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ON SOME APPLICATIONS OF CONVOLUTION TO LINEAR DIFFERENTIAL EQUATIONS WITH LEVITAN ALMOST PERIODIC COEFFICIENTS

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Abstract. We investigate some properties of Levitan almost periodic functions with particular emphasis on their behavior under convolution. These considerations allow us to establish the main result concerning Levitan almost periodic solutions to linear differential equations of the first order. In particular, we state a condition, which guarantees that a special linear equation possesses a Levitan almost periodic solution. We also compare the class of Levitan almost periodic functions and the class of almost periodic functions with respect to the Lebesgue measure, and simultaneously, give an answer to the open question posed by Basit and Günzler in the paper [2].

1. Introduction

In the years 1924–1926, Bohr introduced the class of uniformly almost periodic functions (see [4]–[6]). The basic notion of almost periodicity was extended in many directions. This led to creation of many classes of almost periodic functions. Among many classes of almost periodic functions an important role play the classes of Stepanov, Weyl and Besicovitch almost periodic functions

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(see e.g. [1], [3], [17], [25]). Another generalization of almost periodicity was introduced by Bochner who defined the concept of almost automorphic functions (see e.g. [13], [15], [16]). In this paper we are going to investigate Levitan almost periodic functions (briefly: LAP functions) and almost periodic functions in view of the Lebesgue measure (briefly: μ-a.p. functions).

The class of almost periodic functions in view of the Lebesgue measure was introduced by Stepanov in 1926. He called it the class of almost periodic functions of the first type ([23]). The space of these functions is more general than the space of uniformly almost periodic functions. Let us notice that in the case of μ-a.p. functions we consider functions measurable in the Lebesgue sense. Thus, in general, a μ-a.p. function does not have to be continuous or even locally integrable in the Lebesgue sense and this fact leads to many problems. In the paper [10], we obtained many results concerning behavior of such functions under convolution.

With every uniformly almost periodic function one can associate a Fourier series of the form

$$\sum_{n=1}^{+\infty} A_n e^{i\lambda_n x},$$

for some countable set $\{\lambda_n \colon n \in \mathbb{N}\} \subset \mathbb{R}$. One of fundamental theorems of the theory of uniformly almost periodic functions states that if two uniformly almost periodic functions have the same Fourier series, then they are identical. In 1937, Levitan extended this uniqueness theorem to a larger class of functions and thereby introduced the class of N-almost periodic functions which now we use to call Levitan almost periodic functions (see e.g. [17], [18], [25]). Let us emphasize that in the extended version of the Fourier series uniqueness theorem, Levitan considered LAP functions satisfying the additional condition

$$\limsup_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|\,dt < +\infty. \tag{1.1}$$

This condition is natural when we would like to consider a Fourier series of an LAP function. Nevertheless, LAP functions do not have to be bounded or even do not have to satisfy condition (1.1), although they are continuous (see Example 2.19). Therefore investigation of this class is quite difficult.

The main goal of this paper is to investigate convolution operators generated by L^1 functions in the space of Levitan almost periodic functions. Moreover, we state a certain lemma which explicitly allows to construct many nontrivial examples of LAP functions. This lemma reveals some nature of the phenomenon of N-almost periodicity because examples constructed with the help of this lemma are "purely" LAP functions, that is, in general these functions are not almost periodic in the sense of Stepanov, Weyl, Besicovich as well as in view of the Lebesgue measure (see Example 2.19). We also compare the classes of

LAP functions and μ-a.p. functions with the strong restrictions on both classes. Simultaneously, we give an answer to the open question posed by Basit and Günzler in the paper [2], who wanted to find an *explicit* example of a uniformly continuous and bounded LAP function, which is not uniformly almost periodic.

Almost periodic functions have applications in many fields. Almost periodic patterns, which correspond to almost periodic measures, describe the structure of quasicrystals (see e.g. [22] for more details). In the biological sciences there is considered the so-called leaky integrate-and-fire model (see e.g. [21]). This model describes, roughly speaking, the process of transferring information by neurons. It is also worth to mention, that the restriction of every elliptic function to the real axis is an almost periodic function in view of the Lebesgue measure.

In the theory of differential equations there are many theorems on the existence and uniqueness of almost periodic as well as almost automorphic solutions to some type of differential equations (see e.g. [9], [8], [11]–[20]). Let us consider the linear differential equation

$$y' = A(x)y + f(x), \quad \text{for } x \in \mathbb{R}, \tag{1.2}$$

where $A(x)$ is a matrix and f is a vector-valued function. In the theory of almost periodic functions the classical result for equation (1.2) is Favard's theorem (see [14]). According to this theorem, the linear differential equation (1.2) with Bohr almost periodic coefficients admits at least one Bohr almost periodic solution if it has a bounded solution, and each nontrivial bounded solution ψ to every limiting equation

$$y' = B(x)y, \quad \text{where } B(x) = \lim_{n\to+\infty} A(x + x_n),$$

for some sequence $\{x_n\}$ such that $x_n \in \mathbb{R}$ for $n \in \mathbb{N}$, is separated from zero, that is $\inf_{x\in\mathbb{R}} |\psi(x)| > 0$. Favard's theorem was extended to linear equations with almost automorphic coefficients as well as with Levitan almost periodic coefficients (see e.g. [17]–[20]).

