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EQUIVALENCE BETWEEN UNIFORM $L^{2^\star}(\Omega)$ A-PRIORI BOUNDS AND UNIFORM $L^\infty(\Omega)$ A-PRIORI BOUNDS FOR SUBCRITICAL ELLIPTIC EQUATIONS

Alfonso Castro — Nsoki Mavinga — Rosa Pardo

Abstract. We provide sufficient conditions for a uniform $L^{2^\star}(\Omega)$ bound to imply a uniform $L^\infty(\Omega)$ bound for positive classical solutions to a class of subcritical elliptic problems in bounded C^2 domains in $\mathbb{R}^N$. We also establish an equivalent result for sequences of boundary value problems.

1. Introduction

We consider the existence of $L^\infty(\Omega)$ *a priori* bounds for classical positive solutions to the boundary value problem

$$-\Delta u = f(u), \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded domain with C^2 boundary $\partial\Omega$. We provide sufficient conditions on f for $L^{2^*}(\Omega)$ *a priori* bounds to imply $L^\infty(\Omega)$ *a priori* bounds, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent. The converse is obviously true without any additional hypotheses.

The existence of *a priori* bounds for (1.1) has a rich history. In chronological order, [18], [14], [17], [4], [15], [11], [10] and [2] are some of the main contributors to such a development. We refer the reader to [6] where their roles are discussed.

The ideas for the proof of our main Theorem are similar to those used in [6, Theorem 1.1]. In [6] we give sufficient conditions on the nonlinearity to have $L^\infty(\Omega)$ *a priori* bounds, while here we prove the equivalence between the existence of $L^\infty(\Omega)$ *a priori* bounds and the existence of $L^{2^*}(\Omega)$ *a priori* bounds for subcritical elliptic equations. Unlike the proof in [6], here we do not use Pohozaev or moving planes arguments.

Our main result is the following theorem.

Theorem 1.1. *Assume that the nonlinearity $f\colon \mathbb{R}^+ \to \mathbb{R}$ is a locally Lipschitzian function that satisfies*:

(H1) *There exists a constant $C_0 > 0$ such that*

$$\liminf_{s\to\infty} \frac{1}{f(s)} \min_{[s/2,s]} f \geq C_0.$$

(H2) *There exists a constant $C_1 > 0$ such that*

$$\limsup_{s\to\infty} \frac{1}{f(s)} \max_{[0,s]} f \leq C_1.$$

(F) $\lim_{s\to+\infty} \dfrac{f(s)}{s^{2^*-1}} = 0$; *that is, f is subcritical.*

Then the following conditions are equivalent:

(a) *there exists a uniform constant C (depending only on Ω and f) such that, for every positive classical solution u of (1.1),*

$$\|u\|_{L^\infty(\Omega)} \leq C,$$

(b) *there exists a uniform constant C (depending only on Ω and f) such that for every positive classical solution u of (1.1)*

$$\int_\Omega |f(u)|^{2N/(N+2)}\, dx \leq C, \tag{1.2}$$

(c) *there exists a uniform constant C (depending only on Ω and f) such that, for every positive classical solution u of (1.1),*

$$\|u\|_{L^{2^*}(\Omega)} \leq C. \tag{1.3}$$

In [7] and [8] the associated bifurcation problem for the nonlinearity $f(\lambda, s) = \lambda s + g(s)$ with g subcritical is studied. Sufficient conditions guaranteeing that either for any $\lambda < \lambda_1$ there exists at least a positive solution, or that there exists a $\lambda^* < 0$ and a continuum (λ, u_λ), $\lambda^* < \lambda < \lambda_1$, of positive solutions such that

$$\|\nabla u_\lambda\|_{L^2(\Omega)} \to \infty, \quad \text{as } \lambda \to \lambda^*,$$

are provided. See [8, Theorem 2]. In the case Ω is convex, for any $\lambda < \lambda_1$ there exists at least a positive solution, see [7, Theorem 1.2]. In [9] the concept of regions with *convex-starlike* boundary is introduced and sufficient conditions for the existence of *a priori* bounds in such regions are established. In [16] the existence of *a priori* bounds for elliptic systems is provided.

In this paper, we also provide sufficient conditions for the equivalence of the existence of $L^{2^\star}(\Omega)$ *a priori* bound with that of $L^\infty(\Omega)$ *a priori* bound for sequences of boundary value problems. In fact, we prove the following theorem.

