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Powers of 2

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Abstract It is shown that in **ZF** Martin's $\aleph_0$-axiom together with the axiom of countable choice for finite sets imply that arbitrary powers $\mathbf{2}^X$ of a 2-point discrete space are Baire; and that the latter property implies the following: (a) the axiom of countable choice for finite sets, (b) power sets of infinite sets are Dedekind-infinite, (c) there are no amorphous sets, and (d) weak forms of the Kinna-Wagner principle.

1 Introduction As is well known, in **ZF** (i.e., Zermelo-Fraenkel set theory without the axiom of choice) products of compact Hausdorff spaces may fail to be compact or Baire. In fact the following hold.

Theorem 1.1 (Rubin and Scott [8], Łos and Ryll-Nardzewski [6]) *Products of compact Hausdorff spaces are compact if and only if the Boolean prime ideal theorem holds.*

Theorem 1.2 (Herrlich and Keremedis [4]) *Products of compact Hausdorff spaces are Baire if and only if the axiom of dependent choice holds.*

It is further known that Theorem 1.1 remains valid if attention is restricted to powers of the discrete space **2** whose underlying set is $2 = \{0, 1\}$.

Theorem 1.3 (Mycielski [7]) *Powers $\mathbf{2}^X$ are compact if and only if the Boolean prime ideal theorem holds.*

The natural question whether Theorem 1.2 remains valid, too, if attention is restricted to powers $\mathbf{2}^X$ has been left unanswered. In this paper we will show that the axiom of countable choice for finite sets together with Martin's $\aleph_0$-axiom suffice to prove

Baire $\mathbf{2}^*$: *All powers $\mathbf{2}^X$ are Baire*.

We will further present several set theoretic conditions that are necessary to deduce **Baire**$(\mathbf{2}^*)$; in particular, the axiom of countable choice for finite sets. However, we are not able to present a set theoretic condition that is equivalent to **Baire** $(\mathbf{2}^*)$.

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2 Terminology Let **2** be the discrete topological space with underlying set $2 = \{0, 1\}$. For any set X let $\mathbf{2}^X$ be the topological product of X copies of **2**. Let Y be a subset of X: For $f \in 2^X$ let f_Y be the restriction of f to Y and let $\pi_Y : 2^X \to 2^Y$ be the Yth projection, defined by $\pi_Y(f) = f_Y$. Let $|Y|$ be the cardinality of Y, if Y is finite, and ∞ otherwise. Let $P(2^X)$ be the set of all partial maps $f: X \to 2$. For $f \in P(2^X)$ let $\mathrm{dom}(f)$ be the domain of f and $B(f) = \pi^{-1}_{\mathrm{dom}(f)}(f)$. Let $P_{\mathrm{fin}}(2^X)$ be the set of all $f \in P(2^X)$ with finite domain. Let $\mathfrak{B} = \{B(f) \mid f \in P_{\mathrm{fin}}(2^X)\}$ be the canonical base for $\mathbf{2}^X$.

Definition 2.1 Consider the following axioms in **ZF** (form numbers refer to Howard and Rubin [5]).

Baire($\mathbf{2}^*$) All powers $\mathbf{2}^X$ are Baire.

BPI Boolean prime ideal theorem [Form 14]; that is, every Boolean algebra with $0 \neq 1$ has a prime ideal.

DC Axiom of dependent choice [Form 43]; that is, for each relation ϱ on a nonempty set X that satisfies the condition

$$\forall x \in X \ \exists y \in X \ \ x \varrho y$$

there exists a sequence (x_n) in X with $x_n \varrho x_{n+1}$ for each n.

DMC Axiom of dependent multiple choice [Form 106]; that is, for each relation ϱ on a nonempty set X that satisfies the condition

$$\forall x \in X \ \exists y \in X \ x \varrho y$$

there exists a sequence (F_n) of nonempty finite subsets of X satisfying the condition

$$\forall n \in \omega \ \ \forall x \in F_n \ \ \exists y \in F_{n+1} \ \ x \varrho y.$$

AC($\aleph_0$) Axiom of countable choice [Form 8]; that is, products $\prod_{n \in \mathbb{N}} X_n$ of sequences (X_n) of nonempty sets are nonempty.