In the paper [11], the authors considered equation (1.2) with Levitan almost periodic coefficients $f\colon \mathbb{R} \to E$, $A\colon \mathbb{R} \to B(\mathbb{R})$, where E is a Banach space and $B(E)$ is the space of bounded linear operators. They proved that, if equation (1.2) has at least one bounded solution, and all nontrivial bounded solutions ψ of the homogeneous equation

$$y' = A(x)y$$

are homoclinic, that is $\lim_{|x|\to+\infty} |\psi(x)| = 0$, then equation (1.2) has a unique Levitan almost periodic solution, which is bounded. In general, this solution is not given *explicitly*.

In this article we consider a special case of equation (1.2), namely

$$y' = \lambda y + f(x), \quad \operatorname{Re}\lambda \neq 0, \tag{1.3}$$

where f is a complex-valued Levitan almost periodic function. In this case, by the above result we know that if (1.3) has a bounded solution, then it possesses a unique bounded Levitan almost periodic solution. The solution to (1.3) usually can be expressed by means of convolution. Therefore we are going to investigate the behavior of Levitan almost periodic functions under convolution.

Let us recall that the convolution operator on the space of almost periodic functions was investigated for example in the paper [7]. We prove that in the case of real-valued bounded below (or bounded above) Levitan almost periodic non-homogeneous term, equation (1.3), with $\lambda \in \mathbb{R}$, $\lambda < 0$, has a Levitan almost periodic solution if and only if this equation possesses a bounded solution, and we give some equivalent condition for the existence of a bounded solution to equation (1.3) (Theorems 5.11 and 5.12). Moreover, we give a sufficient condition, under which equation (1.3) has a bounded Levitan almost periodic solution (Theorem 4.4). This condition allows to consider real-valued functions which are neither bounded above nor bounded below (Example 4.7). Furthermore, we show that sometimes equation (1.3) possesses an unbounded LAP solution which can be expressed by means of convolution. We give also an example showing that the convolution is not always a good tool to look for LAP solutions to equation (1.3) (Example 5.9). Let us also add that a more general equation than equation (1.3) in context of almost automorphic solutions was investigated for example in the paper [8].

2. Preliminaries

By μ will be denote the Lebesgue measure on $\mathbb{R}$ and by $L^0(\mathbb{R})$ the family of all equivalence classes of complex-valued Lebesgue measurable functions. For a function $f\colon \mathbb{R} \to \mathbb{C}$, by f_τ, where $\tau \in \mathbb{R}$, we will denote the function $f_\tau\colon \mathbb{R} \to \mathbb{C}$, defined by the formula

$$f_\tau(x) = f(x+\tau) \quad \text{for } x \in \mathbb{R}.$$

For $a \in \mathbb{R}$, we define $\lfloor a \rfloor := z$, where $z \in \mathbb{Z}$ is such that $z \le a < z+1$. For $\eta > 0$ and $f, g \in L^0(\mathbb{R})$ we define

$$D(\eta; f, g) := \sup_{u \in \mathbb{R}} \mu(\{x \in [u, u+1] : |f(x) - g(x)| \ge \eta\}).$$

In the theory of almost periodic functions an important role plays the notion of a relatively dense set.

Definition 2.1 (Bohr). A nonempty set $E \subset \mathbb{R}$ is called *relatively dense* if there exists a positive number ω such that in each open interval $(a, a+\omega)$, $a \in \mathbb{R}$, there exists at least one element of the set E.

Now we recall the definition of a uniformly almost periodic function (or a Bohr almost periodic function).

Definition 2.2 (Bohr). A continuous function $f\colon \mathbb{R} \to \mathbb{C}$ is said to be *uniformly almost periodic* if for every $\varepsilon > 0$ the set

$$E\{\varepsilon; f\}{:=}\Big\{\tau \in \mathbb{R}\colon \sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \le \varepsilon\Big\}$$

is relatively dense.

Theorem 2.3 ([4]). *The limit of a uniformly convergent sequence of almost periodic functions is a uniformly almost periodic function.*

Now we are going to recall the definition of an (ε, η)-almost period of a function $f \in L^0(\mathbb{R})$ and the definition of an almost periodic function in view of the Lebesgue measure.

Definition 2.4 ([24]). Let $f \in L^0(\mathbb{R})$. If $D(\eta; f_\tau, f) \le \varepsilon$, for some $\varepsilon, \eta > 0$, then the real number τ is said to be an (ε, η)-*almost period* (briefly: (ε, η)-a.p.) of the function f.

By $E\{\varepsilon, \eta; f\}$ we will denote the set of all (ε, η)-almost periods of the function f, that is

$$E\{\varepsilon, \eta; f\}{:=}\Big\{\tau \in \mathbb{R} : \sup_{u\in\mathbb{R}} \mu\big(\{x \in [u, u+1] : |f(x+\tau) - f(x)| \ge \eta\}\big) \le \varepsilon\Big\}.$$

Definition 2.5 [24]. A function $f \in L^0(\mathbb{R})$ is said to be *almost periodic in view of the Lebesgue measure* μ (briefly: μ-*a.p.*) if for any numbers $\varepsilon, \eta > 0$, the set $E\{\varepsilon, \eta; f\}$ is relatively dense.

Remark 2.6. For any $\varepsilon > 0$ and $\varepsilon' > \eta > 0$ we have $E\{\varepsilon'; f\} \subset E\{\varepsilon, \eta; f\}$, and by Definitions 2.2 and 2.5 we see that every uniformly almost periodic function is μ-a.p.