THEOREM 1.2. *Consider the following sequence of* BVP*s*

$$-\Delta v = g_k(v) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{1.3$_k$}$$

with $g_k \colon \mathbb{R}^+ \to \mathbb{R}$ locally Lipschitzian. We assume that the following hypotheses are satisfied

(H1)$_k$ *There exists a uniform constant $C_1 > 0$, such that*

$$\liminf_{s\to+\infty} \frac{1}{g_k(s)} \min_{[s/2,s]} g_k \ge C_1.$$

(H2)$_k$ *There exists a uniform constant $C_2 > 0$ such that*

$$\limsup_{s\to+\infty} \frac{1}{g_k(s)} \max_{[0,s]} g_k \le C_2.$$

Let $\{v_k\}$ be a sequence of classical positive solutions to $(1.3)_k$ for $k \in \mathbb{N}$. If

(F)$_k$ $\lim_{k\to+\infty} g_k(\|v_k\|)/\|v_k\|^{2^\star-1} = 0,$

then, the following two conditions are equivalent:

(a) *there exists a uniform constant C, depending only on Ω and the sequence $\{g_k\}$, but independent of k, such that for every $v_k > 0$, classical solution to $(1.3)_k$*

$$\limsup_{k\to\infty} \|v_k\|_{L^\infty(\Omega)} \le C;$$

(b) *there exists a uniform constant C, depending only on Ω and the sequence $\{g_k\}$, but independent of k, such that for every $v_k > 0$, classical solution to $(1.3)_k$*

$$\limsup_{k\to\infty} \int_\Omega |g_k(v_k)|^{2N/(N+2)}\, dx \le C. \tag{1.4}$$

(c) *there exists a uniform constant C (depending only on Ω and the sequence $\{g_k\}$) such that for every positive classical solution v_k of $(1.3)_k$*

$$\|v_k\|_{L^{2^*}(\Omega)} \le C. \tag{1.5}$$

Hypothesis $(\mathrm{H1})_k$, and $(\mathrm{H2})_k$, are not sufficient for the existence of an L^∞ *a priori* bound. Atkinson and Pelletier in [1] show that for $f_\varepsilon(s) = s^{2^\star-1-\varepsilon}$ and Ω a ball in $\mathbb{R}^3$, there exists $x_0 \in \Omega$ and a sequence of solutions u_ε such that $\lim_{\varepsilon\to 0} u_\varepsilon = 0$ in $C^1(\Omega \setminus \{x_0\})$ and $\lim_{\varepsilon\to 0} u_\varepsilon(x_0) = +\infty$. See also Han [13], for non-spherical domains.

Furthermore, hypotheses $(\mathrm{H1})_k$, $(\mathrm{H2})_k$, and $(\mathrm{F})_k$, are not sufficient for the existence of an L^∞ *a priori* bound. In fact, in Section 4 we construct a sequence of BVP satisfying $(\mathrm{H1})_k$, $(\mathrm{H2})_k$, and $(\mathrm{F})_k$, and a sequence of solutions v_k such that $\lim_{k\to\infty} \|v_k\|_\infty = +\infty$. Our example also shows the non-uniqueness of positive solutions.

2. Proof of Theorems 1.1 and 1.2

In this section, we state and prove our main results that hold for general bounded domains, including the non-convex case. We provide a sufficient condition for a uniform $L^{2^\star}(\Omega)$ bound to imply a uniform $L^\infty(\Omega)$ bound for classical positive solutions of the subcritical elliptic equation (1.1). We also give sufficient conditions such that the $L^\infty(\Omega)$ bound of a sequence of classical positive solutions of a sequence of BVPs $(1.3)_k$ is equivalent to the uniform $L^{2^\star}(\Omega)$ bound of the sequence of reaction functions. The arguments rely on the estimation of the radius R of a ball where the function u exceeds half of its L^∞ bound, see Figure 1.

All throughout this paper, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary, and C denotes several constants independent of u, where $u > 0$ is any classical solution to (1.1).

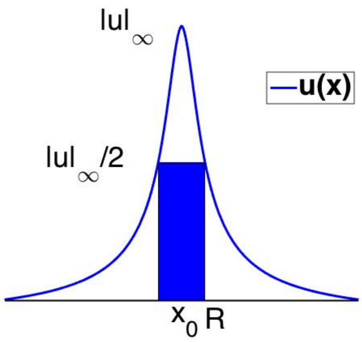

Figure 1. A solution, its L^∞ norm, and the estimate of the radius R such that $u(x) \geq \|u\|_\infty/2$ for all $x \in B(x_0, R)$, where x_0 is such that $u(x_0) = \|u\|_\infty$.

