathfrak{M}$ on an $\mathfrak{F} \in \mathfrak{C}$ if each finite subset Ψ of Φ has a model $\mathfrak{M}'$ on an $\mathfrak{F}' \in \mathfrak{C}$. Let L be any *stit* logic. We say that L is *compact* if $\models_{\mathfrak{C}(\mathsf{L})}$ is compact. The compactness of $\mathsf{L}_{\min}$ and $\mathsf{L}_{\max}$ are shown in [35]. We show in this paper that the following holds for every *stit* logic L.

(4) L is uncompact if $\mathsf{L} \subseteq \mathsf{L}_{refref}$ and L contains a generalized *refref* conditional.

In particular, L_{refref} is uncompact since L_{refref} contains $[\alpha]q \to [\alpha]{\sim}[\alpha]{\sim}[\alpha]q$ and $[\alpha]{\sim}[\alpha]{\sim}[\alpha]q \to [\alpha]q$.

As indicated at the beginning of this paper, we also want to establish a sufficient semantic condition of uncompactness. To that end, we need the notion of busy choice sequences. Let $\mathfrak{F} = \langle T, \leqslant, Agent, Instant, Choice\rangle$ be any structure with $a \in Agent$. A *busy a-choice sequence in* $\mathfrak{F}$ (or *in* $\mathfrak{M}$ on $\mathfrak{F}$) is an upper- and lower-bounded infinite chain of "*a*-choice points" (moments at which *a* has nonvacuous choice), as discussed in [2] and [6]. For our purpose, we define a *busy a-choice sequence in* $\mathfrak{F}$ (or *in* $\mathfrak{M}$ on $\mathfrak{F}$) as an upper- and lower-bounded chain of *a*-choice points that does not terminate in the upward direction, that is, a nonempty chain BC of moments such that

(i) $\exists w \exists m \forall w'(w' \in \mathsf{BC} \to w \leqslant w' < m)$,
(ii) $\forall w(w \in \mathsf{BC} \to Choice(a, w) \neq \{H_{(w)}\})$,
(iii) $\forall w(w \in \mathsf{BC} \to \exists w'(w' \in \mathsf{BC} \wedge w < w'))$.

We will fix a single agent for our discussion, and therefore we will speak of "busy choice sequences" rather than "busy *a*-choice sequences." We will use BC, BC', and so on, to range over busy choice sequences. Let $\mathsf{BC} < \mathsf{BC}'$ if and only if $\forall w \forall w'(w \in \mathsf{BC} \wedge w' \in \mathsf{BC}' \to w < w')$. A *chain* of busy choice sequences in $\mathfrak{F}$ (or in $\mathfrak{M}$) is defined in an obvious way. Let us use C, C', and so on, to range over chains of busy choice sequences. We use $w \leqslant \mathsf{BC}$ for $\forall w'(w' \in \mathsf{BC} \to w \leqslant w')$, $\mathsf{BC} < i$ for $\forall w(w \in \mathsf{BC} \to w < i)$, $w \leqslant \mathsf{C}$ for $\forall \mathsf{BC}(\mathsf{BC} \in \mathsf{C} \to w \leqslant \mathsf{BC})$, $\mathsf{C} < i$ for $\forall \mathsf{BC}(\mathsf{BC} \in \mathsf{C} \to \mathsf{BC} < i)$, and $\mathsf{BC} < \mathsf{C}$ for $\forall \mathsf{BC}'(\mathsf{BC}' \in \mathsf{C} \to \mathsf{BC} < \mathsf{BC}')$. $w < \mathsf{BC}$, $w < \mathsf{C}$, $\mathsf{BC} < w$, and $\mathsf{C} < w$, and so on, are defined in an obvious way. A *past* in a structure (or a model) is a nonempty set p of moments that is upper-bounded and closed downward. Let p be any past. $p < w$ if and only if $\forall w'(w' \in p \to w' < w)$, $p < i$ if and only if $p < m$ for some $m \in i$. We use $i|_{>p}$ for $\{m: m \in i \wedge p < m\}$ (provided $p < i$), and use, for each BC, p_{BC} for the smallest past including BC, that is, $p_{\mathsf{BC}} = \{w: \exists w'(w' \in \mathsf{BC} \wedge w \leqslant w')\}$. For each $n \geqslant 0$, $\|\mathfrak{F}\| = n$ if and only if there is a chain C of busy choice sequences in $\mathfrak{F}$ such that $|\mathsf{C}| = n$ and there is no chain C' of busy choice sequences in $\mathfrak{F}$ such that $|\mathsf{C}'| > n$.[10]

It has been shown in [2], [29], and [30] that the following hold for every structure $\mathfrak{F}$, where q is any propositional variable.

(5) $\|\mathfrak{F}\| = 0$ iff $\mathfrak{F} \models [\alpha]q \longleftrightarrow [\alpha]{\sim}[\alpha]{\sim}[\alpha]q$
iff $\mathfrak{F} \models [\alpha]q \to [\alpha]{\sim}[\alpha]{\sim}[\alpha]q$
iff $\mathfrak{F} \models [\alpha]{\sim}[\alpha]{\sim}[\alpha]q \to [\alpha]q$.

Let $\mathfrak{F} = \langle T, \leqslant, Instant, Agent, Choice\rangle$ be any structure. We say that $\mathfrak{F}$ is a *finite structure* if T is finite. A model $\mathfrak{M}$ on $\mathfrak{F}$ is a *finite model* if $\mathfrak{F}$ is a finite structure. Let $\mathfrak{C}_f$ be the class of all finite structures, and for each $n \geqslant 0$, let $\mathfrak{C}_n = \{\mathfrak{F}: \|\mathfrak{F}\| \leqslant n\}$. Obviously, $\mathfrak{C}_f \subseteq \mathfrak{C}_0$. In addition to establishing (4) above, we will establish the following for every *stit* logic L.

(6) L is uncompact if $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}_n$ for some $n \geqslant 0$.

Although (6) may not be equivalent to (4) above,[11] (6) does guarantee (4) if we can show that the following hold for every *stit* logic L.

(7) for each $n \geqslant 0$, $\mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}_n$ if L contains a *refref* generalized conditional with degree n;

and

(8) $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L})$ if $\mathsf{L} \subseteq \mathsf{L}_{refref}$.

But (8) is trivial, for by (5), $\mathfrak{C}_f \subseteq \mathfrak{C}_0 = \mathfrak{C}(\mathsf{L}_{refref})$. We thus only need to show (6) and (7) in order to establish (4). In Section 3 we prove that a generalized *refref* conditional with degree $n \geqslant 0$ is valid in a structure $\mathfrak{F}$ only if $\|\mathfrak{F}\| \leqslant n$, from which (7) follows. Then in Section 4 we prove (6) (Theorem 4.6) and then (4) (Theorem 4.7).

3 Busy choice sequences and generalized refref conditionals In this section, we prove that each chain C of busy choice sequences with $|\mathsf{C}| \geqslant n+1$ will suffice to invalidate all generalized *refref* conditionals $[\alpha]{\sim}[\alpha]{\sim}A \to A$ and $A \to [\alpha]{\sim}[\alpha]{\sim}A$ with q-refraining formulas A such that $\mathsf{Rdg}^q(A) \leqslant n$, from which (7) above follows. The main lemma in this section is Lemma 3.7. Since we have only one agent term α, we will use $m' \equiv_w m$ for $m' \equiv_w^{\bar{\alpha}} m$. The fact below has been shown in [30] and is useful later.

Fact 3.1 *Let $\mathfrak{M}$ be any model in which $p < i$ and let A be any formula. Suppose that $\mathfrak{M} \models_{i|_{>p}} A$, and that for every $w \in p$, there is a $w' \in p$ such that $w < w'$ and $\mathfrak{M} \not\models_{i|_{>w'}} A$. Then $\mathfrak{M} \models_{i|_{>p}} {\sim}[\alpha]A$.*

The following has been established as §2.3 in [30] which constitutes the base step of the induction in our proof of Lemma 3.7.

Fact 3.2 *Suppose that $\mathfrak{F}$ is any structure in which there is a busy choice sequence $\mathsf{BC} < m^* \in i$. Then there is a model $\mathfrak{M}$ on $\mathfrak{F}$ such that $\mathfrak{M} \models_i {\sim}[\alpha]{\sim}[\alpha]q$, and $\mathfrak{M} \not\models_i B$ and $\mathfrak{M} \not\models_i {\sim}B$ for every subformula B of $[\alpha]q$.*

Our proof of the main result in this section depends on a certain relation between structures and their substructures and between models and their submodels. A *substructure* of a structure $\mathfrak{F} = \langle T, \leqslant, \mathit{Instant}, \mathit{Agent}, \mathit{Choice}\rangle$ is any *structure* $\mathfrak{F}' = \langle T', \leqslant', \mathit{Instant}', \mathit{Agent}', \mathit{Choice}'\rangle$ satisfying the conditions $T' \subseteq T$, $\forall w \forall w'(w \in T' \wedge w' \in T \wedge w \leqslant w' \to w' \in T')$, ${\leqslant'} = {\leqslant} \cap (T' \times T')$, $\mathit{Instant}' = \{i' \colon \exists i(i \in \mathit{Instant} \wedge i' = i \cap T' \neq \varnothing)\}$, $\mathit{Agent}' = \mathit{Agent}$, and for every $a \in \mathit{Agent}'$ and $w \in T'$,

$$\begin{array}{rcl} \mathit{Choice}'(a, w) & = & \{f(H) \colon H \in \mathit{Choice}(a, w)\} \\ \text{where } f(H) & = & \{h' \colon \exists h(h \in H \wedge h' = h \cap T')\}. \end{array}$$

Note that if $\mathfrak{F}'$ is a substructure of $\mathfrak{F}$, then for any $a \in \mathit{Agent}' = \mathit{Agent}$ and any moment $w \in T'$, $m' \equiv_w^a m$ if and only if $m' \equiv_w'^a m$ for all $m, m' \in T$, where $m' \equiv_w'^a m$ means that in $\mathfrak{F}'$, m' and m are choice equivalent for a at w.