Theorem 2.7 ([24]). *Let (λ_n) be an arbitrary sequence of positive numbers convergent to zero. If f is a μ-a.p., then*

$$\lim_{n\to+\infty} \sup_{u\in\mathbb{R}} \mu\big(\{x \in [u, u+1] : \lambda_n |f(x)| \ge 1\}\big) = 0.$$

The following lemma will be used in the sequel.

Lemma 2.8. *If $f \in L^0(\mathbb{R})$ is a μ-a.p. function, then for every $\varepsilon, \eta, \alpha > 0$ the set $E\{\varepsilon, \eta; f\} \cap \alpha\mathbb{Z}$ is relatively dense.*

The proof of the above lemma is similar to the proof of the theorem stating that the sum of μ-a.p. functions is a μ-a.p. function (see e.g. [24] and [17]) and therefore we omit it.

The following result gives nontrivial examples of μ-a.p. functions.

THEOREM 2.9 ([24]). *Let $F\colon \Omega \to \mathbb{C}$, where $\Omega = \{x + iy \in \mathbb{C}\colon -a < y < a\}$, $a > 0$, be a bounded holomorphic function. Assume that the function $g\colon \mathbb{R} \to \mathbb{R}$, given by the formula $g(x) = F(x)$ for $x \in \mathbb{R}$, is uniformly almost periodic. Then the function f, defined by the formula*

$$f(x) = \begin{cases} \dfrac{1}{g(x)} & \text{for } x \in \mathbb{R} \text{ such that } g(x) \neq 0, \\ 0 & \text{for } x \in \mathbb{R} \text{ such that } g(x) = 0, \end{cases}$$

is μ-a.p.

One of several equivalent definitions of Levitan almost periodic functions uses the notion of an $[N,\varepsilon]$-almost period of a function which is defined as follows.

DEFINITION 2.10 ([17]). A real number τ is said to be an $[N,\varepsilon]$-*almost period of a function* $f\colon \mathbb{R} \to \mathbb{C}$ (briefly: $[N,\varepsilon]$-a.p.) if $|f(x+\tau) - f(x)| < \varepsilon$ for $|x| < N$.

Using the notion of an $[N,\varepsilon]$-almost period of a function one can define Levitan almost periodic function as follows.

DEFINITION 2.11 ([17]). A continuous function $f\colon \mathbb{R} \to \mathbb{C}$ is said to be *Levitan almost periodic* (briefly: LAP) if for every $N, \varepsilon > 0$ there exist $\lambda_1, \dots, \lambda_p \in \mathbb{R}$ and $\delta > 0$ such that every number τ, satisfying inequalities $|\lambda_r \tau| < \delta \pmod{2\pi}$ for $r = 1, \dots, p$, is an $[N,\varepsilon]$-a.p. of the function f.

REMARK 2.12. The inequality $|x| < \delta \pmod{2\pi}$ means that there exists $k \in \mathbb{Z}$ such that $-\delta < x - 2k\pi < \delta$.

REMARK 2.13. It is easy to show that $f\colon \mathbb{R} \to \mathbb{C}$ is LAP if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are LAP. A similar fact is true for uniformly almost periodic functions and almost periodic functions in view of the Lebesgue measure. This is a consequence of the following obvious inequalities:

$$|\operatorname{Re} z|, |\operatorname{Im} z| \le |z| \quad \text{and} \quad |z| \le |\operatorname{Re} z| + |\operatorname{Im} z|, \quad \text{for } z \in \mathbb{C}.$$

REMARK 2.14 ([17]). The set of all LAP functions is an algebra.

REMARK 2.15 ([17], [18]). Every uniformly almost periodic function is an LAP function.

Now we recall some basic results concerning LAP functions which will be useful in the sequel.

THEOREM 2.16. *The limit of a uniformly convergent sequence of* LAP *functions is an* LAP *function.*

PROOF. The proof follows from the inequality

$$|f(x+\tau) - f(x)| \le |f(x+\tau) - f_{n_0}(x+\tau)| + |f_{n_0}(x+\tau) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)|. \quad \square$$

Theorem 2.17 ([25]). *If $g\colon \mathbb{R} \to \mathbb{R}$ is uniformly almost periodic and $f\colon g(\mathbb{R}) \to \mathbb{R}$ is continuous, then the superposition $f \circ g$ is Levitan almost periodic.*

Now we state a very useful lemma which describes another way to construct examples of LAP functions.

Lemma 2.18. *Let (f_n) be a sequence of continuous $2\cdot 3^{n+1}$-periodic functions such that*

$$\operatorname{supp} f_n \subset \big[3^n, 3^n+1\big] + 2\cdot 3^{n+1}\mathbb{Z} \quad \text{for } n \in \mathbb{N}. \tag{2.1}$$

Let, moreover,

$$f(x) = \begin{cases} f_n(x) & \text{for } x \in [3^n, 3^n+1] + 2\cdot 3^{n+1}\mathbb{Z},\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the function f is well defined and it is Levitan almost periodic.