REMARK 2.1. By (1.2), elliptic regularity and the Sobolev embeddings imply that

$$\|u\|_{H_0^1(\Omega)} \le \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2} \le C. \tag{2.1}$$

Hence, for any classical solutions to (1.1), we have

$$\int_\Omega u f(u) \, dx = \|u\|^2_{H_0^1(\Omega)} \le C. \tag{2.2}$$

PROOF OF THEOREM 1.1. Since Ω is bounded (a) implies (b) and (c). From elliptic regularity and condition (1.2), we deduce that $\|u\|_{W^{2,2N/(N+2)}} \le C$. It follows using twice the Sobolev embedding that a uniform bound in $W^{2,2N/(N+2)}$ implies a uniform bound in $H^1(\Omega)$ and a uniform bound in $L^{2^\star}(\Omega)$, that is,

$$\|u\|_{L^{2^*}(\Omega)} \le C, \tag{2.3}$$

for all classical positive solution u of equation (1.1). Therefore, (b) implies (c).

Now, assume that (c) holds. It follows from the subcriticality condition (F) that $|f(s)|^{2N/(N+2)} \le s^{2^*}$ for all s large enough. Thus, for any classical solution to (1.1), we have

$$\int_\Omega |f(u)|^{2N/(N+2)} \, dx \le \int_\Omega |u|^{2N/(N-2)} \, dx + C < C.$$

Thus (b) and (c) are equivalent.

Next, we concentrate our attention in proving that (b) implies (a). Since $2N/(N+2) = 1 + 1/(2^\star - 1)$, the hypothesis (1.2) can be written

$$\int_\Omega |f(u)|^{1+1/(2^\star-1)} \, dx \le C. \tag{2.4}$$

Therefore,

$$\begin{aligned} \int_\Omega |f(u(x))|^q \, dx &\le \int_\Omega |f(u(x))|^{1+1/(2^\star-1)} |f(u(x))|^{q-1-1(2^\star-1)} \, dx \\ &\le C \|f(u(\,\cdot\,))\|_\infty^{q-1-1/(2^\star-1)}, \end{aligned} \tag{2.5}$$

for any $q > N/2$.

From the elliptic regularity (see [3] and [12, Lemma 9.17]), it follows that

$$\|u\|_{W^{2,q}(\Omega)} \le C \|\Delta u\|_{L^q(\Omega)} \le C \ \|f(u(\,\cdot\,))\|_\infty^{1-1/q-1/((2^\star-1)q)}. \tag{2.6}$$

Let us restrict $q \in (N/2, N)$. From the Sobolev embeddings, for $1/q^* = 1/q - 1/N$ with $q^* > N$ we can write

$$\|u\|_{W^{1,q^*}(\Omega)} \le C \|u\|_{W^{2,q}(\Omega)} \le C \|f(u(\,\cdot\,))\|_\infty^{1-1/q-1/((2^\star-1)q)}. \tag{2.7}$$

From Morrey's Theorem, (see [3, Theorem 9.12 and Corollary 9.14]), there exists a constant C (depending only on Ω, q and N) such that, for all $x_1, x_2 \in \Omega$,

$$|u(x_1) - u(x_2)| \le C |x_1 - x_2|^{1-N/q^*} \|u\|_{W^{1,q^*}(\Omega)}. \tag{2.8}$$

Therefore, for all $x \in B(x_1, R) \subset \Omega$,

$$|u(x) - u(x_1)| \le C R^{2-N/q} \|u\|_{W^{2,q}(\Omega)}. \tag{2.9}$$

Now, we shall argue by contradiction. Suppose that there exists a sequence $\{u_k\}$ of classical positive solutions of (1.1) such that

$$\lim_{k\to\infty} \|u_k\| = +\infty, \quad \text{where } \|u_k\| := \|u_k\|_\infty. \tag{2.10}$$

Let $x_k \in \Omega$ be such that $u_k(x_k) = \max_\Omega u_k$. Let us choose R_k such that $B_k = B(x_k, R_k) \subset \Omega$, and

$$u_k(x) \ge \frac{1}{2}\, \|u_k\| \quad \text{for any } x \in B(x_k, R_k).$$

and there exists $y_k \in \partial B(x_k, R_k)$ such that

$$u_k(y_k) = \frac{1}{2}\, \|u_k\|. \tag{2.11}$$

Let us denote by

$$m_k := \min_{[\|u_k\|/2, \|u_k\|]} f, \qquad M_k := \max_{[0,\|u_k\|]} f.$$

Therefore, we obtain

$$m_k \le f(u_k(x)) \quad \text{if } x \in B_k, \qquad f\big(u_k(x)\big) \le M_k \quad \text{for all } x \in \Omega. \tag{2.12}$$

Then, reasoning as in (2.5), we obtain

$$\int_\Omega \big|f\big(u_k\big)\big|^q \, dx \quad \le \quad C \; {M_k}^{q-1-1/(2^\star-1)}. \tag{2.13}$$