CMC Axiom of countable multiple choice [Form 126]; that is, for each sequence (X_n) of nonempty sets X_n there exists a sequence (F_n) of nonempty finite subsets F_n of X_n.

AC($\aleph_0, < \aleph_0$) Axiom of countable choice for finite sets [Form 10]; that is, products $\prod_{n \in \mathbb{N}} F_n$ of sequences (F_n) of nonempty finite sets are nonempty.

MA($\aleph_0$) Martin's $\aleph_0$-axiom: for every nonempty partially ordered set $(P, \leq)$ with the property that any subset in which any 2 different elements have no common lower bound is at most countable, and for any sequence (D_n) of subsets of P such that $\forall n \ \ \forall x \in P \ \ \exists y \in D_n \ \ y \leq x$ holds there exists a filter on P that meets every D_n [Form 8F; cf. Remark 3.2 below].

Ded($\mathbf{2}^*$) If X is infinite then 2^X is Dedekind infinite [Form 82].

$\mathrm{KW}(\aleph_0, \infty)$	For every sequence (X_n) of infinite sets there exists a sequence (Y_n) of nonempty proper subsets Y_n of X_n [cf. Form 357].
PKW	Partial Kinna-Wagner Selection Principle [Form 379]; that is, for each infinite family $(X_i)_{i \in I}$ of sets X_i with $\lvert X_i \rvert \geq 2$ there exist an infinite subset K of I and a family $(Y_k)_{k \in K}$ of nonempty proper subsets Y_k of X_k.
A	There are no amorphous sets [Form 64]; that is, each infinite set is the disjoint union of two infinite sets.

Definition 2.2 A topological space X is called *Baire* provided that X is either empty or in X countable intersections of dense, open sets are nonempty.

3 Results

Theorem 3.1 *The conjunction of* $\mathbf{AC}(\aleph_0, < \aleph_0)$ *and* $\mathbf{MA}(\aleph_0)$ *implies* $\mathbf{Baire}(\mathbf{2}^*)$.

Proof: Assume $\mathbf{AC}(\aleph_0, < \aleph_0)$ and $\mathbf{MA}(\aleph_0)$, let X be a set, and let (D_n) be a sequence of dense, open subsets of $\mathbf{2}^X$. Call elements f and g of $P(\mathbf{2}^X)$ *compatible* provided that they have a common extension: that is, if and only if $f_{\mathrm{dom}\, f \cap \mathrm{dom} g} = g_{\mathrm{dom}\, f \cap \mathrm{dom} g}$. We claim

(∗) Any set A of pairwise incompatible elements of $P_{\mathrm{fin}}(\mathbf{2}^X)$ is at most countable.

In fact, if the sequence (k_n) of natural numbers is defined by $k_0 = 1$ and $k_{n+1} = 1 + (n+1) \cdot k_n$, then for any set A of pairwise incompatible elements of $P_{\mathrm{fin}}(\mathbf{2}^X)$ and for any natural number n the set $A_n = \{ f \in A \mid \lvert \mathrm{dom}(f) \rvert = n\}$ has at most k_n elements. This follows via induction, since for $f \in A_{n+1}$, and each $x \in \mathrm{dom}(f)$ the set $\{g \in A_{n+1} \mid g(x) \neq f(x)\}$ has at most k_n elements, hence A_{n+1} at most $1 + (n+1) \cdot k_n$ elements. Thus, by $\mathbf{AC}(\aleph_0, < \aleph_0)$, the set A is—as a countable union of finite sets A_n—at most countable. Thus (*) holds.