Proof. The function f is well defined because the sets $[3^n, 3^n+1]+2\cdot 3^{n+1}\mathbb{Z}$, for $n \in \mathbb{N}$, are pairwise disjoint. Let us fix arbitrary $N, \varepsilon > 0$. Let us choose $n_0 \in \mathbb{N}$ such that $3^{n_0} \geq N+1$. We define

$$\begin{aligned} A_{n_0} &= \big[-3^{n_0}, 3^{n_0}\big] + 2\cdot 3^{n_0+1}\mathbb{Z}, \\ B_n &= \big(3^n, 3^n+1\big) + 2\cdot 3^{n+1}\mathbb{Z} \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

First, we will show that $A_{n_0} \cap B_n = \emptyset$ for $n > n_0$. Since $2\cdot 3^{n+1} + A_{n_0} = A_{n_0}$ and $2\cdot 3^{n+1} + B_n = B_n$, for $n > n_0$, it will be sufficient to show that

$$A_{n_0} \cap B_n \cap \big[3^n + 1 - 2\cdot 3^{n+1}, 3^n+1\big] = \emptyset.$$

We have

$$B_n \cap \big[3^n + 1 - 2\cdot 3^{n+1}, 3^n+1\big] = \big(3^n, 3^n+1\big).$$

We will show that for $k \in \mathbb{Z}$ and $n > n_0$ we have

$$\big(\big[-3^{n_0}, 3^{n_0}\big] + 2k\cdot 3^{n_0+1}\big) \cap \big(3^n, 3^n+1\big) = \emptyset. \tag{2.2}$$

For $n = n_0+1$ we have $3^{n_0} < 3^n$ and $-3^{n_0} + 2\cdot 3^{n_0+1} > 3^n + 1$. Thus condition (2.2) is satisfied.

For $n > n_0+1$ we have

$$3^{n_0} + 2\cdot 3^{n_0} + (2\cdot 3^{n_0+1} + \ldots + 2\cdot 3^{n-1}) = 3^n. \tag{2.3}$$

Thus we have

$$3^{n_0} + (2\cdot 3^{n_0+1} + \ldots + 2\cdot 3^{n-1}) < 3^n$$

and adding to both sides of equality (2.3) the number $2\cdot 3^{n_0+1} - 4\cdot 3^{n_0}$ we obtain

$$-3^{n_0} + 2\cdot 3^{n_0+1} + (2\cdot 3^{n_0+1} + \ldots + 2\cdot 3^{n-1}) = 3^n + 2\cdot 3^{n_0+1} - 4\cdot 3^{n_0} > 3^n + 1.$$

We found a number $s \in \mathbb{N}$ such that

$$3^{n_0} + 2s\cdot 3^{n_0+1} < 3^n \quad \text{and} \quad -3^{n_0} + 2(s+1)\cdot 3^{n_0+1} > 3^n + 1.$$

This means that (2.2) is satisfied and thereby sets A_{n_0}, B_n are disjoint for $n > n_0$. For $n \le n_0$ and $k \in \mathbb{Z}$ we have

$$x \in B_n \ \Leftrightarrow \ x + 2k \cdot 3^{n_0+1} \in B_n.$$

Thus, for $|x| \le N+1$ and $k \in \mathbb{Z}$, we have

$$f(x + 2k \cdot 3^{n_0+1}) - f(x) = 0$$

because $x, x + 2k \cdot 3^{n_0+1} \in A_{n_0}$ and according to (2.1) we have

$$x \notin B_n \ \Rightarrow \ f_n(x) = 0, \quad \text{for } n \in \mathbb{N} \ \ (f_n \text{ are continuous}).$$

The function f is uniformly continuous on the interval $[-N-1, N+1]$. Thus there exists $\delta \in (0,1)$ such that for $x, y \in [-N-1, N+1]$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Let $\tau = 2k \cdot 3^{n_0+1} + h$ for some $k \in \mathbb{Z}$, $|h| < \delta$. Then, for $|x| \le N$, we have

$$|f(x+\tau) - f(x)| = |f(x + h + 2k \cdot 3^{n_0+1}) - f(x)| = |f(x+h) - f(x)| < \varepsilon.$$

We showed that every number τ which satisfies the inequality

$$\left| \frac{2\pi}{2 \cdot 3^{n_0+1}} \, \tau \right| < \frac{2\pi\delta}{2 \cdot 3^{n_0+1}} \pmod{2\pi}$$

is an $[N, \varepsilon]$-a.p. of the function f. This means that f is LAP. □

Example 2.19. Let

$$f(x) = \begin{cases} n \cdot 3^{n+1} \sin(2\pi x) & \text{for } x \in [3^n, 3^n+1] + 2 \cdot 3^{n+1}\mathbb{Z}, \ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, from Lemma 2.18, f is LAP. Moreover, we have

$$\frac{1}{2 \cdot 3^{n+1}} \int_{-3^{n+1}}^{3^{n+1}} |f(t)| \, dt \ge \frac{1}{2 \cdot 3^{n+1}} \int_{3^n}^{3^n+1} |f(t)| \, dt = \frac{1}{2 \cdot 3^{n+1}} \cdot \frac{2n}{\pi} \cdot 3^{n+1} = \frac{n}{\pi},$$

for $n \in \mathbb{N}$. Therefore

$$\limsup_{T \to +\infty} \ \frac{1}{2T} \int_{-T}^{T} |f(t)| \, dt = +\infty.$$

The function f is not almost periodic in the sense of Stepanov, Weyl and Besicovitch, because it is well known in the theory of almost periodic functions that each almost periodic function of such type satisfies the condition

$$\limsup_{T \to +\infty} \ \frac{1}{2T} \int_{-T}^{T} |f(t)| \, dt < +\infty \quad \text{(see e.g. [1]).}$$

Moreover, the function f satisfies

$$\lim_{n \to +\infty} \sup_{u \in \mathbb{R}} \mu\big(\{x \in [u, u+1] \colon |f(x)| \ge n\}\big) = 1,$$

and, from Theorem 2.7, f is not μ-a.p.