From the elliptic regularity, see (2.6), we deduce

$$\|u_k\|_{W^{2,q}(\Omega)} \le C M_k^{1-1/q-1/((2^\star-1)q)}. \tag{2.14}$$

Therefore, from Morrey's Theorem, see (2.9), for any $x \in B(x_k, R_k)$

$$|u_k(x) - u_k(x_k)| \le C (R_k)^{2-N/q} M_k^{1-1/q-1/((2^\star-1)q)}. \tag{2.15}$$

Taking $x = y_k$ in the above inequality and from (2.11) we obtain

$$C\ (R_k)^{2-N/q} M_k^{1-1/q-1/((2^\star-1)q)} \ge |u_k(y_k) - u_k(x_k)| = \frac{1}{2}\|u_k\|, \tag{2.16}$$

which implies

$$(R_k)^{2-N/q} \ge \frac{1}{2C}\, \frac{\|u_k\|}{M_k^{1-1/q-1/((2^\star-1)q)}}, \tag{2.17}$$

or equivalently,

$$R_k \ge \left(\frac{1}{2C}\, \frac{\|u_k\|}{M_k^{1-1/q-1/((2^\star-1)q)}} \right)^{1/(2-N/q)}. \tag{2.18}$$

Consequently,

$$\int_{B(x_k,R_k)} u_k^{2^*} \ge \left(\frac{1}{2}\,\|u_k\|\right)^{2^*} \omega(R_k)^N,$$

where $\omega = \omega_N$ is the volume of the unit ball in $\mathbb{R}^N$.

Due to $B(x_k, R_k) \subset \Omega$, substituting inequality (2.18), taking into account hypothesis (H2), and rearranging terms, we obtain

$$\begin{aligned}
\|u_k\|^{2^*}_{L^{2^*}(\Omega)} = \int_\Omega u_k^{2^*} &\ge \left(\frac{1}{2}\,\|u_k\|\right)^{2^*} \omega\left(\frac{1}{2C}\,\frac{\|u_k\|}{M_k^{1-1/q-1/((2^\star-1)q)}}\right)^{N/(2-N/q)} \\
&\ge \left(\frac{1}{2}\,\|u_k\|\right)^{2^*} \omega\left(\frac{1}{2C}\,\frac{\|u_k\|}{[f(\|u_k\|)]^{1-1/q-1/((2^\star-1)q)}}\right)^{1/(2/N-1/q)} \\
&= C\|u_k\|^{2^\star-1}\left([\|u_k\|]^{2/N-1/q}\,\frac{\|u_k\|}{[f(\|u_k\|)]^{1-1/q-1/((2^\star-1)q)}}\right)^{1/(2/N-1/q)} \\
&= C\,\frac{\|u_k\|^{2^\star-1}}{f(\|u_k\|)}\,\left(\frac{\|u_k\|^{1+2/N-1/q}}{[f(\|u_k\|)]^{1-2/N-1/(2^\star-1)q}}\right)^{1/(2/N-1/q)} \\
&\ge C\,\frac{\|u_k\|^{2^\star-1}}{f(\|u_k\|)}\left(\frac{\|u_k\|^{(N+2)[1/N-1/((N+2)q)]}}{[f(\|u_k\|)]^{(N-2)[1/N-1/((N+2)q)]}}\right)^{1/(2/N-1/q)}.
\end{aligned}$$

Finally, from (2.10) and the hypothesis (F) we deduce

$$\begin{aligned}
\int_\Omega u_k^{2^*} &\ge C\,\frac{\|u_k\|^{2^\star-1}}{f(\|u_k\|)}\left(\frac{\|u_k\|^{2^\star-1}}{f(\|u_k\|)}\right)^{(N-2)[1/N-1/((N+2)q)](2/N-1/q)} \\
&= \left(\frac{\|u_k\|^{2^\star-1}}{f(\|u_k\|)}\right)^{1+(N-2)[1/N-1/((N+2)q)]/(2/N-1/q)} \to \infty \quad \text{as } k\to\infty,
\end{aligned}$$

which contradicts (2.3). Thus (b) implies (a). □

Remark 2.2. One can easily see that condition (1.4) implies that there exists a uniform constant $C_4 > 0$ such that

$$\limsup_{k\to\infty} \int_\Omega v_k\, g_k(v_k)\, dx \le C_4, \tag{2.19}$$

for all classical positive solutions $\{v_k\}$ to $(1.3)_k$.

Proof of Theorem 1.2. Clearly, condition (a) implies (b) and (c). By the elliptic regularity and condition (1.4), we have that $\|v_k\|_{W^{2,2N/(N+2)}} \le C$. Therefore, $\|v_k\|_{H^1(\Omega)} \le C$. Hence, by the Sobolev embedding, we deduce that

$$\|v_k\|_{L^{2^*}(\Omega)} \le C \quad \text{for all } k. \tag{2.20}$$

Using similar arguments as in Theorem 1.1 and condition $(\mathrm{F})_\mathrm{k}$, one can show that (b) and (c) are equivalent. We shall concentrate our attention in proving that (b) implies (a). All throughout this proof C denotes several constants independent of k.