A model $\mathfrak{M}' = \langle \mathfrak{F}', V'\rangle$ is a *submodel* of a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ *with respect to an instant i'* in $\mathfrak{M}'$ if $\mathfrak{F}'$ is a substructure of $\mathfrak{F}$ and for each agent term α, $V'(\alpha) = V(\alpha)$; and for each $m \in i'$, each h in $\mathfrak{F}$ with $m \in h$, and each propositional variable q, $\langle m, h'\rangle \in V'(q)$ if and only if $\langle m, h\rangle \in V(q)$, where h' is the history in $\mathfrak{F}'$ such that $h' = h \cap T'$. The following has been obtained as §2.5 in [30].

Fact 3.3 *Let $\mathfrak{M}' = \langle T', <', Instant', Agent', Choice' \rangle$ be a submodel of $\mathfrak{M}$ with respect to an instant i' in $\mathfrak{M}'$, and let Γ be any set of formulas closed under subformulas. Suppose that for each agent term α and each formula A, $[\alpha]A \in \Gamma$ only if $\mathfrak{M}' \not\models_{i'} A$. Then for every formula $A \in \Gamma$, every $m \in i'$, and every h in $\mathfrak{M}$ with $m \in h$, $\mathfrak{M} \models_{m/h} A$ if and only if $\mathfrak{M}' \models_{m/h'} A$, where h' is in $\mathfrak{M}'$ with $h' = h \cap T'$.*

Let $\mathfrak{F}$ be any structure, and let T' be a subset of T and $\leqslant' = \leqslant \cap (T' \times T')$ such that *historical connection* and *closed-upwardness* are satisfied. Then T' determines a unique substructure $\mathfrak{F}'$ of $\mathfrak{F}$. Suppose that c is a *nonempty* chain in T. We use T_c for $\{w\colon w \in T \wedge \exists w'(w' \in c \wedge w' \leqslant w)$. It is easy to see that T_c determines a unique substructure $\mathfrak{F}'$ of $\mathfrak{F}$ such that $T' = T_c$. In this case, we use $\mathfrak{F}_c$ for $\mathfrak{F}'$. Similarly, given an instant i in $\mathfrak{F}$, we use i_c for $i \cap T_c$, and use $\mathfrak{M}_c$ for the submodel $\mathfrak{M}' = \langle \mathfrak{F}_c, V_c \rangle$ of $\mathfrak{M}$ with respect to i_c (ignoring the values of formulas at any m/h with $m \notin i$). It is easy to see by *no downward branching* that if $p < c$ (i.e., $\forall w(w \in c \to p < w)$), $i_c = i|_{>p} \cap T_c \subseteq i|_{>p}$.

In the induction step of our proof of the main lemma, there are three respects we need to consider. Given that $|\mathsf{C}| \geqslant n$, $\mathsf{BC} < \mathsf{C} < m^* \in i$ and $p = p_{\mathsf{BC}}$. It is easy to see that $i = i|_{>p} \cup s \cup s'$ and that $i|_{>p}$, s and s' are mutually disjoint, where $s = \{m\colon m \in i - i|_{>p} \wedge \forall w(w \in p \wedge w < m \to m \equiv_w m^*)\}$ and $s' = \{m\colon m \in i - i|_{>p} \wedge \exists w(w \in p \wedge w < m \wedge m \not\equiv_w m^*)\}$. In order to work out the desired values that formulas should have at moments in i, we will consider the desired values they have at moments in $i|_{>p}$, s, and s' separately. Lemma 3.4 handles the first, Lemma 3.5 the second, and Fact 3.6 the third.

Lemma 3.4 *Let A be any formula in which the only agent term occurring is α, let $\mathfrak{M}$ be any model in which $p < w^* < i$, and let $c = \{w\colon p < w \leqslant w^*\}$, $i_c = i \cap T_c$ and $s = \{m\colon m \in i|_{>p} - i_c \wedge \forall m' \forall w(m' \in i_c \wedge w \in p \to m' \equiv_w m)\}$. Suppose that (a) $\mathfrak{M} \not\models_{i_c} B$ for every subformula B of A, and (b) $\mathfrak{M} \models_s q$ or $\mathfrak{M} \models_s \sim q$ for each propositional variable q occurring in A. Then the following hold:*

(i) $\mathfrak{M} \models_s B$ *or* $\mathfrak{M} \models_s \sim B$ *for every subformula B of A;*
(ii) $\mathfrak{M} \models_s \sim[\alpha]B$ *for every subformula B of A.*[12]

Proof: By induction on the construction of B: the base case for (i) is provided by (b). The induction steps for $\sim$ and $\wedge$ are straightforward. It is thus sufficient to suppose that (i) holds for B and show that (ii) holds for B. Note first that $i_c \subseteq i|_{>p}$. Suppose for reductio that $\mathfrak{M} \models_m [\alpha]B$ for some $m \in s$ with witness w. Since $w < m$ and $m \in i|_{>p}$, we have by *no downward branching* that either $w \in p$ or $p < w$.

Case 1 ($w \in p$): By definition of s, $m' \equiv_w m$ for all $m' \in i_c$, and hence $\mathfrak{M} \models_{i_c} B$ since $\mathfrak{M} \models_m [\alpha]B$, contrary to (a).

Case 2 ($p < w$): Consider any counter m^* to $[\alpha]B$ at m. Then

(9) $\mathfrak{M} \not\models_{m^*} B$.

We show as follows that $m^* \in s$. First, since $p < w < m^*$, $m^* \in i|_{>p}$. Next, suppose for reductio that $m^* \in i_c$. Since $p < w < m^*$ and since $w_1 < m^*$ for some $w_1 \in c$, either $w \in c$ or $c < w$ by *no downward branching*. Hence in either case, $w \in T_c$. It follows that $m \in i_c$ since $w < m \in i|_{>p}$ and $i_c = i|_{>p} \cap T_c$, contrary to our assumption that $m \in s$. Hence $m^* \notin i_c$. Finally, consider any $m' \in i_c$ and any $w' \in p$. Since $m \equiv_{w'} m^*$ by Fact 2.1 and the fact that $w' \in p < w < m^*$, it is then clear that if $m^* \not\equiv_{w'} m'$, we would have $m \not\equiv_{w'} m'$, contrary to our assumption that $m \in s$. It follows that $\forall m' \forall w'(m' \in i_c \wedge w' \in p \to m' \equiv_{w'} m^*)$ holds, and hence $m^* \in s$. But we know, by (i) and the assumption that $\mathfrak{M} \models_m [\alpha]B$ with $m \in s$, that $\mathfrak{M} \models_s B$, and hence $\mathfrak{M} \models_{m^*} B$, contrary to (9).

Since we have contradictions in both cases, it follows that our first assumption, that is, $\mathfrak{M} \models_m [\alpha]B$ for $m \in s$, must be false. Hence, (ii) must hold for B. □

Lemma 3.5 *Let A be any formula in which the only agent term occurring is α, let $\mathfrak{M}$ be any model in which $p < m^* \in i$, and let $s = \{m\colon m \in i - i|_{>p} \wedge \forall w(w \in p \wedge w < m \to m \equiv_w m^*)\}$. Suppose that for every propositional variable q occurring in A, either $\mathfrak{M} \models_s q$ or $\mathfrak{M} \models_s {\sim}q$, and*

(10) for each $w \in p$, there is a $w' \in p$ such that $w < w'$ and $\mathfrak{M} \not\models_{i|_{>w'}} B$ for every subformula B of A.

Then the following hold:

(i) for every subformula B of A, either $\mathfrak{M} \models_s B$ or $\mathfrak{M} \models_s {\sim}B$;

(ii) for every subformula B of A, $\mathfrak{M} \models_s {\sim}[\alpha]B$.

Proof: Similar to our proof in Lemma 3.4, we suppose that (i) holds for B and prove that (ii) holds for B. Suppose for reductio that $\mathfrak{M} \models_m [\alpha]B$ for some $m \in s$ with witness w. There are two cases.

Case 1 ($w \in p$): Since $m \notin i|_{>p}$, it is then clear that there is a $w' \in p$ such that $w' \nless m$. It follows from our case assumption that $w < w'$, and hence by (10), there is an $m' \in i|_{>p}$ such that $w' < m'$ and $\mathfrak{M} \not\models_{m'} B$. But Fact 2.1 implies that $m' \equiv_w m^*$ and then, since $m \equiv_w m^*$ by definition of s, $m' \equiv_w m$, and hence $\mathfrak{M} \models_{m'} B$, a contradiction.

Case 2 ($w \notin p$): Consider any counter m'' to $[\alpha]B$ at m. Then

(11) $\mathfrak{M} \not\models_{m''} B$.

We show as follows that $m'' \in s$. First, since $w < m''$ and $w \notin p$, we have the following by *no downward branching*:

(12) for every $w'' \in p$ with $w'' < m''$, $w'' < w$.

Then by Fact 2.1, for every $w'' \in p$ with $w'' < m''$, $m'' \equiv_{w''} m$ since $w < m$ and $w < m''$, and hence, $m'' \equiv_{w''} m^*$ since $m \equiv_{w''} m^*$ ($m \in s$). Next, if $m'' \in i|_{>p}$, we would have $p < m''$, and hence by (12), $p < w$, which implies $m \in i|_{>p}$, contrary to our assumption that $m \in s$. Hence, $m'' \notin i|_{>p}$. It follows that $m'' \in s$. But we know, by (i) and the fact that $\mathfrak{M} \models_m [\alpha]B$ with $m \in s$, that $\mathfrak{M} \models_s B$, and hence $\mathfrak{M} \models_{m''} B$, contrary to (11). From this reductio we conclude that $\mathfrak{M} \models_s {\sim}[\alpha]B$. □

The following has been shown as §4.4 in Xu [33].