We will also need the following

LEMMA 2.20 ([18]). *For any numbers* $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$, $\delta > 0$ *the set*

$$\big\{\tau \in \mathbb{R} : |\tau\lambda_r| < \delta \pmod{2\pi} \text{ for } r = 1, \ldots, p\big\}$$

is relatively dense.

REMARK 2.21. From the above lemma if $f\colon \mathbb{R} \to \mathbb{C}$ is LAP, then

$$\liminf_{x\to-\infty} |f(x)| < +\infty.$$

3. The comparison of LAP and μ-a.p. functions

In this section we want to establish a relation between the class of Levitan almost periodic functions and the class of μ-a.p. functions. The class of LAP functions is defined on the space continuous functions, while the class of μ-a.p. functions is defined on the space of functions measurable in the Lebesgue sense. Obviously, if we want to compare these classes we should compare the classes of Levitan almost periodic functions and continuous μ-a.p. functions.

The intersection of these two classes contains uniformly almost periodic functions, which are uniformly continuous and bounded ([4]). Moreover, this intersection contains some continuous and unbounded functions. Indeed, let g be a generalized trigonometric polynomial of constant sign and such that $\inf_{x\in\mathbb{R}} |f(x)| = 0$. Then, by Theorems 2.9 and 2.17, we know that the function $f\colon \mathbb{R} \to \mathbb{R}$, defined by the formula $f(x) = 1/g(x)$ for $x \in \mathbb{R}$, is simultaneously LAP and μ-a.p. In particular, the function defined as follows:

$$f(x) = \frac{1}{2 + \cos x + \cos\left(\sqrt{2}x\right)} \quad \text{for } x \in \mathbb{R},$$

is LAP and μ-a.p.

The first example below shows that there exists a uniformly continuous and bounded Levitan almost periodic function, which is not μ-a.p.

EXAMPLE 3.1. Let

$$f(x) = \begin{cases} \sin(2\pi x) & \text{for } x \in \displaystyle\bigcup_{n=1}^{+\infty} \big([3^n, 3^n+1] + 2\cdot 3^{n+1}\mathbb{Z}\big), \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.18, the function f is Levitan almost periodic. Moreover, the function f is uniformly continuous and bounded. We will show that this function is not μ-a.p.

Let us assume that f is μ-a.p. Let us denote $A_n = 3^n + 2 \cdot 3^{n+1}\mathbb{Z}$, for $n \in \mathbb{N}$. For $z \in \bigcup_{n=1}^{+\infty} A_n$ we have

$$\mu\left(\left\{x \in [z, z+1] : |f(x)| \geq \frac{1}{2}\right\}\right) = \frac{2}{3}.$$

By Lemma 2.8, the set $E\{1/3, 1/2; f\} \cap 18\mathbb{Z}$ is relatively dense.

If $\tau \in E\{1/3, 1/2; f\} \cap 18\mathbb{Z}$ and $z \in \bigcup_{n=1}^{+\infty} A_n$, then $\tau + z \in \bigcup_{n=1}^{+\infty} A_n$. Indeed, let us suppose that $\tau + z \notin \bigcup_{n=1}^{+\infty} A_n$. Then

$$\begin{aligned}\mu\left(\left\{x \in [z, z+1] : |f(x+\tau) - f(x)| \geq \frac{1}{2}\right\}\right)& \\ = \mu\left(\left\{x \in [z, z+1] : |f(x)| \geq \frac{1}{2}\right\}\right) &= \frac{2}{3}.\end{aligned}$$

This means that $\tau \notin E\{1/3, 1/2; f\}$.

Since the set $E\{1/3, 1/2; f\}$ is symmetric, for $\tau \in E\{1/3, 1/2; f\}$ and $z \in \bigcup_{n=1}^{+\infty} A_n$ we have $-\tau + z \in \bigcup_{n=1}^{+\infty} A_n$. This leads to the equality

$$\bigcup_{n=1}^{+\infty} (\tau + A_n) = \bigcup_{n=1}^{+\infty} A_n. \tag{3.1}$$

Let us fix $\tau \in 18\mathbb{Z} \setminus \{0\}$. Let $\tau = 2 \cdot 3^{s+1} m$, for some $s \in \mathbb{N}$, $m \notin 3\mathbb{Z}$. Every set A_n, where $n \leq s$, satisfies $2 \cdot 3^{s+1} m + A_n = A_n$. Therefore

$$\bigcup_{n=1}^{s} (\tau + A_n) = \bigcup_{n=1}^{s} A_n.$$

The sets $\tau + A_n$, for $n \in \mathbb{N}$, are pairwise disjoint, and the sets A_n, for $n \in \mathbb{N}$, also are pairwise disjoint, so (3.1) is equivalent to the equality

$$\bigcup_{n=s+1}^{+\infty} (\tau + A_n) = \bigcup_{n=s+1}^{+\infty} A_n$$

or equivalently to the equality

$$\bigcup_{i=1}^{+\infty} (2m + 3^{i-1} + 2 \cdot 3^i \mathbb{Z}) = \bigcup_{i=1}^{+\infty} (3^{i-1} + 2 \cdot 3^i \mathbb{Z}).$$

We will show that the above equality leads to a contradiction.

