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Dependent Choices and Weak Compactness

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Abstract We work in set theory without the Axiom of Choice **ZF**. We prove that the Principle of Dependent Choices (**DC**) implies that *the closed unit ball of a uniformly convex Banach space is weakly compact* and, in particular, that *the closed unit ball of a Hilbert space is weakly compact*. These statements are not provable in **ZF** and the latter statement does not imply **DC**. Furthermore, **DC** does not imply that the closed unit ball of a reflexive space is weakly compact.

1 Introduction We work in set theory without Axiom of Choice **ZF** and we denote by ω the set of natural numbers. In this paper, *normed spaces* (as defined, for example, in [2], Definition 1.2, p. 63) are *real* normed spaces and they are endowed with the norm metric. A metric space is said to be *complete* when every Cauchy filter of this space converges (see Remarks 2.9 and 2.10). A *Banach space* is a normed space which is complete. The *continuous dual* of a normed space $(E, \|.\|)$ is the vector space E^* of real linear functionals on E which are bounded on the closed unit ball of E, and E^* is endowed with the dual norm $\|.\|_*$: for every $f \in E^*$, $\|f\|_* := \sup\{f(x) : \|x\| \leq 1\}$. The *second dual* of E is the normed space E^{**}. For every $x \in E$, we denote by $\hat{x}$ the *evaluation* at point x, that is, the mapping $E^* \to \mathbb{R}$ such that for every $f \in E^*$, $\hat{x}(f) = f(x)$. The *natural map* $j_E : E \to E^{**}$, given by $j_E(x) = \hat{x}$, is linear and continuous since $\|j_E(x)\|_{**} \leq \|x\|$. Using the *Hahn-Banach* axiom, j_E can be proved *isometric*, that is, $\forall x \in E \ \|j_E(x)\|_{**} = \|x\|$ (see [2], Corollary 6.7, p. 79), but this is not provable in **ZF**, since there are models of **ZF** with infinite dimensional normed spaces E such that $E^{**} = \{0\}$ (see Remark 2.8). The usual definition of "reflexivity" for a normed space E (see [2], p. 89, Definition 11.2) relies on the fact that j_E is isometric, so we will formulate this definition in **ZF** and we will call it *simple reflexivity* or *reflexivity*. The *weak topology* of E is the coarsest topology on E for which every $f \in E^*$ is continuous: it is generated by the sets $\{x \in E : f(x) < \lambda\}$, $\lambda \in \mathbb{R}$, and $f \in E^*$, and it is denoted by $\sigma(E, E^*)$ (see [2], Definition 1.1, p. 124). The *weak** topology of E^* (see [2], Definition 1.1, pp. 124–5) is the coarsest topology on E^* such that for every $x \in E$, $\hat{x}$ is continuous: it is generated by the sets $\{f \in E^* : f(x) < \lambda\}$,

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$\lambda \in \mathbb{R}$, and $x \in E$, and it is denoted by $\sigma(E^*, E)$. A topological space X is *compact* if every nonempty set of closed subsets of X with the finite intersection property has a nonempty intersection. In set theory with the Axiom of Choice **ZFC**, the reflexivity of E is known to be equivalent to the compactness of its closed unit ball for its weak topology (see [2], Theorem 4.2, p. 132), but this equivalence is not provable in **ZF** (see Remarks 1.2, 2.7, and 2.8), so we shall consider another notion of reflexivity which we call *compact reflexivity*.

Let us state these two notions of reflexivity for a normed space E:

(Simple) Reflexivity: *The natural mapping j_E from E to its second dual E^{**} is onto and isometric.*

Compact Reflexivity: *The closed unit ball of E is compact for the weak topology.*

Note that the classical proof of the following statement of *Reflexive Compactness* relies on *Alaoglu's Theorem* (see [2], Theorem 3.1, pp. 130–1) which is equivalent (within **ZF**) to many other classical statements, for example, the *Boolean Prime Ideal Theorem* (see Howard and Rubin [4], pp. 21–7); this last statement is not provable in **ZF** (see Jech [5]), hence *Alaoglu's Theorem* is not provable in **ZF** either.