Observe that $1 + 1/(2^* - 1) = 2N/(N + 2)$. From hypothesis (b), see (1.4), there exists a fixed constant $C > 0$, (independent of k) such that

$$\begin{aligned}\int_\Omega |g_k(v_k(x))|^q \, dx &\le \int_\Omega |g_k(v_k(x))|^{1+1/(2^\star-1)} |g_k(v_k(x))|^{q-1-1/(2^\star-1)} \, dx \\ &\le C \|g_k(v_k(\,\cdot\,))\|_\infty^{q-1-1/(2^\star-1)},\end{aligned} \tag{2.21}$$

for k big enough, and for any $q > N/2$. Therefore, from the elliptic regularity, see [12, Lemma 9.17]

$$\|v_k\|_{W^{2,q}(\Omega)} \le C \|\Delta v_k\|_{L^q(\Omega)} \le C \ \|g_k(v_k(\,\cdot\,))\|_\infty^{1-1/q-1/((2^\star-1)q)}, \tag{2.22}$$

for k big enough.

Let us restrict $q \in (N/2, N)$. From Sobolev embeddings, for $1/q^* = 1/q - 1/N$ with $q^* > N$ we can write

$$\|v_k\|_{W^{1,q^*}(\Omega)} \le C \|v_k\|_{W^{2,q}(\Omega)} \le C \ \|g_k(v_k(\,\cdot\,))\|_\infty^{1-1/q-1/((2^\star-1)q)}, \tag{2.23}$$

for k big enough. From Morrey's Theorem, (see [3, Theorem 9.12 and Corollary 9.14]), there exists a constant C only dependent on Ω, q and N such that

$$|v_k(x_1) - v_k(x_2)| \le C |x_1 - x_2|^{1-N/q^*} \|v_k\|_{W^{1,q^*}(\Omega)}, \tag{2.24}$$

for all $x_1, x_2 \in \Omega$ and for any k. Therefore, for all $x \in B(x_1, R) \subset \Omega$

$$|v_k(x) - v_k(x_1)| \le C \ R^{2-N/q} \|v_k\|_{W^{2,q}(\Omega)}, \tag{2.25}$$

for any k.

From now on, we argue by contradiction. Let $\{v_k\}$ be a sequence of classical positive solutions to $(1.3)_k$ and assume that

$$\lim_{k\to\infty} \|v_k\| = +\infty, \quad \text{where } \|v_k\| := \|v_k\|_\infty. \tag{2.26}$$

Let $x_k \in \Omega$ be such that $v_k(x_k) = \max_\Omega v_k$. Let us choose R_k such that $B_k := B(x_k, R_k) \subset \Omega$, and

$$v_k(x) \ge \frac{1}{2} \|v_k\| \quad \text{for any } x \in B_k.$$

and there exists $y_k \in \partial B_k$ such that

$$v_k(y_k) = \frac{1}{2} \ \|v_k\|. \tag{2.27}$$

Let us denote by

$$m_k := \min_{[\|v_k\|/2, \|v_k\|]} g_k, \qquad M_k := \max_{[0, \|v_k\|]} g_k.$$

Therefore, we obtain

$$m_k \le g_k(v_k(x)) \quad \text{if } x \in B_k, \qquad g_k(v_k(x)) \le M_k \quad \text{for all } x \in \Omega. \tag{2.28}$$

Then, reasoning as in (2.21), we obtain

$$\int_\Omega |g_k(v_k)|^q\,dx \le C{M_k}^{q-1-1/(2^\star-1)}. \tag{2.29}$$

From the elliptic regularity, see (2.22), we deduce

$$\|v_k\|_{W^{2,q}(\Omega)} \le CM_k^{1-1/q-1/((2^\star-1)q)}. \tag{2.30}$$

Therefore, from Morrey's Theorem, see (2.25), for any $x\in B_k$,

$$|v_k(x)-v_k(x_k)| \le C(R_k)^{2-N/q}M_k^{1-1/q-1/((2^\star-1)q)}. \tag{2.31}$$

Particularizing $x=y_k$ in the above inequality and from (2.27) we obtain

$$C(R_k)^{2-N/q}M_k^{1-1/q-1/((2^\star-1)q)} \ge |v_k(y_k)-v_k(x_k)| = \frac{1}{2}\,\|v_k\|, \tag{2.32}$$

which implies

$$(R_k)^{2-N/q} \ge \frac{1}{2C}\,\frac{\|v_k\|}{M_k^{1-1/q-1/((2^\star-1)q)}}, \tag{2.33}$$

or equivalently

$$R_k \ge \left(\frac{1}{2C}\,\frac{\|v_k\|}{M_k^{1-1/q-1/((2^\star-1)q)}}\right)^{1/(2-N/q)}. \tag{2.34}$$

Consequently, taking into account (2.28),

$$\int_{B_k} v_k|g_k(v_k)|\,dx \ge \frac{1}{2}\,\|v_k\|m_k\omega(R_k)^N,$$

where $\omega=\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$, see Figure 2 (b).