Fact 3.6 *Let D be any formula, let A be any D-refraining formula, and let $\mathfrak{M}$ be any model in which $p < m^* \in i$ and $s = \{m : m \in i - i|_{>p} \wedge \exists w(w \in p \wedge w < m \wedge m \not\equiv_w m^*\}$. Suppose that $\mathfrak{M} \not\models_{i|_{>p}} C$ and $\mathfrak{M} \not\models_{i|_{>p}} {\sim}C$ for each subformula C of*

A. Then, if $\mathfrak{M} \models_s D$ and $\mathsf{Rdg}^D(A)$ is odd, $\mathfrak{M} \models_s [\alpha]{\sim}A$; and if $\mathfrak{M} \models_s {\sim}D$ and $\mathsf{Rdg}^D(A)$ is even, $\mathfrak{M} \models_s [\alpha]{\sim}A$.

Now we are ready for the main lemma.

Lemma 3.7 *Let $\mathfrak{F}$ be any structure in which there is a chain C of busy choice sequences such that $|\mathsf{C}| \geqslant n+1$ and let A be a q-refraining formula such that $\mathsf{Rdg}^q(A) = n$. Then there is a model $\mathfrak{M}$ on $\mathfrak{F}$ such that for some i in $\mathfrak{M}$, $\mathfrak{M} \models_i {\sim}[\alpha]{\sim}A$, and $\mathfrak{M} \not\models_i B$ and $\mathfrak{M} \not\models_i {\sim}B$ for every subformula B of A.*

Proof: Assume that $|\mathsf{C}| = n+1$ (or else select a subchain of C with cardinality $n+1$). By definitions of busy choice sequences and instants, there is an i in $\mathfrak{F}$ such that $\mathsf{C} < i$. Our proof is by induction on n. Fact 3.2 has provided the base step for $n = 0$. We assume that $n \geqslant 1$ and the lemma holds for $n-1$ and show that it holds for n. Since $|\mathsf{C}| = n+1$ and $\mathsf{C} < i$, there are w^*, m^*, BC, and C_1 such that $\mathsf{BC} < w^* \leqslant \mathsf{C}_1 < m^* \in i$, $\{\mathsf{BC}\} \cup \mathsf{C}_1 = \mathsf{C}$, and $|\mathsf{C}_1| = n$. Let $p = p_{\mathsf{BC}}$ and $c = \{w\colon p < w \leqslant w^*\}$. Then c is a nonempty chain and $\mathfrak{F}_c$ is a substructure of $\mathfrak{F}$. It is easy to see that C_1 is a chain of busy choice sequences in $\mathfrak{F}_c$ and $\mathsf{C}_1 < m^* \in i_c = i \cap T_c$. Setting $A = [\alpha]{\sim}B$ with B to be a q-refraining formula such that $\mathsf{Rdg}^q(B) = n-1$, we know by $|\mathsf{C}_1| = n$ and the induction hypothesis that there is a model $\mathfrak{M}_c$ on $\mathfrak{F}_c$ such that $\mathfrak{M}_c \models_{i_c} {\sim}[\alpha]{\sim}B$, and $\mathfrak{M}_c \not\models_{i_c} C$ and $\mathfrak{M}_c \not\models_{i_c} {\sim}C$ for every subformula C of B, and hence

(13) $\mathfrak{M}_c \models_{i_c} {\sim}[\alpha]{\sim}B$, and $\mathfrak{M}_c \not\models_{i_c} C$ and $\mathfrak{M}_c \not\models_{i_c} {\sim}C$ for every subformula C of ${\sim}B$.

We can define a model $\mathfrak{M}$ on $\mathfrak{F}$ in such a way that for each $m \in i$ and each h in $\mathfrak{F}$ with $m \in h$, if $m \in i_c$, $\langle m, h\rangle \in V(q)$ if and only if $\langle m, h'\rangle \in V_c(q)$, where $h' = h \cap T_c$; if $m \in i|_{>p} - i_c$, $\langle m, h\rangle \in V(q)$; and if $m \in i - i|_{>p}$, $\langle m, h\rangle \in V(q)$ if and only if $\mathsf{Rdg}^q(A)$ is even. Then by (13) and Fact 3.3, we have $\mathfrak{M} \models_{i_c} {\sim}[\alpha]{\sim}B$, and $\mathfrak{M} \not\models_{i_c} C$ and $\mathfrak{M} \not\models_{i_c} {\sim}C$ for every subformula C of ${\sim}B$. Let $s_0 = i|_{>p} - i_c$. Consider any $m \in s_0$ ($\subseteq i|_{>p}$), any $m' \in i_c$ ($\subseteq i|_{>p}$) and any $w \in p$. Since $p = p_{\mathsf{BC}}$, there is a $w' \in p$ such that $w < w'$, and hence, since $w' < m$ and $w' < m'$, $m' \equiv_w m$ by Fact 2.1. It follows that $s_0 = \{m\colon m \in i|_{>p} - i_c \wedge \forall m' \forall w(m \in i_c \wedge w \in p \rightarrow m' \equiv_w m)\}$. By definition of V and Lemma 3.4 (substituting ${\sim}B$ here for A there), we have $\mathfrak{M} \models_{s_0} {\sim}[\alpha]{\sim}B$. It follows from $i|_{>p} = i_c \cup s_0$ that

(14) $\mathfrak{M} \models_{i|_{>p}} {\sim}[\alpha]{\sim}B$, and $\mathfrak{M} \not\models_{i|_{>p}} C$ and $\mathfrak{M} \not\models_{i|_{>p}} {\sim}C$ for every subformula C of ${\sim}B$.

Let $s = \{m\colon m \in i - i|_{>p} \wedge \exists w(w \in p \wedge w < m \wedge m \not\equiv_w m^*)\}$ and $s' = (i - i|_{>p}) - s' = \{m\colon m \in i - i|_{>p} \wedge \forall w(w \in p \wedge w < m \rightarrow m \equiv_w m^*)\}$. It is easy to see that $i = i|_{>p} \cup s \cup s'$. Since $\mathsf{Rdg}^q(A)$ is even if and only if $\mathsf{Rdg}^q(B)$ is odd, it is then clear by definition of V, (14), and Fact 3.6 that

(15) $\mathfrak{M} \models_s [\alpha]{\sim}B$ and $\mathfrak{M} \models_s {\sim}[\alpha]{\sim}[\alpha]{\sim}B$ (by truth definition).

Since $p = p_{\mathsf{BC}}$, it then follows, from (14), (15), and definition of busy choice sequences, that for each $w \in p$, there is a $w' \in p$ such that $w < w'$ and $\mathfrak{M} \not\models_{i|_{>w'}} C$ for every subformula C of ${\sim}[\alpha]{\sim}B$. Then by Lemma 3.5, we have $\mathfrak{M} \models_{s'} {\sim}[\alpha]{\sim}[\alpha]{\sim}B$; and by (14) and Fact 3.1, $\mathfrak{M} \models_{i|_{>p}} {\sim}[\alpha]{\sim}[\alpha]{\sim}B$, and hence, $\mathfrak{M} \models_i {\sim}[\alpha]{\sim}[\alpha]{\sim}B$

since $i = i|_{>p} \cup s \cup s'$. It is easy to verify that $\mathfrak{M} \not\models_i C$ and $\mathfrak{M} \not\models_i \sim C$ for every subformula C of $[\alpha]\sim B$. □

The following is our main result in this section.

Theorem 3.8 *Let $\mathfrak{F}$ be any structure with $\|\mathfrak{F}\| \geqslant n+1$ and let A be any q-refraining formula with $\mathrm{Rdg}^q(A) \leqslant n$. Then*

(i) $\mathfrak{F} \not\models B \to [\alpha]\sim[\alpha]\sim A$ and $\mathfrak{F} \not\models \sim B \to [\alpha]\sim[\alpha]A$ for every subformula B of A,

(ii) $\mathfrak{F} \not\models [\alpha]\sim[\alpha]\sim A \to B$ and $\mathfrak{M} \not\models [\alpha]\sim[\alpha]A \to \sim B$ for every subformula B of A.

Proof: Since $\|\mathfrak{F}\| \geqslant n+1$, there is a chain C of busy choice sequences in $\mathfrak{F}$ such that $|\mathsf{C}| \geqslant n+1$. (i) By Lemma 3.7, there is a model $\mathfrak{M}$ on $\mathfrak{F}$ with an instant i in $\mathfrak{F}$ such that, on the one hand, for each subformula B of A, there are some $m, m' \in i$ with $\mathfrak{M} \not\models_m B$ and $\mathfrak{M} \not\models_{m'} \sim B$; and on the other hand, $\mathfrak{M} \models_i \sim[\alpha]\sim A$, and hence $\mathfrak{M} \models \sim[\alpha]\sim[\alpha]\sim A$. It follows that (i) holds.

(ii) We may assume that $\mathsf{C} = \{\mathsf{BC}\} \cup \mathsf{C}'$ and $\mathsf{BC} < \mathsf{C}'$ and $\mathsf{BC} = \{w_0, w_1, \ldots\}$. Let $\mathsf{BC}' = \mathsf{BC} - \{w_0\}$, $\mathsf{C}'' = \{\mathsf{BC}'\} \cup \mathsf{C}'$, and $c = \{w\colon w_0 < w \leqslant w_1\}$. It is easy to see that C' is a chain of busy choice sequences in $\mathfrak{F}_c$ and $\left|\mathsf{C}''\right| \geqslant n+1$. By Lemma 3.7, there is a model $\mathfrak{M}_c$ on $\mathfrak{F}_c$ such that for some i_c in $\mathfrak{F}_c$,

(16) $\mathfrak{M}_c \models_{i_c} \sim[\alpha]\sim A$, and $\mathfrak{M}_c \not\models_{i_c} B$ and $\mathfrak{M}_c \not\models_{i_c} \sim B$ for each subformula B of A.

