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Rejection and Truth-Value Gaps

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Abstract A theorem due to Shoesmith and Smiley that axiomatizes two-valued multiple-conclusion logics is extended to partial logics.

Rumfitt [1] extends Smiley's [3] discussion of rejection by axiomatizing a calculus where truth values of sentences are given by truth tables that admit truth-value gaps. "The Smiley multiple-conclusion consequence relation" for the calculus is defined over assertions and rejections. Rumfitt gives a complex Henkin-style proof of completeness for this calculus. Our goal is to show that there is a simple procedure for axiomatizing calculi of the sort that he considers. We do this by imitating Shoesmith and Smiley's [2] proof of a similar result (their Theorem 18.1) where truth tables do not admit truth-value gaps and the consequence relation is defined without using rejections.

Let $A_1, A_2, \ldots$ be *sentences*. And let $+p$ and $*p$ be *assertions* and *rejections*, respectively, given that p is a sentence. Assertions and rejections are *judgments*. We let $+J, \ldots, *J, \ldots$, and $J, \ldots$ range over sets of assertions, sets of rejections, and sets of judgments, respectively.

Let a valuation v be a function that maps sentences into $\{t, n, f\}$ (true, neither true nor false, and false) and judgments into $\{c, i\}$ (correct and incorrect), where $v(+p) = c$ if and only if $v(p) = t$ and $v(*p) = c$ if and only if $v(p) = f$. The Smiley multiple-conclusion consequence relation, $\models$, is defined as follows: $J \models K$ if and only if, for every valuation v, v assigns i to a member of J or c to a member of K (so $\models$ preserves correctness).

Assume a language with connectives $c_1, \ldots, c_n$ where valuations are determined by truth tables for the connectives. To define $J \vdash K$ (K is deducible from J) we use the following *structural rules* together with the *truth-table rules*:

Structural Rules

Overlap:	$J \vdash K$ if J and K have a common member .
Dilution:	$J \vdash K$ if $J' \vdash K'$ given that $J' \subseteq J$ and $K' \subseteq K$.

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Cut: $J \vdash K$ if for every partition L_1, L_2 of a set L of judgments, $J, L_1 \vdash L_2, K$.
Ex falso quodlibet (EFQ): $+p, *p \vdash \varnothing$.

Truth-table Rules

t-rules: If $v(+c_r(p_1, \ldots, p_m)) = t$ then $\{+p_i : v(p_i) = t\}$, $\{*p_j : v(p_j) = f\} \vdash +c_r(p_1, \ldots, p_m)$, $\{+p_k : v(p_k) = n\}, \{*p_k : v(p_k) = n\}$.

n-rules: If $v(+c_r(p_1, \ldots, p_m)) = n$ then $\{+p_i : v(p_i) = t\}$, $\{*p_j : v(p_j) = f\}, +c_r(p_1, \ldots, p_m) \vdash$ $\{+p_k : v(p_k) = n\}, \{*p_k : v(p_k) = n\}$, and $\{+p_i : v(p_i) = t\}, \{*p_j : v(p_j) = f\}, *c_r(p_1, \ldots, p_m) \vdash$ $\{+p_k : v(p_k) = n\}, \{*p_k : v(p_k) = n\}$.

f-rules: If $v(+c_r(p_1, \ldots, p_m)) = f$ then $\{+p_i : v(p_i) = t\}$, $\{*p_j : v(p_j) = f\} \vdash *c_r(p_1, \ldots, p_m)$, $\{+p_k : v(p_k) = n\}, \{*p_k : v(p_k) = n\}$.

$L \vdash M$ if and only if the relationship between L and M is generated by using the structural rules or the truth-table rules.

Theorem 1 *$J \models K$ if and only if $J \vdash K$.*

Proof: (If) Straightforward. For example, for EFQ, note that $v(+p) = i$ or $v(*p) = i$.
(Only if) Suppose $J \nvdash K$. Then, by Cut, there is a partition $+L_1, *L_2, +L_3, *L_4$ of the universal set of judgments such that $J, +L_1, *L_2 \nvdash +L_3, *L_4, K$. By Overlap, $J \subseteq +L_1 \cup *L_2$ and $K \subseteq +L_3 \cup *L_4$. So, it will suffice to show that $+L_1, *L_2 \not\models +L_3, *L_4$. □

Let v be a valuation that assigns t, n, or f to an atomic sentence A depending upon whether $+A \in +L_1$, $+A \in +L_3$ or $*A \in *L_4$, or $*A \in *L_2$, respectively.

Lemma 2 *For any sentence p,*

(i) If $v(p) = t$, then $+L_1, *L_2 \vdash +p, +L_3, *L_4$.
(ii) If $v(p) = n$, then $+L_1, *L_2, +p \vdash +L_3, *L_4$.
(iii) If $v(p) = n$, then $+L_1, *L_2, *p \vdash +L_3, *L_4$.
(iv) If $v(p) = f$, then $+L_1, *L_2 \vdash *p, +L_3, *L_4$.