Due to $B_k\subset\Omega$, substituting inequality (2.34), and rearranging terms, we obtain

$$\begin{aligned}\int_\Omega v_k|g_k(v_k)|\,dx &\ge \frac{1}{2}\,\|v_k\|m_k\omega\left(\frac{1}{2C}\,\frac{\|v_k\|}{M_k^{1-1/q-1/((2^\star-1)q)}}\right)^{N/(2-N/q)}\\
&= C\,m_k\left([\|v_k\|]^{2/N-1/q}\frac{\|v_k\|}{M_k^{1-1/q-1/((2^\star-1)q)}}\right)^{1/(2/N-1/q)}\\
&= C\,m_k\left(\frac{\|v_k\|^{1+2/N-1/q}}{M_k^{1-1/q-1/((2^\star-1)q)}}\right)^{1/(2/N-1/q)}\\
&= C\,\frac{m_k}{M_k}\left(\frac{\|v_k\|^{1+2/N-1/q}}{M_k^{1-2/N-1/((2^\star-1)q)}}\right)^{1/(2/N-1/q)}\end{aligned}$$

At this moment, let us observe that from hypothesis $(\mathrm{H1})_k$ and $(\mathrm{H2})_k$

$$\frac{m_k}{M_k}\ge C,\quad \text{for all } k \text{ big enough.} \tag{2.35}$$

Hence, taking again into account hypothesis $(\mathrm{H2})_k$, and rearranging exponents, we can assert that

$$\begin{aligned}(2.36)\qquad \int_\Omega v_k|g_k(v_k)|\,dx &\geq C\bigg(\frac{\|v_k\|^{1+2/N-1/q}}{M_k^{1-2/N-1/((2^\star-1)q)}}\bigg)^{1/(2/N-1/q)}\\ &\geq C\bigg(\frac{\|v_k\|^{1+2/N-1/q}}{[g_k(\|v_k\|)]^{1-2/N-1/((2^\star-1)q)}}\bigg)^{1/(2/N-1/q)}\\ &\geq C\bigg(\frac{\|v_k\|^{(N+2)[1/N-1/((N+2)q)]}}{[g_k(\|v_k\|)]^{(N-2)[1/N-1/((N+2)q)]}}\bigg)^{1/(2/N-1/q)}.\end{aligned}$$

Finally, from hypothesis $(\mathrm{F})_k$ we deduce

$$\int_\Omega v_k|g_k(v_k)|\,dx \geq C\bigg(\frac{\|v_k\|^{2^*-1}}{g_k(\|v_k\|)}\bigg)^{(N-2)[1/N-1/((N+2)q)]/(2/N-1/q)} \to \infty,$$

as $k\to\infty$, which contradicts (2.19). □

3. Radial problems with almost critical exponent

In this section, we build an example of a sequence of functions $\{g_k\}$ growing subcritically, and satisfying the hypotheses $(\mathrm{H1})_k$, $(\mathrm{H2})_k$, and $(\mathrm{F})_k$, such that the corresponding sequence of BVP

$$(3.1)\qquad \begin{cases}\Delta w_k + g_k(w_k) = 0 & \text{in } |x|\leq 1,\\ w_k(x) = 0 & \text{for } |x| = 1.\end{cases}$$

has an unbounded (in the $L^\infty(\Omega)$-norm) sequence $\{w_k\}$ of positive solutions. As a consequence of Theorem 1.2, this sequence $\{w_k\}$ is also unbounded in the $L^{2^*}(\Omega)$-norm.

Let $N\geq 3$ be an integer. For each positive integer $k>2$ let

$$g_k(s) = \begin{cases} 0 & \text{for } s<0,\\ s^{(N+2)/(N-2)} & \text{for } s\in[0,k],\\ k^{(N+2)/(N-2)} & \text{for } s\in[k,k^{(N+2)/(N-2)}],\\ k^{(N+2)/(N-2)} + (s-k^{(N+2)/(N-2)})^{(N+1)/(N-2)} & \\ & \text{for all } s>k^{(N+2)/(N-2)}.\end{cases}$$

For the sake of simplicity in notation, we write $g_k := g$.