RC (Reflexive Compactness): *The closed unit ball of a reflexive normed space is compact for the weak topology.*

A (Alaoglu): *The closed unit ball of the continuous dual of a normed space is compact for the weak* topology.*

We now consider some *geometric* properties of normed spaces. A normed space $(E, \|.\|)$ is a *prehilbert space* when there exists an inner product $\langle ., . \rangle : E \times E \to \mathbb{R}$ such that for every $x \in E$, $\|x\| = \sqrt{\langle x, x \rangle}$. A *Hilbert space* is a complete prehilbert space. A normed space E is *uniformly convex* (see [1], p. 189) if the *modulus of uniform convexity of E*, $\delta_E : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+$ defined below, satisfies $\varepsilon > 0 \Rightarrow \delta_E(\varepsilon) > 0$.

$$\delta_E(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \;:\; \|x\| \le 1, \|y\| \le 1 \text{ and } \|x - y\| \ge \varepsilon\right\}$$

Every prehilbert space is uniformly convex (see [1], pp. 189–90). We now consider Reflexive Compactness particularized to uniformly convex Banach spaces and particularized further to Hilbert spaces:

RCuc (Reflexive compactness for uniformly convex Banach spaces): *The closed unit ball of a uniformly convex Banach space is weakly compact.*

RCh (Reflexive Compactness for Hilbert spaces): *The closed unit ball of a Hilbert space is weakly compact.*

Remark 1.1 Using projections on closed convex subsets in a Hilbert space (see Lemma 3 in [3]), one can prove in **ZF** that every Hilbert space is simply reflexive. Hence **RC** implies **RCh**; in particular, **RCh** does not imply **DC**.

In Fossy and Morillon [3], it is proved that **RCh** implies the following set-theoretic axiom $\mathbf{AC}_\omega^{\text{fin}}$ which is not provable in **ZF** (see [5]); in particular, *the statements* **RC** *and* **RCh** *are not provable in* **ZF** *either*.

$\mathbf{AC}_\omega^{\text{fin}}$ (Countable Axiom of Choice for finite sets): *If $(A_n)_{n\in\omega}$ is a sequence of nonempty finite sets, then $\prod_{n\in\omega} A_n \neq \varnothing$.*

Remark 1.2 Thus, though in **ZF** every Hilbert space is simply reflexive, there are models of **ZF** in which some Hilbert spaces are not compact reflexive. Hence *simple reflexivity does not imply compact reflexivity*.

Now the following question is natural.

Question 1.3 *Is there a principle of "Countable Choice" which implies the axiom* **RCh***?*

In this paper, we prove that the following *Principle of Dependent Choices* implies **RCuc** (thus it implies **RCh** too):

DC (Principle of Dependent Choices): *If E is a nonempty set and R is a binary relation on E satisfying*

$$\forall x \in E\ \exists y \in E\ xRy,$$

then there exists a sequence $(x_n)_{n\in\omega}$ such that for every $n \in \omega$, $x_n R x_{n+1}$.

We shall also observe that **DC** does not imply **RC** (see Remark 2.7). Note that **BPI** does not imply **DC** and that **DC** does not imply **BPI** (see [4] or [5]).

2 *The Principle of Dependent Choices implies* RCuc

Notation 2.1 Consider a (real) normed space $(E, \|\cdot\|)$. For each nonnegative real number r, $\Gamma(0,r)$ denotes the closed ball of center 0 and radius r, that is, $\{z \in E : \|z\| \leq r\}$; the closed unit ball $\Gamma(0,1)$ is denoted by Γ_E. Given two real numbers r and r' such that $0 \leq r \leq r'$, the crown $\{z \in E : r \leq \|z\| \leq r'\}$ is denoted by $D(0; r, r')$.

Given a normed space E, we denote by T_E the set of finite unions of closed convex subsets of Γ_E. Notice that $(T_E, \cap, \cup)$ is a lattice of subsets of Γ_E and that each closed set of Γ_E for the weak topology is an intersection of elements of T_E. A *filter* of T_E is any nonempty set $\mathcal{F}$ of nonempty elements of T_E such that the intersection of any two elements of $\mathcal{F}$ is in $\mathcal{F}$ and such that any element of T_E which is a superset of an element of $\mathcal{F}$ is in $\mathcal{F}$ too.

For each set $\mathcal{F}$ of subsets of Γ_E, let

$$R(\mathcal{F}) := \inf\Big\{r \in \mathbb{R} \ :\ 0 \leq r \leq 1 \text{ and } \forall F \in \mathcal{F},\ \Gamma(0,r) \cap F \neq \varnothing\Big\}.$$

When $\mathcal{F}$ has the finite intersection property, $\mathcal{F} \cup \{\Gamma(0,r) : R(\mathcal{F}) < r \leq 1\}$ generates a filter $\mathcal{F}_c$ of T_E called the *circled filter* associated to $\mathcal{F}$.

The following lemma is an easy consequence of the definitions.