Proof by induction: For the basis step, where p is an atomic sentence, use Overlap. For the induction step, use Dilution and Cut. Suppose $v(+(c_i(p, q, r)) = t$, where p, q, and r may or may not be atomic. Suppose $v(p) = n, v(q) = t$ and $v(r) = f$. By the t-rules $+q, *r \vdash +c_i(p, q, r), +p, *p$. By the induction hypothesis $+L_1, *L_2, +p \vdash +L_3, *L_4$; $+L_1, *L_2, *p \vdash +L_3, *L_4$; $+L_1, *L_2 \vdash +q, +L_3, *L_4$, and $+L_1, *L_2 \vdash *r, +L_3, *L_4$. So, by Dilution and Cut, $+L_1, *L_2 \vdash +c(p, q, r)$, $+L_3, *L_4$. □

Lemma 3 *For any sentence p,*

(i) $v(p) = t$ *if and only if* $+p \in +L_1$*;*
(ii) $v(p) = n$ *if and only if* $+p \in +L_3$ *or* $*p \in *L_4$*; and*
(iii) $v(p) = f$ *if and only if* $*p \in *L_2$.

Proof: For (i), suppose $v(p) = t$. If $+p \in +L_3$, then, by Overlap, $+L_1, *L_2 \vdash +L_3, *L_4$. Suppose $+p \in +L_1$. Suppose $v(p) = f$. Then $+L_1, *L_2 \vdash *p, +L_3, *L_4$ by Lemma 1. If $*p \in *L_2$ then, by EFQ and Dilution, $+L_1, *L_2 \vdash +L_3, *L_4$. If $*p \in *L_4$ then $+L_1, *L_2 \vdash +L_3, *L_4$. For (iii) use similar reasoning. (ii) follows given (i) and (iii). So, given valuation v, $+L_1, *L_2 \not\models +L_3, *L_4$. □

Example 4 We illustrate the above theorem by axiomatizing a partial logic axiomatized by Rumfitt. Valuations of sentences are given by the following truth tables.

	$\neg$
t	f
n	n
f	t

t-rule:	(1.1)	$+p \vdash *\neg p$.
n-rules:	(1.2)	$+\neg p \vdash +p, *p$.
and	(1.3)	$*\neg p \vdash +p, *p$.
f-rule:	(1.4)	$*p \vdash +\neg p$.

So, for example, Rumfitt's 'From $+\neg p$ infer $*p$' is generated as follows. $+\neg p, +p \vdash *\neg p$ by (1.1) and Dilution. $+p, +\neg p, *\neg p \vdash \varnothing$ by EFQ and Dilution. So $+p, +\neg p \vdash *p$ by Cut and Dilution. $+\neg p \vdash +p, *p$ by (1.3). So $+\neg p \vdash *p$ by Cut.

&	t	n	f
t	t	n	f
n	n	n	n
f	f	n	f

t-rule:	(2.1)	From $+p, +q$ infer $+(p\&q)$.
n-rules:	(2.2)	From $+p, +(p\&q)$ infer $+q, *q$.
	(2.3)	From $+p, *(p\&q)$ infer $+q, *q$.
	(2.4)	From $+q, +(p\&q)$ infer $+p, *p$.
	(2.5)	From $+q, *(p\&q)$ infer $+p, *p$.
	(2.6)	From $+(p\&q)$ infer $+p, *p, +q, *q$.
	(2.7)	From $*(p\&q)$ infer $+p, *p, +q, *q$.
	(2.8)	From $*q, +(p\&q)$ infer $+p, *p$.
	(2.9)	From $*q, *(p\&q)$ infer $+p, *p$.
	(2.10)	From $*p, +(p\&q)$ infer $+q, *q$.
	(2.11)	From $*p, *(p\&q)$ infer $+q, *q$.
f-rules:	(2.12)	From $+p, *q$ infer $*(p\&q)$.
	(2.13)	From $*p, +q$ infer $*(p\&q)$.
	(2.14)	From $*p, *q$ infer $*(p\&q)$.

The rules may be simplified by replacing the ten n-rules with the following four rules:

(2.1′) $+(p\&q) \vdash +p$.
(2.2′) $+(p\&q) \vdash +q$.
(2.3′) $*(p\&q) \vdash +p, *p$.
(2.4′) $*(p\&q) \vdash +q, *q$.

Rumfitt uses the first two of these rules to give his axiomatization.

The proof is simplified by using the following derived meta-rule: (Reversal) If $J \vdash +p, K$ then $J, *p \vdash K$; and if $J \vdash *p, K$ then $J, +p \vdash K$. Prove Reversal by using EFQ, Dilution, and Cut.

That (2.1′) is a derived rule is shown as follows. $+(p\&q) \vdash +q, *q$ by (2.2), (2.10), (2.6), Dilution, and Cut. $*p, +q, +(p\&q) \vdash \varnothing$ by (2.13) and Reversal. So, $*p, +(p\&q) \vdash *q$ (α) by Dilution and Cut. $+(p\&q) \vdash +p, *p$ by (2.4), (2.8), (2.6), Dilution, and Cut. $*p, *q, +(p\&q) \vdash \varnothing$ by (2.14) and Reversal. So $*q, +(p\&q) \vdash +p$ (β) by Dilution and Cut. So, $*p, +(p\&q) \vdash +p$ by α, β, Dilution, and Cut. $+(p\&q) \vdash +p, *p$ as noted above. So $+(p\&q) \vdash +p$ by Cut.

(2.3′) is derived from (2.5), (2.8), and (2.7) by using Dilution and Cut. Reasoning that shows that (2.2′) and (2.4′) are derived rules parallels the reasoning for (2.1′) and (2.3′), respectively.

By Dilution, the n-rules (2.1) to (2.11) follow from (2.1′) to (2.4′).

	T
t	t
n	f
f	f

t-rule: (3.1) From $+p$ infer $+Tp$.
f-rules: (3.2) From $\varnothing$ infer $*Tp, +p, *p$; and
(3.3) From *p infer *Tp.

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