Let i be the instant in $\mathfrak{F}$ such that $i_c = i \cap T_c$. We define a model $\mathfrak{M}$ on $\mathfrak{F}$ in such a way that for each $m \in i$ and each h in $\mathfrak{F}$ with $m \in h$, if $m \in i_c$, $\langle m, h\rangle \in V(q)$ if and only if $\left\langle m, h'\right\rangle \in V'(q)$, where $h' = h \cap T_c$; and if $m \notin i_c$, $\langle m, h\rangle \in V(q)$ if and only if $\mathrm{Rdg}^q(A)$ is odd. Then by (16) and Fact 3.3,

(17) $\mathfrak{M} \models_{i_c} \sim[\alpha]\sim A$, and $\mathfrak{M} \not\models_{i_c} B$ and $\mathfrak{M} \not\models_{i_c} \sim B$ for each subformula B of A.

Let us fix an $m^* \in i_c$. Consider any $m \in i_c \subseteq i|_{>w_0}$. We know, by *no downward branching* and definitions of c and T_c, that there is a $w \in c$ such that $w_0 < w < m$ and $w < m^*$, and hence by Fact 2.1, $m \equiv_{w_0} m^*$. It follows that

(18) $i_c \subseteq s = \{m:\ m \in i|_{>w_0} \wedge m \equiv_{w_0} m^*\}$.

Let $s' = s - i_c = \{m\colon m \in i|_{>w_0} - i_c \wedge m \equiv_{w_0} m^*\}$. Consider any $m \in s'$, any $m' \in i_c$, and any $w' \leqslant w_0$. By (18), $m' \equiv_{w_0} m^*$, and hence, since $m \equiv_{w_0} m^*$, $m' \equiv_{w_0} m$. Fact 2.1 and $w' \leqslant w_0$ imply that $m' \equiv_{w'} m$. It follows that $s' = \{m\colon m \in i|_{>w_0} - i_c \wedge \forall m' \forall w'(m' \in i_c \wedge w' \leqslant w_0 \to m' \equiv_{w'} m)\}$, and hence by (17) and Lemma 3.4 (substituting $\{w'\colon w' \leqslant w_0\}$ for p there), $\mathfrak{M} \models_{s'} \sim[\alpha]\sim A$, and hence by (17) and $s = i_c \cup s'$,

(19) $\mathfrak{M} \models_s \sim[\alpha]\sim A$.

Let $s_1 = \{m\colon m \in i|_{>w_0} \wedge m \not\equiv_{w_0} m^*\}$. We show as follows that $\mathfrak{M} \models_{s_1} [\alpha]\sim A$. Our definition of $\mathfrak{M}$ implies that $\mathfrak{M} \models_{s_1} q$ if and only if $\mathrm{Rdg}^q(A)$ is odd, and $\mathfrak{M} \models_{s_1} \sim q$ if and only if $\mathrm{Rdg}^q(A)$ is even. Assume that $\mathrm{Rdg}^q(A)$ is odd. Then by (17) and truth definition, $\mathfrak{M} \models_{s_1} [\alpha]q$. It is sufficient to show by induction on $\mathrm{Rdg}^q(B)$ that for each *q-refraining* subformula B of $[\alpha]\sim A$, $\mathfrak{M} \models_{s_1} B$ if $\mathrm{Rdg}^q(B)$ is even. The base step has just been shown. Let $B = [\alpha]{\sim}C$ for some q-refraining formula C. Suppose that $\mathsf{Rdg}^q(B)$ is even. Then $\mathsf{Rdg}^q(C)$ is odd, and hence by induction hypothesis, $\mathfrak{M} \models_{s_1} {\sim}C$. Since C is a subformula of A, we know by (17) that $\mathfrak{M} \not\models_{i|_{>w_0}} {\sim}C$. It follows that $\mathfrak{M} \models_{s_1} B$. Suppose that $\mathsf{Rdg}^q(B)$ is odd. Then $\mathsf{Rdg}^q(C)$ is even, and then by induction hypothesis, $\mathfrak{M} \models_{s_1} C$. It follows from truth definition that $\mathfrak{M} \models_{s_1} {\sim}B$. A similar induction handles the case that $\mathsf{Rdg}^q(A)$ is even, which starts with the base case that $\mathfrak{M} \models_{s_1} {\sim}[\alpha]q$.

By definition of busy choice sequences, $Choice(\bar{\alpha}, w_0) \neq \{H_{(w_0)}\}$. Then $s_1 \neq \varnothing$, and hence, since $\mathfrak{M} \models_{s_1} [\alpha]{\sim}A$, $\mathfrak{M} \not\models_{i|_{>w_0}} {\sim}[\alpha]{\sim}A$. It follows from (19) that $\mathfrak{M} \models_s [\alpha]{\sim}[\alpha]{\sim}A$. It is easy to see by (17) and (18) that $\mathfrak{M} \not\models_s B$ and $\mathfrak{M} \not\models_s {\sim}B$ for every subformula B of A. $\square$

Theorem 3.8 implies that if $\|\mathfrak{F}\| \geqslant n+1$, and if A is a q-refraining formula with degree n, then $\mathfrak{F} \not\models [\alpha]{\sim}[\alpha]{\sim}A \to A$ and $\mathfrak{F} \not\models A \to [\alpha]{\sim}[\alpha]{\sim}A$. That is to say, no generalized *refref* conditional with degree n is valid in any $\mathfrak{F} \in \mathfrak{C}_{n+1}$. Thus we have the following.

Corollary 3.9 *Let* L *be any stit logic containing a generalized refref conditional with degree* $n \geqslant 0$*. Then* $\mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}_n$.

4 Uncompactness of some stit logics In this section we prove two sufficient conditions of uncompactness. The main idea of our proof is to use a feature of "companions to *stit* formulas." Let us briefly describe what a companion to a *stit* formula is. Let $\mathfrak{M} \models_m [\alpha]A$ with witness w, and let $s = \{m' \colon m' \in i_{(m)} \wedge m' \equiv_w m\}$. It is easy to see that $\mathfrak{M} \models_s [\alpha]A$. Consider any formula C. Since *stit* formulas are either settled true or settled false at every moment, we know that either $\mathfrak{M} \models_m [\alpha](A \wedge C^\alpha)$ or $\mathfrak{M} \models_m {\sim}[\alpha](A \wedge C^\alpha)$. In fact,

(20) if $\mathfrak{M} \models_m [\alpha](A \wedge C^\alpha)$, $\mathfrak{M} \models_s [\alpha](A \wedge C^\alpha)$; and if $\mathfrak{M} \models_m {\sim}[\alpha](A \wedge C^\alpha)$, $\mathfrak{M} \models_s {\sim}[\alpha](A \wedge C^\alpha)$.

In the former case, we call $[\alpha](A \wedge C^\alpha)$ a *pos-companion to* $[\alpha]A$ *at* m, and C a *pos-companion root of* $[\alpha]A$ *at* m; and in the latter case, we call ${\sim}[\alpha](A \wedge C^\alpha)$ a *neg-companion to* $[\alpha]A$ *at* m, and C a *neg-companion root of* $[\alpha]A$ *at* m. Both pos-companions and neg-companions to $[\alpha]A$ at m are *companions to* $[\alpha]A$ *at* m. Because (20) holds for every formula C, we know that $[\alpha]A$ must be true together with all its companions at every $m' \in i_{(m)}$ choice equivalent to m for α at w. One feature of companions to $[\alpha]A$ is that they are in general not consequences (semantic or deductive) of $[\alpha]A$ but are nevertheless true together with $[\alpha]A$. Thus when we study syntactical characterizations of *stit* theories, we should not only consider consequences of *stit* formulas, but rather, we should also take these companions into consideration, as can be seen in [29], [32], [34], and [35]. Another feature of companions to *stit* formulas is the following, which we will use in this section. Although in our language there is no explicit tense operator, companions to *stit* formulas provide a sufficient condition for determining the temporal order between witnesses to *stit* formulas. To be more precise, let $\mathfrak{M} \models_m [\alpha]A$ with witness w, and let $m' \in i|_{>w}$ (possibly $m = m'$) and $\mathfrak{M} \models_{m'} [\alpha]B$ with witness w'. Then, if any formula C is a pos-companion root of $[\alpha]A$ at m but a neg-companion root of $[\alpha]B$ at m', then we must have $w' < w$.

In this section, we use the second feature of companions mentioned above to construct a set Φ of formulas in such a way that all formulas in Φ are companions to some *stit* formulas, which will, when joined together, force each model of Φ to contain a chain C of busy choice sequences with $|\mathsf{C}| \geqslant n$ for every $n \geqslant 0$ (Lemma 4.2). After showing that each finite subset of Φ has a finite model (Lemma 4.5), we arrive at a sufficient semantic condition of uncompactness (Theorem 4.6), and combining the result presented in the previous section, a sufficient syntactical condition of uncompactness (Theorem 4.7).

The following is useful and has been established in [29], where Fact 4.1(iv) is called the Companion Theorem in [29], and Fact 4.1(v) presents the feature of companions we will use in this section.