For $i = 2$ and arbitrary $z_1 \in \mathbb{Z}$ there exist $j \geq 1$, $z_2 \in \mathbb{Z}$ such that

$$2m + 3 + 18z_1 = 3^{j-1} + 2 \cdot 3^j z_2.$$

Suppose that $j > 1$. Then

$$2m = -3 - 18z_1 + 3^{j-1} + 2 \cdot 3^j z_2 = 3(-1 - 6z_1 + 3^{j-2} + 2 \cdot 3^{j-1} z_2).$$

So $2m \in 3\mathbb{Z}$, but this is impossible, because $m \notin 3\mathbb{Z}$. Therefore for $i = 2$ we have $j = 1$. Then

$$2m + 3 + 18z_1 = 1 + 6z_2$$

and $m + 1 = -9z_1 + 3z_2$. Thus the number m satisfies $m + 1 \in 3\mathbb{Z}$.

Moreover, for $i = 1$ and arbitrary $z_1 \in \mathbb{Z}$ there exist $j \geq 1$, $z_2 \in \mathbb{Z}$ such that

$$2m + 1 + 6z_1 = 3^{j-1} + 2 \cdot 3^j z_2.$$

Suppose that $j > 1$. Then

$$2m + 1 = -6z_1 + 3^{j-1} + 2 \cdot 3^j z_2 = 3(-2z_1 + 3^{j-2} + 2 \cdot 3^{j-1} z_2),$$

so $2m + 1 \in 3\mathbb{Z}$. This implies that $m \in 3\mathbb{Z}$, because we know that $m + 1 \in 3\mathbb{Z}$. Therefore for $i = 1$ we have $j = 1$ and

$$2m + 1 + 6z_1 = 1 + 6z_2.$$

We obtain that $m = -3z_1 + 3z_2$. This is a contradiction, because $m \notin 3\mathbb{Z}$. We have shown that for $\tau \in 18\mathbb{Z} \setminus \{0\}$ we have

$$\bigcup_{n=1}^{+\infty} (\tau + A_n) \neq \bigcup_{n=1}^{+\infty} A_n,$$

so $\tau \notin E\{1/3, 1/2; f\}$. Thus f is not μ-a.p.

Remark 3.2. In the paper [2], Basit and Günzler posed the following problem:

Give an explicit example showing that AP *is strictly contained in* BAA_u,

where AP denotes the class of uniformly almost periodic functions and BAA_u denotes the class of uniformly continuous, almost automorphic functions in the sense of Bochner. In this article we do not recall the definition of almost automorphic functions, but even from remarks in the paper [2] we know that the class of uniformly continuous and bounded almost automorphic functions coincides with the class of uniformly continuous and bounded LAP functions. Furthermore, every uniformly almost periodic function is μ-a.p. (see Remark 2.6). Therefore the above example gives a stronger answer.

The next example shows that there exists a continuous and bounded function μ-a.p., which is not a Levitan almost periodic function.

Example 3.3. Let us define the sets $A_n = 2^{n-1} + 2^n\mathbb{Z}$ for $n \in \mathbb{N}$. Let us observe that

$$\bigcup_{n=1}^{+\infty} A_n = \mathbb{Z} \setminus \{0\}.$$

Indeed, we have $A_1 = 1 + 2\mathbb{Z}$. Let $z = 2^k(1 + 2l)$, where $k \in \mathbb{N}$, $l \in \mathbb{Z}$. Then

$$z = 2^k + 2^{k+1}l \in A_{k+1}.$$

Suppose that $0 \in A_n$ for some $n \in \mathbb{N}$. Then there exists $k \in \mathbb{Z}$ such that

$$0 = 2^{n-1} + 2^n k \quad \Rightarrow \quad 0 = 2^{n-1}(1+2k).$$

This is impossible, because the right-hand side of the above equation is the product of two nonzero numbers. Moreover,

$$A_{n+1} \subset \mathbb{Z} \setminus \left(\bigcup_{i=1}^{n} A_n \right) = 2^n \mathbb{Z} \quad \text{for } n \in \mathbb{N}.$$

Let

$$f(x) = \begin{cases} \cos(4\pi n x) & \text{for } x \in \left[2^{n-1} + \frac{1}{2} - \frac{1}{8n}, 2^{n-1} + \frac{1}{2} + \frac{1}{8n}\right] + 2^n\mathbb{Z},\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The function f is well defined, because the sets

$$\left[2^{n-1} + \frac{1}{2} - \frac{1}{8n}, 2^{n-1} + \frac{1}{2} + \frac{1}{8n}\right], \quad \text{for } n \in \mathbb{N},$$

are pairwise disjoint. We will show that for every $\varepsilon, \eta > 0$ and for $1/n \le \varepsilon$ we have $2^n\mathbb{Z} \subset E\{\varepsilon, \eta; f\}$.