For each $n \in \mathbb{N}$, define $E_n = \{ f \in P_{\mathrm{fin}}(\mathbf{2}^X) \mid B(f) \subset D_n\}$. Then for each $n \in \mathbb{N}$ and for each $f \in P_{\mathrm{fin}}(\mathbf{2}^X)$ there exists an extension of f in E_n. This fact, together with (∗) implies via $\mathbf{MA}(\aleph_0)$ that there exists a subset F of $P_{\mathrm{fin}}(\mathbf{2}^X)$ with the following properties:

(1) Any two elements of F are compatible.

(2) F meets each E_n.

In view of (1), there exists in $P(\mathbf{2}^X)$ a common extension f of all elements of F. In view of (2), $B(f) \subset D_n$ for each n. Thus $g \colon X \to 2$, defined by

$$g(x) = \begin{cases} f(x), & \text{if } x \in \mathrm{dom}(f) \\ 0, & \text{otherwise} \end{cases}$$

is an element of each D_n. □

Remark 3.2 It has been claimed that $\mathbf{AC}(\aleph_0)$ implies $\mathbf{MA}(\aleph_0)$. (See Shannon [9], p. 382 and [5], Form 8 $\Longleftrightarrow$ Form 8F.) However, the relation between $\mathbf{AC}(\aleph_0)$

and **MA**($\aleph_0$) is still unknown. Form 8F has been changed to Form 339. (See http:www.math.purdue.edu/~jer/cgi-bin/changes.html, changes and additions.)

Theorem 3.3 **Baire**$(\mathbf{2}^*)$ *implies* **Ded**$(\mathbf{2}^*)$.

Proof: Let X be an infinite set. Choose $Y = X \times \mathbb{N}$ and consider $\mathbf{2}^Y$. For each $n \in \mathbb{N}$, the set

$$D_n = \{ f \in 2^Y \mid \exists k > n \ \ \forall m \le n \ \ \exists x \in X \ \ f(x,k) \ne f(x,m) \}$$

is open and dense in $\mathbf{2}^Y$. By **Baire**$(\mathbf{2}^*)$ there exists some f in $\cap D_n$. Define, via induction, a strictly increasing sequence (k_n) of natural numbers as follows:

$$\begin{aligned} k_0 &= 0 \\ k_{n+1} &= \min\{k > k_n \mid \forall m \le k_n \ \ \exists x \in X \ \ f(x,k) \ne f(x,m)\}. \end{aligned}$$

Then the map $\varphi\colon \mathbb{N} \to 2^X$, defined by $\varphi(n) = f(-, k_n)$, is injective. Thus, 2^X is Dedekind-infinite. □

Remark 3.4 The above theorem implies that the conditions **Baire**$(\mathbf{2}^*)$ (= powers of **2** are Baire) and **Tych**$(\mathbf{2}^*)$ (= powers of **2** are compact; = **BPI**) are independent of each other. In Feferman's model (M2 in [5]) **DC** hence **Baire**$(\mathbf{2}^*)$ hold, but **BPI** and thus **Tych**$(\mathbf{2}^*)$ fail. In Mostowski's Linearly Ordered Model (N3 in [5]) **BPI** and thus **Tych**$(\mathbf{2}^*)$ hold, but **Ded**$(\mathbf{2}^*)$ and thus **Baire**$(\mathbf{2}^*)$ fail.

Theorem 3.5 **Baire**$(\mathbf{2}^*)$ *implies* **AC**$(\aleph_0, < \aleph_0)$.

Proof: Let (X_n) be a sequence of nonempty, finite sets. Assume, without loss of generality, that the X_n are pairwise disjoint, and form $X = \cup X_n$. For each $n \in \mathbb{N}$, the set

$$D_n = \{ f \in 2^X \,\Big|\, \exists m \ge n \ |X_m \cap f^{-1}(1)| = 1\}$$

is open and dense in $\mathbf{2}^X$. By **Baire**$(\mathbf{2}^*)$ there exists some f in $\cap D_n$. The set $M = \{m \in \mathbb{N} \,\Big|\, |X_m \cap f^{-1}(1)| = 1\}$ is unbounded and the product $\prod_{m \in M} X_m$ is nonempty. Thus **PAC** $(\aleph_0, < \aleph_0)$, the partial axiom of countable choice for finite sets, holds. Thus (Brunner [2]) **AC**$(\aleph_0, < \aleph_0)$ holds. □