Fact 4.1 *Let $\mathfrak{M} = \langle T, \leqslant, Instant, Agent, Choice, V\rangle$ be any model in which $w < m$ and $i = i_{(m)}$. Then the following hold, where $s = \{m' \colon m' \in i \wedge m' \equiv_w m\}$:*

- *(i) if $\mathfrak{M} \models_s B$ and $\mathfrak{M} \models_m [\alpha]A$ with witness w, $\mathfrak{M} \models_m [\alpha](A \wedge B)$;*
- *(ii) if $\mathfrak{M} \models_s A$ and $\mathfrak{M} \not\models_s [\alpha]A$, $\mathfrak{M} \models_{i|_{>w}} A^\alpha$; and thus, if $\mathfrak{M} \models_m [\alpha](A \wedge B)$ with witness w and $\mathfrak{M} \not\models_s [\alpha]B$, $\mathfrak{M} \models_{i|_{>w}} B^\alpha$ and $\mathfrak{M} \models_s [\alpha](A \wedge B^\alpha)$;*
- *(iii) if $\mathfrak{M} \models_m [\alpha](A \wedge B^\alpha)$ with witness w, $\mathfrak{M} \models_m [\alpha]A$ with the same witness; and if $\mathfrak{M} \models_m [\alpha](A \wedge {\sim}[\alpha](A \wedge B^\alpha))$ with witness w, $\mathfrak{M} \models_m [\alpha]A$ with the same witness;*
- *(iv) if $\mathfrak{M} \models_m [\alpha]A$ with witness w, then for each B, either $\mathfrak{M} \models_m [\alpha](A \wedge B^\alpha)$ with witness w or $\mathfrak{M} \models_m [\alpha](A \wedge {\sim}[\alpha](A \wedge B^\alpha))$ with witness w;*
- *(v) if $\mathfrak{M} \models_m [\alpha](A \wedge C^\alpha)$ with witness w, and if $m' \in i|_{>w}$ and $\mathfrak{M} \models_{m'} [\alpha](B \wedge {\sim}[\alpha](B \wedge C^\alpha))$ with witness w', then $w' < w$.*[13]

Let us arrange all propositional variables into two disjoint sets $\Sigma = \{p_\xi \colon 0 \leqslant \xi < \omega \times \omega\}$ and $\Pi = \{q_\xi \colon 0 \leqslant \xi < \omega \times \omega\}$. For every ξ with $0 \leqslant \xi < \omega \times \omega$, let $A_\xi = [\alpha](p_\xi \wedge q_\xi^\alpha)$, and let $B_\xi = {\sim}[\alpha](p_\xi \wedge q_{\xi+1}^\alpha)$. Let $\Phi_0 = \{A_k \colon 0 \leqslant k < \omega\} \cup \{B_k \colon 0 \leqslant k < \omega\}$, and for each $n \geqslant 0$, let

$$\begin{aligned}\Phi_{n+1} \;=\;& \Phi_n \cup \{A_\xi : \ \omega \times (n+1) \leqslant \xi < \omega \times (n+2)\}\\ & \cup \{B_\xi : \ \omega \times (n+1) \leqslant \xi < \omega \times (n+2)\}\\ & \cup \{{\sim}[\alpha](p_\zeta \wedge q^\alpha_{\omega\times(n+1)}) : \ 0 \leqslant \zeta < \omega \times (n+1)\}.\end{aligned}$$

Finally, let us fix $\Phi = \bigcup_{0\leqslant n<\omega} \Phi_n$.

Lemma 4.2 *Let $n \geqslant 0$, and let $\mathfrak{M}$ be any model in which $\mathfrak{M} \models_m \Phi_n$. Then there is a chain $w_0 < w_1 < \cdots < w_\xi < w_{\xi+1} < \cdots < m$ (where $0 \leqslant \xi < \omega \times (n+1)$) of moments in $\mathfrak{M}$ such that for each ξ with $0 \leqslant \xi < \omega \times (n+1)$, $\mathfrak{M} \models_m [\alpha](p_\xi \wedge q_\xi^\alpha)$ with witness w_ξ.*

Proof: Our proof is by induction on n.

Case 1 ($n = 0$): Since $\mathfrak{M} \models_m \Phi_0$, we have by hypothesis and the definition of Φ_0 that there is a w_0 such that $\mathfrak{M} \models_m [\alpha](p_0 \wedge q_0^\alpha)$ with witness w_0. Suppose that $k \geqslant 0$, and that we have

(21) $w_0 < \cdots < w_k$ such that for each j with $0 \leqslant j \leqslant k$, $\mathfrak{M} \models_m [\alpha](p_j \wedge q_j^\alpha)$ with witness w_j.

We show below how to select w_{k+1} in such a way that (21) holds with k replaced by $k+1$. It follows from (21) and Fact 4.1(iii) that

(22) $\mathfrak{M} \models_m [\alpha]p_k$ with witness w_k.

By hypothesis and the definition of Φ_0, we know that

(23) $\mathfrak{M} \models_m \sim[\alpha](p_k \wedge q^\alpha_{k+1})$

and

(24) there is a w_{k+1} such that $\mathfrak{M} \models_m [\alpha](p_{k+1} \wedge q^\alpha_{k+1})$ with witness w_{k+1}.

Applying Fact 4.1(iv) to (22) and (23), we have

(25) $\mathfrak{M} \models_m [\alpha](p_k \wedge \sim[\alpha](p_k \wedge q^\alpha_{k+1}))$ with witness w_k,

and applying Fact 4.1(v) to (24) and (25), we have $w_k < w_{k+1}$. It follows that there is a chain $w_0 < w_1 < \cdots < m$ such that for each k with $0 \leqslant k < \omega$, $\mathfrak{M} \models_m [\alpha](p_k \wedge q^\alpha_k)$ with witness w_k.

Case 2 ($n+1$): Assume that $\mathfrak{M} \models_m \Phi_{n+1}$. Since $\Phi_n \subseteq \Phi_{n+1}$, we know by induction hypothesis that there is a chain $w_0 < w_1 < \cdots < w_\xi < w_{\xi+1} < \cdots < m$ such that for each ξ with $0 \leqslant \xi < \omega \times (n+1)$, $\mathfrak{M} \models_m [\alpha](p_\xi \wedge q^\alpha_\xi)$ with witness w_ξ. In particular, we know by Fact 4.1(iii) that for each ζ with $\omega \times n \leqslant \zeta < \omega \times (n+1)$, $\mathfrak{M} \models_m [\alpha]p_\zeta$ with witness w_ζ. Since $\mathfrak{M} \models_m \Phi_{n+1}$, we have by the definition of Φ_{n+1} that $\mathfrak{M} \models_m \sim[\alpha](p_\zeta \wedge q^\alpha_{\omega\times(n+1)})$ for each ζ with $0 \leqslant \zeta < \omega \times (n+1)$. It follows from Fact 4.1(iv) that

(26) for each ζ with $0 \leqslant \zeta < \omega \times (n+1)$, $\mathfrak{M} \models_m [\alpha](p_\zeta \wedge \sim[\alpha](p_\zeta \wedge q^\alpha_{\omega\times(n+1)}))$ with witness w_ζ.

Applying the definition of Φ_{n+1}, we have that $\mathfrak{M} \models_m [\alpha](p_{\omega\times(n+1)} \wedge q^\alpha_{\omega\times(n+1)})$ with some witness $w_{\omega\times(n+1)}$, and hence by (26) and Fact 4.1(v), $w_\zeta < w_{\omega\times(n+1)}$ for all ζ with $0 \leqslant \zeta < \omega \times (n+1)$. The same argument in *Case* $n = 0$ will handle the rest of our proof, except that we need to replace 0, j, k, and so on, by $\omega \times (n+1)$, $\omega \times (n+1) + j$, $\omega \times (n+1) + k$, and so on. □

Corollary 4.3 *Let $n \geqslant 0$ and let $\mathfrak{M} = \langle T, \leqslant, Instant, Agent, Choice, V\rangle$ be any model in which $\mathfrak{M} \models_m \Phi_n$. Then there is a chain C of busy choice sequences in $\mathfrak{M}$ such that $|\mathsf{C}| = n+1$.*

Proof: By Lemma 4.2 there is a chain $w_0 < w_1 < \cdots < w_\xi < w_{\xi+1} < \cdots < m$ of moments in $\mathfrak{M}$ such that for each ξ with $0 \leqslant \xi < \omega \times (n+1)$, $\mathfrak{M} \models_m [\alpha](p_\xi \wedge q^\alpha_\xi)$ with witness w_ξ. It follows from Fact 2.2(iii) that $Choice(\bar{\alpha}, w_\xi) \neq \{H_{(w_\xi)}\}$ for each ξ with $0 \leqslant \xi < \omega \times (n+1)$. For each k with $0 \leqslant k \leqslant n$, let us define $\mathsf{BC}_k = \{w_{\omega\times k}, w_{\omega\times k+1}, \ldots\}$. By definition, each BC_k is clearly a busy choice sequence and for each k with $0 \leqslant k < n$, $\mathsf{BC}_k < \mathsf{BC}_{k+1}$. Setting $\mathsf{C} = \{\mathsf{BC}_0, \ldots, \mathsf{BC}_n\}$, we have that $|\mathsf{C}| = n+1$. □

Applying this corollary and our definitions of Φ, $\|\mathfrak{F}\|$, and $\mathfrak{C}_n$, we obtain the following.

Corollary 4.4 *For each model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, $\mathfrak{M}$ is a model for Φ only if $\|\mathfrak{F}\| > n$ for every $n \geqslant 0$, that is, only if $\mathfrak{F} \notin \mathfrak{C}_n$ for every $n \geqslant 0$.*

Our next step is to show that every finite subset of Φ has a finite model (Lemma 4.5 below). To that end, we use the following easily verifiable fact: for each finite subset Φ' of Φ, there are $\xi_1, \ldots, \xi_n$ with $n > 1$ such that $0 \leqslant \xi_1 < \cdots < \xi_n < \omega \times \omega$, and, setting $\Psi = \{[\alpha](p_{\xi_1} \wedge q^{\alpha}_{\xi_1}), \ldots, [\alpha](p_{\xi_n} \wedge q^{\alpha}_{\xi_n})\} \cup \{\sim[\alpha](p_{\xi_j} \wedge q^{\alpha}_{\xi_k})\colon 0 \leqslant j < k \leqslant n\}$, Ψ is a finite extension of Φ'.