Lemma 2.2 *Let E be a normed space and $\mathcal{F}$ be a filter of T_E.*

1. *For every real numbers r and r' such that $0 \leq r < R(\mathcal{F}) < r'$, there exists $F \in \mathcal{F}_c$ such that $F \subseteq D(0; r, r')$.*
2. *For any filter $\mathcal{F}'$ of T_E extending $\mathcal{F}_c$: $R(\mathcal{F}') = R(\mathcal{F})$ and $(\mathcal{F}')_c = \mathcal{F}'$.* □

Lemma 2.3 *Given a uniformly convex normed space E with modulus of uniform convexity δ_E, consider real numbers $\eta > 0$ and r, r' such that $0 < r < r'$ and $r \geq \big(1 - \delta_E(\frac{\eta}{r'})\big)r'$. Then the diameter of any convex subset C of the crown $D(0; r, r')$ is less than or equal to η.*

Proof: Assume by contradiction that some convex subset C of $D(0; r, r')$ contains two points x and y such that $\|x - y\| > \eta$. Then, from the definition of δ_E, it follows that $\|\frac{x+y}{2}\| < \big(1 - \delta_E(\frac{\eta}{r'})\big)r'$; but $\|\frac{x+y}{2}\| \geq r$, since C is convex. □

Theorem 2.4 *Given a uniformly convex Banach space E, let $\mathcal{F}$ be a filter of T_E. The Principle of Dependent Choices* **DC** *implies the nonemptiness of the set $\cap\mathcal{F}$.*

Proof: We prove the existence of a Cauchy filter $\mathcal{G}$ of T_E (i.e., a filter containing sets of arbitrary small diameter) extending the circled filter $\mathcal{F}_c$ associated to $\mathcal{F}$ (thus, since the elements of $\mathcal{G}$ are closed and E is complete, $\cap\mathcal{G} \neq \varnothing$ and a fortiori, $\cap\mathcal{F} \neq \varnothing$). Denote by δ_E the modulus of uniform convexity of E. Let $R := R(\mathcal{F})$. If $R = 0$, then $\mathcal{F}_c$ is already Cauchy. Now assuming that $R > 0$, for each $n \in \omega$, let r_n and r'_n be positive real numbers such that $r_n < R < r'_n$ and $r_n \geq \big(1 - \delta_E(\frac{1}{(n+1)r'_n})\big)r'_n$ (e.g., consider $\alpha_n > 0$ such that $\frac{2\alpha_n}{\alpha_n + R} \leq \delta_E(\frac{1}{2(n+1)R})$ and let $r_n := R - \alpha_n$ and $r'_n := R + \alpha_n$). Let $\mathcal{S}$ denote the set of finite mappings $s \subset \omega \times T_E$ such that

(i) for every $n \in \text{domain}(s)$, $s(n) \subseteq D(0; r_n, r'_n)$,

(ii) $\mathcal{F}_c \cup \text{range}(s)$ has the finite intersection property.

Every element of $\mathcal{S}$ admits a proper extension in $\mathcal{S}$: given $s \in \mathcal{S}$ and $n \notin \text{domain}(s)$, it follows from Lemma 2.2 that some element F of the filter $\mathcal{F}_s$ of T_E generated by $\mathcal{F}_c \cup \text{range}(s)$ is a subset of $D(0; r_n, r'_n)$; then, given closed convex subsets $C_1, \ldots, C_m$ of E such that $F = C_1 \cup \cdots \cup C_m$, observe that for some $i \in \{1, \ldots, m\}$, C_i meets every element of $\mathcal{F}_s$, so that $s \cup \{(n, C_i)\}$ is a proper extension of s in $\mathcal{S}$. Now, invoking **DC**, get an increasing sequence (with respect to proper extension) $(s_n)_{n\in\omega}$ of T_E, and observe that, given $s := \cup\{s_n : n \in \omega\}$, Lemma 2.3 implies that $\mathcal{F}_c \cup \text{range}(s)$ generates a Cauchy filter of T_E. □

Corollary 2.5 **DC** $\Longrightarrow$ **RCuc**.

Proof: Given a uniformly convex Banach space E, let $\mathcal{H}$ be a nonempty set of weakly closed sets of Γ_E with the finite intersection property. Since each such closed set is an intersection of elements of T_E, $\cap\mathcal{H} = \cap\mathcal{F}$, where $\mathcal{F} = \{F \in T_E \ : \ F \supseteq H, \text{for some } H \in \mathcal{H}\}$; but $\mathcal{F}$ is a filter of T_E, hence $\cap\mathcal{F} \neq \varnothing$, according to Theorem 2.4. □

Remark 2.6 **RCuc** *does not imply* **A**, since **RCuc** follows from **DC**, which does not imply **BPI**, and hence, does not imply **A** either.

Remark 2.7 Pincus and Solovay [6] have built a model $\mathcal{M}$ of (**ZF** + **DC**) in which every finitely additive measure is trivial. This means that, given any set I, for every finitely additive mapping $m : \mathcal{P}(I) \to \mathbb{R}$, there exists a family $(\lambda_i)_{i\in I}$ of real numbers such that, for every subset A of I, $m(A) = \sum_{i\in A} \lambda_i$. It follows that the continuous dual of $\ell^\infty(I)$ is equal to $\ell^1(I)$. Thus, in this model $\mathcal{M}$, every $\ell^1(I)$ is a reflexive normed space.