Corollary 3.6 *The following conditions are equivalent:*

(1) Products of compact Hausdorff spaces are Baire.
(2) Compact Hausdorff spaces and spaces of the form $\mathbf{2}^X$ *are Baire.*

Proof: Obviously (1) implies (2). The reverse implication follows immediately from Theorem 3.5 and the following facts: (1) is equivalent to **DC** ([4]. Compact Hausdorff spaces are Baire if and only if **DMC** holds (Fossy and Morillon [3]). **DC** is equivalent to the conjunction of **DMC** and **AC**$(\aleph_0, < \aleph_0)$ (Blass [1]). □

Remark 3.7 In $\mathbf{ZF}^0$, set theory with atoms, **Baire**$(\mathbf{2}^*)$ is not implied by **DMC**. This follows from Theorem 3.5 since in the second Fraenkel Model (N2 in [5]) **DMC** (and thus **CMC**) holds, but **AC**$(\aleph_0, < \aleph_0)$ and thus **Baire**$(\mathbf{2}^*)$ fail.

Theorem 3.8 **Baire**$(\mathbf{2}^*)$ *implies* **KW** $(\aleph_0, \infty)$.

Proof: Let (X_n) be a sequence of infinite sets. Assume, without loss of generality that the X_n are pairwise disjoint, and form $X = \cup X_n$. For each $n \in \mathbb{N}$, the set

$$D_n = \{ f \in \mathbf{2}^X \mid f[X_n] = 2\}$$

is open and dense in $\mathbf{2}^X$. By **Baire**$(\mathbf{2}^*)$ there exists some f in $\cap D_n$. Then for each $n \in \mathbb{N}$, $Y_n = f^{-1}[0]$ is a nonempty, proper subset of X_n. □

Theorem 3.9 **Baire**$(\mathbf{2}^*)$ *implies* **PKW**.

Proof: Let $(X_i)_{i\in I}$ be an infinite family of sets X_i with $|X_i| \geq 2$. Assume without loss of generality that the X_i are pairwise disjoint. Form $X = \bigcup_{i\in I} X_i$. For each $n \in \mathbb{N}$ the set

$$D_n = \left\{ f \in \mathbf{2}^X \middle| \, |\{i \in I \mid f[X_i] = 2\}| \geq n \right\}$$

is open and dense in $\mathbf{2}^X$. By **Baire**$(\mathbf{2}^*)$ there exists some f in $\cap D_n$. Then $J = \{i \in I \mid f[X_i] = 2\}$ is infinite, and for each $i \in J$ the set $f^{-1}(1) \cap X_i$ is a nonempty, proper subset of X_i. □

In view of the fact that **Ded**$(\mathbf{2}^*)$ implies **A** our next result follows immediately from Theorem 3.3. However, we supply a simple direct proof as well.

Theorem 3.10 **Baire**$(\mathbf{2}^*)$ *implies* **A**.

Proof: Let X be an infinite set. For each $n \in \mathbb{N}$ the set

$$D_n = \left\{ f \in \mathbf{2}^X \middle| \, |f^{-1}(0)| \geq n \text{ and } |f^{-1}(1)| \geq n \right\}$$

is open and dense in $\mathbf{2}^X$. By **Baire**$(\mathbf{2}^*)$ there exists some f in $\cap D_n$. Thus $Y = f^{-1}(0)$ and $X \backslash Y = f^{-1}(1)$ are both infinite. □

Remark 3.11 Observe that all the consequences of **Baire**$(\mathbf{2}^*)$, exhibited in our results 3.3, 3.5, 3.6, 3.8, 3.9, and 3.10 are also consequences of **AC**$(\aleph_0)$.

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