Lemma 4.5 *Each finite subset of $\Phi = \bigcup_{0 \leqslant n < \omega} \Phi_n$ has a finite model.*

Proof: By the fact mentioned above, it is sufficient to let $0 \leqslant \xi_1 < \cdots < \xi_n < \omega \times \omega$ and show that there is a model $\mathfrak{M}$ with a moment m^* in it such that

(27) $\mathfrak{M} \models_{m^*} [\alpha](p_{\xi_k} \wedge q^{\alpha}_{\xi_k})$ for each k with $1 \leqslant k \leqslant n$,

and

(28) $\mathfrak{M} \models_{m^*} \bigwedge_{1 \leqslant j \leqslant k} \sim[\alpha](p_{\xi_j} \wedge q^{\alpha}_{\xi_{k+1}})$ for each k with $1 \leqslant k < n$.

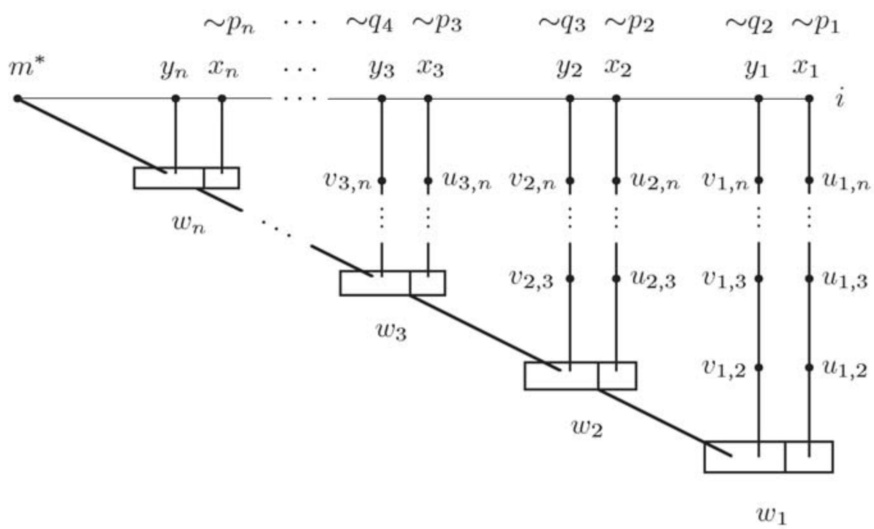
For convenience, let us use p_k for p_{ξ_k}, q_k for q_{ξ_k}, and so on, in the following discussion. Let $\mathfrak{F} = \langle T, \leqslant, \mathit{Instant}, \mathit{Agent}, \mathit{Choice} \rangle$ be defined as follows, where m^*, x_ks, y_ks, w_ks, $u_{k,j}$s, and $v_{k,j}$s are all different.

$$
\begin{array}{lcl}
T & = & \{m^*\} \cup \{w_k\colon 1 \leqslant k \leqslant n\} \\
 & & \cup \{y_k\colon 1 \leqslant k \leqslant n\} \cup \{x_k\colon 1 \leqslant k \leqslant n\} \\
 & & \cup \{u_{k,j}\colon 1 \leqslant k < j \leqslant n\} \cup \{v_{k,j}\colon 1 \leqslant k < j \leqslant n\}. \\
\leqslant & = & \{\langle w, w \rangle\colon w \in T\} \cup \{\langle w_k, w_j \rangle\colon 1 \leqslant k < j \leqslant n\} \\
 & & \cup \{\langle w_k, m^* \rangle\colon 1 \leqslant k \leqslant n\} \\
 & & \cup \{\langle u_{k,j}, u_{k,j'} \rangle\colon 1 \leqslant k < j < j' \leqslant n\} \\
 & & \cup \{\langle v_{k,j}, v_{k,j'} \rangle\colon 1 \leqslant k < j < j' \leqslant n\} \\
 & & \cup \{\langle w_k, u_{k',j} \rangle\colon 1 \leqslant k \leqslant k' < j \leqslant n\} \\
 & & \cup \{\langle w_k, v_{k',j} \rangle\colon 1 \leqslant k \leqslant k' < j \leqslant n\} \\
 & & \cup \{\langle w_k, x_{k'} \rangle\colon 1 \leqslant k \leqslant k' \leqslant n\} \\
 & & \cup \{\langle w_k, y_{k'} \rangle\colon 1 \leqslant k \leqslant k' \leqslant n\} \\
 & & \cup \{\langle u_{k,j}, x_k \rangle\colon 1 \leqslant k < j \leqslant n\} \\
 & & \cup \{\langle v_{k,j}, y_k \rangle\colon 1 \leqslant k < j \leqslant n\}. \\
\mathit{Instant} & = & \{\{w_k\} \cup \{u_{j,k}\colon 1 \leqslant j < k\} \cup \{v_{j,k}\colon 1 \leqslant j < k\}\colon 1 \leqslant k \leqslant n\} \\
 & & \cup \{\{m^*\} \cup \{x_k\colon 1 \leqslant k \leqslant n\} \cup \{y_k\colon 1 \leqslant k \leqslant n\}\}. \\
\mathit{Agent} & = & \{a\}.
\end{array}
$$

Let $i = \{m^*\} \cup \{x_k\colon 1 \leqslant k \leqslant n\} \cup \{y_k\colon 1 \leqslant k \leqslant n\}$. It is easy to check that i is the last instant in *Instant*. For each $m \in i$, let us use h_m for the unique history passing through m. Thus $h_{m^*} = \{w_1, \ldots, w_n, m^*\}$, and for each k with $1 \leqslant k \leqslant n$, $h_{x_k} = \{w\colon w \in T \wedge w \leqslant x_k\} = \{w_1, \ldots, w_k, u_{k,k+1}, \ldots, u_{k,n}, x_k\}$, and $h_{y_k} = \{w\colon w \in T \wedge w \leqslant y_k\} = \{w_1, \ldots, w_k, v_{k,k+1}, \ldots, v_{k,n}, y_k\}$. We define *Choice* as follows:

$$
\begin{array}{lcl}
\mathit{Choice}(a, m) & = & \{\{h_m\}\} \text{ for each } m \in i; \\
\mathit{Choice}(a, u_{k,j}) & = & \{\{h_{x_k}\}\} \text{ for each } k,\ j \text{ with } 1 \leqslant k < j \leqslant n; \\
\mathit{Choice}(a, v_{k,j}) & = & \{\{h_{y_k}\}\} \text{ for each } k,\ j \text{ with } 1 \leqslant k < j \leqslant n; \\
\mathit{Choice}(a, w_k) & = & \{\{h_{x_k}\}, H_{(w_k)} - \{h_{x_k}\}\} \text{ for each } k \text{ with } 1 \leqslant k \leqslant n.
\end{array}
$$

It is easy to see that for each k with $1 \leqslant k \leqslant n$, $H_{\langle w_k \rangle} - \{h_{x_k}\} = \{h_{y_k}, h_{x_{k+1}}, h_{y_{k+1}}, \ldots, h_{x_n}, h_{y_n}, h_{m^*}\}$. Finally, we define $\mathfrak{M}$ by letting V be an assignment such that $V(\alpha) = a$, and for each k with $1 \leqslant k \leqslant n$, $V(p_k) = \{\langle m, h\rangle : m \in h \cap i \wedge m \neq x_k\}$, $V(q_1) = \{\langle m, h\rangle : m \in i \wedge m \in h\}$, and for each k with $1 < k \leqslant n$, $V(q_k) = \{\langle m, h\rangle : m \in h \cap i \wedge m \neq y_{k-1}\}$. In the following diagram illustrating $\mathfrak{M}$, we only indicate at which moment p_k or q_k is (settled) false. That is to say, at each moment $m \in i$, if p_k or q_k with $1 \leqslant k \leqslant n$ is not indicated to be (settled) false at m, it should be understood that it is (settled) true at m.



It is easy to see from our definition of $\mathfrak{M}$ that for each k with $1 \leqslant k \leqslant n$, $x_k \not\equiv_{w_k} m^*$, $\mathfrak{M} \models_{i-\{x_k\}} p_k$, and $\mathfrak{M} \not\models_{x_k} p_k$. It follows that

(29) for each k with $1 \leqslant k \leqslant n$, $\mathfrak{M} \models_{m^*} [\alpha]p_k$ with witness w_k.

It is also easy to see from the definition of $\mathfrak{M}$ that

(30) for each k with $1 \leqslant k \leqslant n$, $\mathfrak{M} \models_{i|_{>w_k}} q_k$.

Consider any k with $1 \leqslant k \leqslant n$. If $k = 1$, we know by the definition of $\mathfrak{M}$ that $\mathfrak{M} \models_i q_k$ and hence $\mathfrak{M} \models_{m^*} {\sim}[\alpha]q_k$. Assume that $k > 1$. We know by the definition of $\mathfrak{F}$ that $y_{k-1} \equiv_{w_{k-1}} m^*$ and $\mathfrak{M} \not\models_{y_{k-1}} q_k$, and hence $\mathfrak{M} \not\models_s q_k$ where $s = \{m : m \in i \wedge m \equiv_{w_{k-1}} m^*\}$. It follows from (30), Fact 2.1, and our definition above that for each $w < m^*$, either $\mathfrak{M} \models_{i|_{>w}} q_k$ or $\mathfrak{M} \not\models_{s'} q_k$ where $s' = \{m : m \in i \wedge m \equiv_w m^*\}$, and hence $\mathfrak{M} \models_{m^*} {\sim}[\alpha]q_k$. We thus have

(31) for each k with $1 \leqslant k \leqslant n$, $\mathfrak{M} \models_{m^*} {\sim}[\alpha]q_k$.