Let $u_k := u$ denote the solution to

$$(3.2)\qquad \begin{cases} u'' + \dfrac{N-1}{r}u' + g(u) = 0 & \text{for } r\in(0,1],\\ u(0) = k^{N/(N-2)} & \text{for } u'(0) = 0.\end{cases}$$

Let $r_1 = \sup\{r > 0 : u_k(s) \geq k \text{ on } [0,r]\}$. Since $g \geq 0$, u is decreasing, consequently for $r \in [0, r_1]$, $k \leq u(r) \leq k^{N/(N-2)}$, and

$$\begin{aligned} -r^{N-1}u'(r) &= \int_0^r s^{N-1} g(u(s))\, ds \\ &= \int_0^r s^{N-1} k^{(N+2)/(N-2)}\, ds = \frac{k^{(N+2)/(N-2)}}{N}\, r^N, \end{aligned} \tag{3.3}$$

so

$$u'(r) = \frac{k^{(N+2)/(N-2)}}{N}\, r. \tag{3.4}$$

Hence

$$u(r) = k^{N/(N-2)} - \frac{k^{(N+2)/(N-2)}}{2N}\, r^2, \quad \text{for } r \in [0, r_1]. \tag{3.5}$$

Thus, $u(r) \geq k^{N/(N-2)}/2$, for all $0 \leq r \leq r_0 := \sqrt{N}/k^{1/(N-2)}$, and $u(r_0) = k^{N/(N-2)}/2$.

By well established arguments based on the Pohozaev identity, see [5], we have

$$P(r) := r^N E(r) + \frac{N-2}{2}\, r^{N-1}\, u(r)\, u'(r) = \int_0^r s^{N-1}\, \Gamma\big(u(s)\big)\, ds, \tag{3.6}$$

where

$$E(r) = \frac{1}{2}(u'(r))^2 + G(u(r)), \quad \Gamma(s) = NG(s) - \frac{N-2}{2}\, sg(s), \quad G(s) = \int_0^s g(t)\, dt.$$

For $s \in [k, k^{N/(N-2)}]$,

$$\Gamma(s) = -\frac{N+2}{2}\, k^{2N/(N-2)} + \frac{N+2}{2}\, s\, k^{(N+2)/(N-2)} \geq 0. \tag{3.7}$$

Hence

$$\Gamma(u(r)) \geq \frac{N+2}{8}\, k^{(2N+2)/(N-2)} \quad \text{for all } r \leq r_0,\ k \geq 4^{(N-2)/2}.$$

Due to $\Gamma(s) = 0$ for all $s \leq k$, (3.6) and (3.7), for $r \geq r_0$,

$$P(r) \geq P(r_0) \geq \frac{N+2}{8N}\, k^{(2N+2)/(N-2)}\, r_0^N \geq \frac{N+2}{8}\, N^{(N-2)/2}\, k^{(N+2)/(N-2)}.$$

Due to (3.7), for $r \geq r_0$, we have

$$P(r) \geq P(r_0) \geq \frac{N+2}{8}\, N^{(N-2)/2} k^{(N+2)/(N-2)}.$$

From (3.5) $u(r_1) = k$ with

$$r_1 = \sqrt{2N\left[\left(\frac{1}{k}\right)^{2/(N-2)} - \left(\frac{1}{k}\right)^{4/(N-2)}\right]} = \sqrt{2N}\left(\frac{1}{k}\right)^{1/(N-2)} + o\left(\left(\frac{1}{k}\right)^{1/(N-2)}\right).$$

From the definition of g, $-u'(r_1) = k^{(N+2)/(N-2)}\, r_1/N$ (see (3.4)), which implies

$$\begin{aligned} P(r_1) &\geq r_1^{N+2} O\big(k^{2(N+2)/(N-2)}\big) - r_1^N O\big(k^{2N/(N-2)}\big) \\ &\geq O\big(k^{(N+2)/(N-2)}\big) - O\big(k^{N/(N-2)}\big) \geq O\big(k^{(N+2)/(N-2)}\big). \end{aligned}$$

For $r \geq r_1$,

$$\begin{aligned} -\frac{N-2}{2}\, r^{N-1}\, u(r)\, u'(r) &\geq \frac{(N-2)r^N}{2N}\, u(r)\, u(r)^{(N+2)/(N-2)} \\ &= \frac{(N-2)r^N}{2N}\, u(r)^{2N/(N-2)} = r^N G(u(r)). \end{aligned} \tag{3.8}$$

This and Pohozaev's identity imply

$$[(u'(r)]^2 \geq O\big(k^{(N+2)/(N-2)}\big)\frac{1}{r^N} \quad \text{or} \quad -u'(r) \geq O\big(k^{(N+2)/(2(N-2))}\big)\frac{1}{r^{N/2}}.$$

Integrating on $[r_1, r]$ we have

$$u(r) \leq k - O\big(k^{(N+2)/(2(N-2))}\big)\bigg(\frac{1}{r_1^{(N-2)/2}} - \frac{1}{r^{(N-2)/2}}\bigg),$$

which implies that there exists k_0 such that if $k \geq k_0$ then $u(r) = 0$ for some $r \in (r_1, 2r_1]$. Since (3.8), $r_1 = r_1(k) \to 0$ as $k \to \infty$.