Fix $\varepsilon, \eta > 0$. Let $\tau \in 2^n\mathbb{Z}$. If $z \in \mathbb{Z}$, then $z \in \bigcup_{i=1}^{n} A_i$ if and only if $\tau + z \in \bigcup_{i=1}^{n} A_i$. For $z \in \bigcup_{i=1}^{n} A_i$, we have

$$\mu(\{x \in [z, z+1] : f(x+\tau) \neq f(x)\}) = 0.$$

Further, for $z \notin \bigcup_{i=1}^{n} A_i$, the following inequality holds:

$$\mu(\{x \in [z, z+1] : f(x) \neq 0\}) \le \frac{1}{4n}.$$

Therefore for $z \notin \bigcup_{i=1}^{n} A_i$ we get

$$\begin{aligned} &\mu(\{x \in [z, z+1] : f(x+\tau) \neq f(x)\}) \\ &\quad \le \mu(\{x \in [z, z+1] : f(x+\tau) \neq 0\}) + \mu(\{x \in [z, z+1] : f(x) \neq 0\}) \le \frac{1}{2n}. \end{aligned}$$

This implies that for arbitrary $u \in \mathbb{R}$ we have

$$\begin{aligned} \mu(\{x \in [u, u+1] &: |f(x+\tau) - f(x)| \ge \eta\}) \\ &\le \mu(\{x \in [u, u+1] : f(x+\tau) \neq f(x)\}) \\ &\le \mu(\{x \in [\lfloor u \rfloor, \lfloor u \rfloor + 1]\ f(x+\tau) \neq f(x)\}) \\ &\quad + \mu(\{x \in [\lfloor u \rfloor + 1, \lfloor u \rfloor + 2] : f(x+\tau) \neq f(x)\}) \le \frac{1}{n} \le \varepsilon. \end{aligned}$$

Every set $E\{\varepsilon,\eta;f\}$ contains the relatively dense set $2^n\mathbb{Z}$, so $E\{\varepsilon,\eta;f\}$ is also relatively dense. Thus f is a μ-a.p. function. Moreover, f is continuous because it is continuous on every interval $[z,z+1]$, for $z\in\mathbb{Z}$.

By the construction of the function f we deduce, that for $x\in[0,1]$ we have $f(x)=0$, and for $z\in\mathbb{Z}\setminus\{0\}$ we have $f(z+1/2)=1$. Let us fix $\tau\geq 1$. There are two cases.

(i) If $-1<\lfloor\tau\rfloor-\tau<-1/2$, let $x=3/2+\lfloor\tau\rfloor-\tau$. Then $x\in[0,1]$, $f(x)=0$ and $f(x+\tau)=1$. Thus $|f(x+\tau)-f(x)|=1>1/2$.

(ii) If $-1/2\leq\lfloor\tau\rfloor-\tau\leq 0$, then for $x=1/2+\lfloor\tau\rfloor-\tau$ we obtain the analogous conclusion.

Therefore if $\tau\geq 1$, then τ is not $[1,1/2]$-a.p. of the function f. By Lemma 2.20, the function f is not LAP.

Remark 3.4. Every uniformly continuous and bounded μ-a.p. function is uniformly almost periodic (see [24]); it is also is an LAP function (Remark 2.15). Thus there is no uniformly continuous and bounded function μ-a.p., which is not LAP.

4. Convolution of Levitan almost periodic functions

In this section we are going to investigate the convolution of LAP functions with some functions integrable in the Lebesgue sense. First we recall a result for bounded LAP functions.

Theorem 4.1 ([9]). *Let $f\colon\mathbb{R}\to\mathbb{R}$ be a bounded Levitan almost periodic function and $g\colon\mathbb{R}\to\mathbb{R}$ be a Lebesgue integrable function. Then the convolution $f*g\colon\mathbb{R}\to\mathbb{R}$, defined by the following formula:*

$$(f*g)(x)=\int_{-\infty}^{+\infty}f(x-t)g(t)\,dt,$$

is bounded and almost periodic in the sense of Levitan.

For every LAP function we shall prove the following

Theorem 4.2. *Let $f\colon\mathbb{R}\to\mathbb{C}$ be a Levitan almost periodic function and $g\colon\mathbb{R}\to\mathbb{C}$ be a compactly supported Lebesgue integrable function. Then $f*g$ exists and it is* LAP.

Proof. Clearly, for every $x\in\mathbb{R}$ the convolution exists, because g is compactly supported and f is a continuous function. If $\int_{\mathbb{R}}|g(t)|\,dt=0$, then the conclusion of the theorem is obvious. Let us assume that $\eta:=\int_{\mathbb{R}}|g(t)|\,dt>0$. Let, moreover, $\operatorname{supp}g\subset[-M,M]$, for some $M>0$. Let us fix arbitrary $\varepsilon,N>0$. We will show the following implication: if for some $\tau\in\mathbb{R}$ we have

$$|f(x+\tau)-f(x)|<\frac{\varepsilon}{\eta}\quad\text{for }|x|<M+N,$$

then

$$|(f*g)(x+\tau)-(f*g)(x)|<\varepsilon \quad \text{for } |x|<N.$$

Indeed, we have

$$\begin{aligned}|(f*g)(x+\tau)-(f*g)(x)| &\le \int_{-\infty}^{+\infty}|f(x+\tau-t)-f(x-t)||g(t)|\,dt\\ &= \int_{-M}^{M}|f(x+\tau-t)-f(x-t)||g(t)|\,dt < \frac{\varepsilon}{\eta}\int_{-M}^{M}|g(t)|\,dt=\varepsilon.\end{aligned}$$

The continuity of the convolution $f*g$ follows from the above implication. Moreover, since f is Levitan almost periodic, there exist numbers $\lambda_1,\dots,\lambda_p$ and $\delta>0$ such that every number which satisfies inequalities $|\tau\lambda_r|<\delta \pmod{2\pi}$ for $r=1,\dots,p$, is $[(M+N),\varepsilon/\eta]$-a.p. of the function f. We have shown that every $[(M+N),\varepsilon/\eta]$-a.p. of the function f is $[N,\varepsilon]$-a.p. of the convolution $f*g$. □