To show that (27) holds, consider any k with $1 \leqslant k \leqslant n$. Clearly, $\{m : m \in i \wedge m \equiv_{w_k} m^*\} \subseteq i|_{>w_k}$. We then know by (30), (31), and Fact 4.1(ii) that $\mathfrak{M} \models_{i|_{>w_k}} q_k^{\alpha}$, and hence by (29) and Fact 4.1(i), $\mathfrak{M} \models_{m^*} [\alpha](p_k \wedge q_k^{\alpha})$. It follows that (27) holds. To show that (28) holds, let $1 \leqslant k < n$. We show that $\mathfrak{M} \models_{m^*} \bigwedge_{1 \leqslant j \leqslant k} {\sim}[\alpha](p_j \wedge q_{k+1}^{\alpha})$. Suppose for reductio that there is a j such that $1 \leqslant j \leqslant k$ and $\mathfrak{M} \models_{m^*} [\alpha](p_j \wedge q_{k+1}^{\alpha})$ with witness w. Then by Fact 4.1(iii), $\mathfrak{M} \models_{m^*} [\alpha]p_j$ with witness w. It follows from (29) and Fact 2.2(ii) that $w = w_j$. The hypothesis of this reductio implies that

$\mathfrak{M} \models_{s_j} q_{k+1} \wedge \sim[\alpha]q_{k+1}$, where $s_j = \{m: m \in i \wedge m \equiv_{w_j} m^*\}$. Hence by Fact 4.1(ii), $\mathfrak{M} \models_{i|_{>w_j}} q_{k+1}$. Since $j \leqslant k$, $y_k \in i|_{>w_k} \subseteq i|_{>w_j}$. It follows that $\mathfrak{M} \models_{y_k} q_{k+1}$. But by our definition of $\mathfrak{M}$, $\mathfrak{M} \not\models_{y_k} q_{k+1}$, a contradiction. It follows from this reductio that $\mathfrak{M} \models_{m^*} \bigwedge_{1\leqslant j\leqslant k} \sim[\alpha](p_j \wedge q^{\alpha}_{k+1})$. Hence (28) holds. □

Now we establish our two sufficient conditions of uncompactness.

Theorem 4.6 *Let* L *be any stit logic. Then,* L *is uncompact if* $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}_n$ *for some* $n \geqslant 0$*.*

Proof: Suppose that $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}_n$ for some $n \geqslant 0$. Consider the set Φ of formulas. By Lemma 4.5 we know that each finite subset Ψ of Φ has a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ with $\mathfrak{F} \in \mathfrak{C}(\mathsf{L})$, but by Corollary 4.4 Φ has no model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ with $\mathfrak{F} \in \mathfrak{C}(\mathsf{L})$. It follows that $\models_{\mathfrak{C}(\mathsf{L})}$ is uncompact, that is, L is uncompact. □

Theorem 4.7 *Let* L *be any stit logic. Then,* L *is uncompact if* $\mathsf{L} \subseteq \mathsf{L}_{refref}$ *and* L *contains a generalized refref conditional.*

Proof: If $\mathsf{L} \subseteq \mathsf{L}_{refref}$, $\mathfrak{C}(\mathsf{L}_{refref}) \subseteq \mathfrak{C}(\mathsf{L})$. We know that $\mathfrak{C}(\mathsf{L}_{refref}) = \mathfrak{C}_0$ (see [29] and [30]), and it is trivially true that $\mathfrak{C}_f \subseteq \mathfrak{C}_0$. It follows that if $\mathsf{L} \subseteq \mathsf{L}_{refref}$ then $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L})$. If in addition L contains a generalized *refref* conditional, we know by Corollary 3.9 that $\mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}_n$ for some $n \geqslant 0$. It then follows from Theorem 4.6 that L is uncompact. □

5 *Remark* We first show that there are infinitely many *stit* logics that satisfy the antecedent of Theorem 4.7 and hence are uncompact. Let A_0 be the conjunction of $[\alpha]q \to [\alpha]\sim[\alpha]\sim[\alpha]q$ and $[\alpha]\sim[\alpha]\sim[\alpha]q \to [\alpha]q$. For each $n \geqslant 0$, let $A_{n+1} = A_n(\sim[\alpha]q/q)$, that is, the result of substituting $\sim[\alpha]q$ for q in A_n, and let L_n be the smallest *stit* logic containing A_n. Clearly, each A_n is the conjunction of two generalized *refref* conditionals with degree n and each L_{n+1} is a sublogic of L_n. Since L_0 is L_{refref}, it follows from Theorem 4.7 that each L_n is uncompact. But are they all different? By §3.7 in Xu [33], on the one hand, for each structure $\mathfrak{F}$ such that $\mathsf{Cdg}(\mathfrak{F}) \leqslant n$ (where $\mathsf{Cdg}(\mathfrak{F})$ is the complexity degree of $\mathfrak{F}$ defined in [33]), $\mathfrak{F} \models A_m$ for all $m \geqslant n$. By §4.8 in [33], on the other hand, for each $n \geqslant 0$, there is a structure $\mathfrak{F}_n$ such that $\mathsf{Cdg}(\mathfrak{F}_n) = n + 1$ and $\mathfrak{F} \not\models A_n$. Consequently, for each $n \geqslant 0$, there is a structure $\mathfrak{F}$ such that $\mathfrak{F} \not\models A_n$ but $\mathfrak{F} \models A_m$ for all $m > n$. It follows that for each $n \geqslant 0$, $\mathsf{L}_n \neq \mathsf{L}_{n+1}$, and hence L_{n+1} is a proper sublogic of L_n. This completes the verification of our claim that infinitely many *stit* logics satisfy the antecedent of Theorem 4.7.

Consider again our Theorem 4.6 that provides a sufficient semantic condition of uncompactness, that is,

(32) L is uncompact if $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}_n$ for some $n \geqslant 0$.

We can actually generalize (32) in two directions—one is to find some $\mathfrak{C}' \subset \mathfrak{C}_f$ and the other is to find some $\mathfrak{C}$ such that $\mathfrak{C}_n \subset \mathfrak{C}$ for all $n \geqslant 0$—and thus obtain some more general conditions of uncompactness:

(33) L is uncompact if $\mathfrak{C}' \subseteq \mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}$.

For instance, it is easy to see that our proof in Lemma 4.5 actually shows that every finite subset of Φ has a finite model whose background structure is *at-most-binary*, where a structure $\mathfrak{F} = \langle T, \leqslant, Instant, Agent, Choice\rangle$ is *at-most-binary* if for each $a \in Agent$ and each $w \in T$, $|Choice(a, w)| \leqslant 2$. We can thus use $\mathfrak{C}'$ to be the class of all finite at-most-binary structures. Some other suitable proper subclass of $\mathfrak{C}_f$ can be found as well. The other direction in generalizing (32) seems less trivial. So far we have only considered $\mathfrak{C}_n$ for some $n \geqslant 0$, that is, the class of structures $\mathfrak{F}$ with $\|\mathfrak{F}\| \leqslant n$. Since it is easy to see that $\bigcup_{0\leqslant n<\omega} \mathfrak{C}_n$ is the class of all structures containing no infinite chain of busy choice sequences, Corollary 4.4 enables us to obtain at most that

(34) L is uncompact if $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L}) \subseteq \bigcup_{0\leqslant n<\omega} \mathfrak{C}_n$.

There are, nevertheless, two ways to generalize (34) without extending our language. The first is to use "ordinal-isomorphism" to distinguish structures containing infinite chains of busy choice sequences. Let ξ be any ordinal such that $|\xi| = \omega$ and let C be any chain of busy choice sequences. $\mathsf{C} \approx \xi$ if and only if there is an isomorphism between C and ξ with respect to the $<$-relation on C and the ordinary $<$-relation (or $\subset$-relation) on ξ (i.e., the busy choice sequences contained in C are arranged according to the order type specified by ξ). $[\![\mathfrak{F}]\!] = \xi$ if and only if there is a chain C of busy choice sequences in $\mathfrak{F}$ such that $\mathsf{C} \approx \xi$, and there is no chain C' of busy choice sequences in $\mathfrak{F}$ such that $\mathsf{C}' \approx \xi + 1$. Finally, let $\mathfrak{C}_\xi = \{\mathfrak{F}\colon [\![\mathfrak{F}]\!] \leqslant \xi\}$. Clearly, $\bigcup_{0\leqslant n<\omega} \mathfrak{C}_n \subset \mathfrak{C}_\xi$ for every ξ with $|\xi| = \omega$. We can obtain that

(35) L is uncompact if $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}_\xi$ for some ξ with $|\xi| = \omega$

by adjusting our definition of Φ in the following way: given ξ with $|\xi| = \omega$. We first arrange all propositional variables into two disjoint sets $\Sigma_\xi = \{p_\zeta\colon 0 \leqslant \zeta < \omega \times (\xi + 1)\}$ and $\Pi_\xi = \{q_\zeta\colon 0 \leqslant \zeta < \omega \times (\xi + 1)\}$. Then for every ζ with $0 \leqslant \zeta < \omega \times (\xi + 1)$, let $A_\zeta = [\alpha](p_\zeta \wedge q^\alpha_\zeta)$ and $B_\zeta = {\sim}[\alpha](p_\zeta \wedge q^\alpha_{\zeta+1})$. Let $\Phi_0 = \{A_k\colon 0 \leqslant k < \omega\} \cup \{B_k\colon 0 \leqslant k < \omega\}$; and for each ζ with $0 \leqslant \zeta < \omega \times (\xi + 1)$, let

$$\begin{aligned} \Phi_{\zeta+1} \;=\; & \Phi_\zeta \cup \{A_\eta\colon \omega \times (\zeta + 1) \leqslant \eta < \omega \times (\zeta + 2)\} \\ & \cup\{B_\eta\colon \omega \times (\zeta + 1) \leqslant \eta < \omega \times (\zeta + 2)\} \\ & \cup\{{\sim}[\alpha](p_\eta \wedge q^\alpha_{\omega\times(\zeta+1)})\colon 0 \leqslant \eta < \omega \times (\zeta + 1)\}; \end{aligned}$$

and for each limit ordinal ζ with $\omega \leqslant \zeta \leqslant \omega \times (\xi + 1)$, let

$$\begin{aligned} \Phi_\zeta \;=\; & (\textstyle\bigcup_{\eta\in\zeta} \Phi_\eta) \cup \{A_\eta\colon \omega \times \zeta \leqslant \eta < \omega \times (\zeta + 1)\} \\ & \cup\{B_\eta\colon \omega \times \zeta \leqslant \eta < \omega \times (\zeta + 1)\} \\ & \cup\{{\sim}[\alpha](p_\eta \wedge q^\alpha_{\omega\times\zeta})\colon 0 \leqslant \eta < \omega \times \zeta\}. \end{aligned}$$

It is easy to see that an argument similar to that in our proof of Lemma 4.2, with an exception of the case concerning limit ordinals, will show that $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ is a model for Φ_ξ only if $[\![\mathfrak{F}]\!] \geqslant \xi + 1$. Then we obtain (35) above by applying the same argument as that in the proof of Lemma 4.5.