Besides, the closed unit ball Γ of $\ell^1(I)$ is never weakly compact when I is infinite: in fact, denoting by $\mathcal{P}_f(I)$ the set of finite subsets of I, for each $H \in \mathcal{P}_f(I)$, let

$$F_H := \left\{ f \in \Gamma : \sum_{k \in I} f(k) = 1 \quad \text{and} \quad \forall k \in H,\ f(k) = 0 \right\}.$$

Each F_H is a weakly closed set of Γ, and when I is infinite, the family $\{F_H : H \in \mathcal{P}_f(I)\}$ has the finite intersection property, but $\bigcap \{F_H : H \in \mathcal{P}_f(I)\}$ is empty.

Summing up, in the model $\mathcal{M}$, the space $\ell^1(\omega)$ is reflexive and separable but its closed unit ball is not weakly compact. Hence *simple reflexivity does not imply compact reflexivity* even in the case of separable spaces. Moreover, since the model $\mathcal{M}$ satisfies **DC**, **DC** *does not imply* **RC** even for *separable* reflexive spaces.

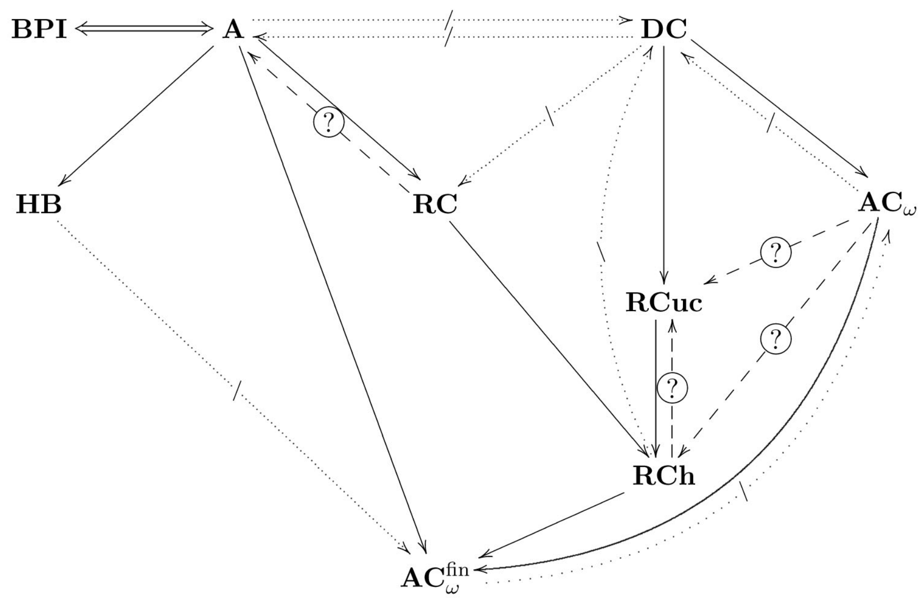
Remark 2.8 *Compact reflexivity does not imply simple reflexivity* since, in every model of **ZF** where Hahn-Banach Theorem fails (for instance, the model $\mathcal{M}$ above), there exists an infinite dimensional normed space E such that $E^* = \{0\}$: such an E fails to be reflexive although Γ_E is weakly compact.

Let us now consider the following consequence of **DC**:

AC$_\omega$ (Countable Axiom of Choice): *If $(A_n)_{n\in\omega}$ is a sequence of nonempty sets, then* $\prod_{n\in\omega} A_n \neq \varnothing$.

Remark 2.9 Say that a metric space (X, d) is *sequentially complete* when every Cauchy sequence converges. So every complete metric space is sequentially complete. In (**ZF**+ **AC**$_\omega$), hence in (**ZF**+ **DC**), every sequentially complete metric space is complete.

Remark 2.10 A set X is *Dedekind-finite* when there exists no one-to-one mapping from ω to X. There are models of **ZF** (e.g., Cohen's first model, see [5]) with a Dedekind-finite dense subset A of $\mathbb{R}$. The metric subspace A is sequentially complete but it is not complete.



Note: Uniformly convex spaces are simply reflexive (work in progress). Thus **RC** implies **RCuc**; in particular, like **RCh**, **RCuc** fails to imply **DC**. (Cf. Abstract and Remark 1.1.)

We know no answer to the following questions:

Question 2.11 *Are the statements* **A** *and* **RC** *equivalent?*

Question 2.12 *Does* $\mathbf{AC}_\omega$ *imply* **RCh** *or* **RCuc**?

Question 2.13 *Are* **RCh** *and* **RCuc** *equivalent?*

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