Let $v := v_k$ denote the solution to

$$\begin{cases} v'' + \dfrac{N-1}{r}\, v' + g(v) = 0, & r \in (0,1], \\ v(0) = k^{(N+2)/(N-2)}, & v'(0) = 0. \end{cases} \tag{3.9}$$

Let $r_1 = \sup\{r > 0 : v_k(s) \geq k \text{ on } [0,r]\}$. For $v(r) \geq k$, i ntegrating (3.4), we deduce

$$v(r) = k^{(N+2)/(N-2)} - \frac{k^{(N+2)/(N-2)}}{2N}\, r^2, \quad \text{for } r \in [0, r_1], \tag{3.10}$$

$$v(r_1) = k^{(N+2)/(N-2)} - \frac{k^{(N+2)/(N-2)}}{2N}\, r_1^2 = k, \tag{3.11}$$

therefore

$$r_1 = \sqrt{2N\bigg(1 - \bigg(\frac{1}{k}\bigg)^{4/(N-2)}\bigg)} > 1, \tag{3.12}$$

therefore $v(r) \geq k$ for all $r \in [0,1]$. So, by continuous dependence on initial conditions, there exists $d_k \in \big(k^{N/(N-2)}, k^{(N+2)/(N-2)}\big)$ such that the solution $w = w_k$ to

$$\begin{cases} w'' + \dfrac{N-1}{r}\, w' + g_k(w) = 0, & r \in (0,1], \\ w(0) = d_k, & w'(0) = 0. \end{cases}$$

satisfies $w(r) \geq 0$ for all $r \in [0,1]$, and $w(1) = 0$. Since k may be taken arbitrarily large, and as a consequence of Theorem 1.2, we have established the following result.

Corollary 3.1. *There exists a sequence of functions $g_k\colon \mathbb{R}\to\mathbb{R}$ and a sequence $\{w_k\}$ of positive solutions to* (3.1)*, such that each function g_k grows subcritically and satisfies the hypotheses* $(\mathrm{H1})_k$*,* $(\mathrm{H2})_k$ *and* $(\mathrm{F})_k$ *of Theorem* 1.2*, and the sequence $\{w_k\}$ of positive solutions to* (3.1)*, is unbounded in the $L^\infty(\Omega)$-norm. Moreover, this sequence $\{w_k\}$ is also unbounded in the $L^{2^*}(\Omega)$-norm.*

Let now $v := v_k$ denote the solution to

$$\begin{cases} v'' + \dfrac{N-1}{r} v' + g(v) = 0, & r\in(0,1],\\ v(0)=k, & v'(0)=0. \end{cases} \tag{3.13}$$

Since $\Gamma(s)=0$ for all $s\le k$, and the solution is decreasing, by Pohozaev's identity

$$r(v'(r))^2 + \frac{N-2}{4N}\, r\, v(r)^{2N/(N-2)} + \frac{N-2}{2}\, v(r)v'(r) = 0, \quad \text{for all } r\in[0,1].$$

Hence, if $v(\widehat{r})=0$ for some $\widehat{r}\in(0,1]$, then $v'(\widehat{r})=0$ and the uniqueness of the solution of the IVP (3.13), implies $v(r)=0$ for all $r\in[0,1]$. Since this contradicts $v(0)=k>0$ we conclude that $v(r)>0$ for all $r\in[0,1]$. Therefore, by continuous dependence on initial conditions, there exists $d_k'\in\big(k, k^{N/(N-2)}\big)$ such that the solution $z=z_k$ to

$$\begin{cases} z'' + \dfrac{N-1}{r}\, z' + g_k(z) = 0, & r\in(0,1],\\ z(0)=d_k', & z'(0)=0. \end{cases}$$

satisfies $z(r)\ge 0$ for all $r\in[0,1]$, and $z(1)=0$.

Corollary 3.2. *For any $k\in\mathbb{N}$, the* BVP (3.1) *has at least two positive solutions.*

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Alfonso Castro
Department of Mathematics
Harvey Mudd College
Claremont, CA 91711, USA

E-mail address: castro@math.hmc.edu

Nsoki Mavinga
Department of Mathematics and Statistics
Swarthmore College
Swarthmore, PA 19081-1390, USA

E-mail address: mavinga@swarthmore.edu

Rosa Pardo
Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040–Madrid, SPAIN

E-mail address: rpardo@mat.ucm.es