The second way to generalize (34), without extending our language, is to use "reversed ordinal-isomorphism" to distinguish structures containing infinite chains of busy choice sequences. Let ξ be any ordinal such that $|\xi| = \omega$ and let C be any chain of busy choice sequences. $\mathsf{C} \cong \xi$ if and only if there is an isomorphism between C

and ξ with respect to the reverse of $<$-relation among busy choice sequences contained in C and the ordinary $<$-relation on ξ. $[\![\mathfrak{F}]\!]^r = \xi$ if and only if there is a chain C of busy choice sequences in $\mathfrak{F}$ such that $\mathsf{C} \cong \xi$ and there is no chain C' of busy choice sequences in $\mathfrak{F}$ such that $\mathsf{C}' \cong \xi + 1$. Finally, let $\mathfrak{C}^r_\xi = \{\mathfrak{F}\colon [\![\mathfrak{F}]\!]^r \leqslant \xi\}$. Clearly, $\bigcup_{n \geqslant 0} \mathfrak{C}_n \subset \mathfrak{C}^r_\xi$ for every ξ with $|\xi| = \omega$. In order to establish that

(36) L is uncompact if $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}^r_\xi$ for some ξ with $|\xi| = \omega$,

we need to define a Ψ_ξ in a way similar to the way we defined Φ_ξ above. More precisely, we use the same strategy but arrange the pos-companion roots and neg-companion roots in such a fashion that forces every model for Ψ_ξ to contain an infinite chain of busy choice sequences that does not terminate toward the direction of past rather than future. Let Σ_ξ, Π_ξ, A_ζ, B_ζ be as specified above, and let $\Psi_0 = \Phi_0$. For each ζ with $0 \leqslant \zeta < \omega \times (\xi + 1)$, let

$$\begin{aligned}\Psi_{\zeta+1} \;=\; & \Psi_\zeta \cup \{A_\eta\colon \omega \times (\zeta+1) \leqslant \eta < \omega \times (\zeta+2)\}\\ & \cup \{B_\eta\colon \omega \times (\zeta+1) \leqslant \eta < \omega \times (\zeta+2)\}\\ & \cup \{[\alpha](p_\eta \wedge q^\alpha_\varsigma)\colon 0 \leqslant \eta < \omega \times (\zeta+1)\\ & \wedge\, \omega \times (\zeta+1) \leqslant \varsigma < \omega \times (\zeta+2)\};\end{aligned}$$

and for each limit ordinal ζ with $\omega \leqslant \zeta \leqslant \omega \times (\xi + 1)$, let

$$\begin{aligned}\Psi_\zeta \;=\; & (\textstyle\bigcup_{\eta \in \zeta} \Psi_\zeta) \cup \{A_\eta\colon \omega \times \zeta \leqslant \eta < \omega \times (\zeta+1)\}\\ & \cup \{B_\eta\colon \omega \times \zeta \leqslant \eta < \omega \times (\zeta+1)\}\\ & \cup \{[\alpha](p_\eta \wedge q^\alpha_\varsigma)\colon 0 \leqslant \eta < \omega \times \zeta \wedge \omega \times \zeta \leqslant \varsigma < \omega \times (\zeta+1)\}.\end{aligned}$$

It can be shown that every model for Ψ_ξ contains a chain C of busy choice sequences such that $\mathsf{C} \cong \xi + 1$, and it can also be shown that each finite subset of Ψ_ξ has a finite model, and hence (36) holds. Details are omitted.

Our method applied in this paper has a limit. As the reader may have realized, although we can handle the situations in which chains of busy choice sequences are countable, there is no way to apply our strategy here to deal with situations in which there are chains of busy choice sequences that are uncountable, unless we extend the language. If we take order types into considerations, the following are among the farthest we can reach by applying the strategy here:

if $|\mathsf{PV}| = \kappa$, then L is uncompact if $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}_\xi$ for some ξ with $|\xi| = \kappa$,

if $|\mathsf{PV}| = \kappa$, then L is uncompact if $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}^r_\xi$ for some ξ with $|\xi| = \kappa$;

where PV is the set of propositional variables in the object language, and κ is any cardinal, and $\mathfrak{C}_\xi$ and $\mathfrak{C}^r_\xi$ are just like what we defined above, but replacing ω by κ.

NOTES

1. See [2] or [23] for a historical review.

2. The names *bstit* and *cstit* are found in [16], [14], and [15], etc. Note that Horty and Belnap used *bstit* and *cstit* as approximations of the operators that Chellas and Brown proposed.

3. This corresponds to the result that over the class of *stit* structures containing no busy choice sequences, there are only ten distinct *stit* modalities, where a *stit* modality is a sequence of $[\alpha]$ and $\sim$, and two such modalities σ and τ are distinct over a class $\mathfrak{C}$ of *stit* structures if $\sigma p \longleftrightarrow \tau p$ is not valid in all *stit* structures in $\mathfrak{C}$. These *stit* modalities, when we write $[\alpha]$ as $\Box$, are exactly the same modalities as those in modal logic S4.2 (see [12]), though the two structures of modalities are different.

4. I am not sure who is the first person who proved that KW is uncompact. One can find such a proof in [18], while in both [18] and [17], Hughes and Cresswell noted that the idea of the proof there was suggested by Fine.

5. For axioms and rules of inference in $\mathsf{L}_{\min}$, see Section 1. $\mathsf{L}_{\max}$ can be axiomatized by taking $\sim[\alpha]p$ as the only (modal) axiom and taking modus ponens and substitution as rules of inference. It has been shown in [35] that every consistent *stit* logic is a sublogic of $\mathsf{L}_{\max}$, which makes $\mathsf{L}_{\max}$ the only *stit* logic that is post complete.

6. That is to say, we use $\bar{\alpha}$ for $V(\alpha)$, where $V(\alpha) \in \mathit{Agent}$.

7. The uniqueness of witness follows immediately from the Witness Identity Lemma in [11]. It also follows from our Fact 2.1 above, as has been shown in [33].

8. In [32], this logic is axiomatized as taking all $\mathsf{A1}$ – $\mathsf{A8}$ as axiom schemata and taking modus ponens, RE and another rule RS as rules of inference. [35] eliminates the rule RS. There is a gap in the proof presented in [32]. A modified proof can be found in [8].

9. In [29], $\mathsf{L}_{\mathit{refref}}$ is shown to be decidable and is axiomatized with an extra axiom $[\alpha](\sim[\alpha](A \wedge [\alpha](B \wedge \sim[\alpha](B \wedge C^{\alpha}))) \wedge C^{\alpha}) \rightarrow [\alpha]B$. Because it is easy to verify that this formula is a theorem of $\mathsf{L}_{\min}$, we conclude that $\mathsf{L}_{\min}$ plus *refref* equivalence is deductively equivalent to the logic given in [29].

10. For a discussion of busy choice sequences and a measure of complexity of chains of busy choice sequences, the reader is referred to §2 in [33].

11. For all we know, a *stit* logic L satisfying the antecedent of (6) may not satisfy that of (4)—there may be, e.g., an L containing $[\alpha]q \rightarrow [\alpha]\sim[\alpha]\sim[\alpha]q$ and some formula not contained in $\mathsf{L}_{\mathit{refref}}$ such that each finite structure is an L-structure, and thus $\mathfrak{C}(\mathsf{L}) \subseteq \mathfrak{C}_0$ (see (5) on p. 6) and $\mathfrak{C}_f \subseteq \mathfrak{C}(\mathsf{L})$. It is not clear now whether the two *refref* conditionals are "deductively equivalent" or whether each proper extension L of $\mathsf{L}_{\mathit{refref}}$ does not take all finite structures as L-structures.

12. Regarding its application in the main lemma, Lemma 3.4 might have been formulated in such a way that we replace "$s = \{m\colon m \in i|_{>p} - i_c \wedge \forall m' \forall w(m' \in i_c \wedge w \in p \rightarrow m' \equiv_w m)\}$" by "$s = i|_{>w} - i_c$ and p has no greatest element". We formulate Lemma 3.4 the way it is now because we need to apply it to Theorem 3.8, under a situation different from that in the main lemma.

13. See §2.3, §2.4, §2.6, §2.8, and §2.9 in [29].

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Department of Philosophy
Indiana University at Bloomington
Bloomington, IN 47405
email: mxu@indiana.edu
mxu@indiana.edu

Cycorp Inc.
3721 Executive Center Dr.
Suite 100
Austin, TX 78731
mingxu@cyc.com
mingxu01@hotmail.com