Now we will consider the convolution of LAP functions with the function $g_\lambda\colon\mathbb{R}\to\mathbb{C}$ ($\operatorname{Re}\lambda<0$), given by the formula

$$g_\lambda(x)=\begin{cases}e^{\lambda x} & \text{for } x\ge 0,\\ 0 & \text{for } x<0.\end{cases}$$

Remark 4.3. Let us notice that

$$(f*g_\lambda)(x)=\int_{-\infty}^{+\infty}f(t)g_\lambda(x-t)\,dt=\int_{-\infty}^{x}f(t)e^{\lambda(x-t)}\,dt=e^{\lambda x}\int_{-\infty}^{x}f(t)e^{-\lambda t}\,dt.$$

Moreover, the existence of the convolution $f*g_\lambda$ (for every $x\in\mathbb{R}$) of an LAP function f with the function g_λ is equivalent to the condition

$$\int_{-\infty}^{0}|f(t)|e^{-\operatorname{Re}\lambda t}\,dt<+\infty.$$

The continuity of the convolution follows from the fact that the function $t\mapsto f(t)e^{-\lambda t}$ is locally integrable and from the form of this convolution.

Theorem 4.4. *Let $f\colon\mathbb{R}\to\mathbb{C}$ be a Levitan almost periodic function which satisfies the condition*

$$\sup_{u\in\mathbb{R}}\int_{u}^{u+1}|f(t)|\,dt<+\infty. \tag{4.1}$$

Then $f*g_\lambda$ exists for all $x\in\mathbb{R}$ and it is bounded, and Levitan almost periodic.

Proof. The convolution $f*g_\lambda$ exists for all $x\in\mathbb{R}$ because

$$\begin{aligned}&\int_{-\infty}^{0}|f(t)|e^{-\operatorname{Re}\lambda t}\,dt=\sum_{r=1}^{+\infty}\int_{-r}^{-r+1}|f(t)|e^{-\operatorname{Re}\lambda t}\,dt\\ &\le\sum_{r=1}^{+\infty}e^{-\operatorname{Re}\lambda(-r+1)}\int_{-r}^{-r+1}|f(t)|\,dt\le\sup_{u\in\mathbb{R}}\int_{u}^{u+1}|f(t)|\,dt\cdot\sum_{r=1}^{+\infty}e^{\operatorname{Re}\lambda(r-1)}<+\infty.\end{aligned}$$

Moreover, it is bounded because we have

$$
\begin{aligned}
|(f * g_\lambda)(x)| &= \left| e^{\lambda x} \int_{-\infty}^{x} f(t) e^{-\lambda t}\, dt \right| = \left| e^{\lambda x} \sum_{r=1}^{+\infty} \int_{x-r}^{x-r+1} f(t) e^{-\lambda t}\, dt \right| \\
&\le e^{\operatorname{Re}\lambda x} \sum_{r=1}^{+\infty} e^{-\operatorname{Re}\lambda (x-r+1)} \int_{x-r}^{x-r+1} |f(t)|\, dt \\
&\le \sup_{u\in\mathbb{R}} \int_{u}^{u+1} |f(t)|\, dt \cdot \sum_{r=1}^{+\infty} e^{\operatorname{Re}\lambda (r-1)} < +\infty.
\end{aligned}
$$

Let us define the sequence (g_n) of functions by the formulae $g_n(x) = g_\lambda \chi_{[0,n]}$ for $n \in \mathbb{N}$, where $\chi_{[0,n]}$ denotes the characteristic function of the interval $[0,n]$. By Theorem 4.2, we know that the convolution $f * g_n$ is LAP. We will show that $f * g_n$ converges uniformly to $f * g_\lambda$. Since

$$
f * g_n(x) = e^{\lambda x} \int_{x-n}^{x} f(t) e^{-\lambda t}\, dt,
$$

we have

$$
\begin{aligned}
|(f * g_\lambda)(x) - (f * g_n)(x)| &= \left| e^{\lambda x} \int_{-\infty}^{x} f(t) e^{-\lambda t}\, dt - e^{\lambda x} \int_{x-n}^{x} f(t) e^{-\lambda t}\, dt \right| \\
&\le e^{\operatorname{Re}\lambda x} \int_{-\infty}^{x-n} |f(t)| e^{-\operatorname{Re}\lambda t}\, dt = e^{\operatorname{Re}\lambda x} \sum_{r=1}^{+\infty} \int_{x-n-r}^{x-n-r+1} |f(t)| e^{-\operatorname{Re}\lambda t}\, dt \\
&\le e^{\operatorname{Re}\lambda x} \sum_{r=1}^{+\infty} e^{-\operatorname{Re}\lambda (x-n-r+1)} \int_{x-n-r}^{x-n-r+1} |f(t)|\, dt \\
&\le \sup_{u\in\mathbb{R}} \int_{u}^{u+1} |f(t)|\, dt \cdot \sum_{r=1}^{+\infty} e^{\operatorname{Re}\lambda (n+r-1)} \to 0
\end{aligned}
$$

as $n \to +\infty$, uniformly in view $x \in \mathbb{R}$. By Theorem 2.16, the convolution $f * g$ is LAP. □

Remark 4.5. The condition

$$
\sup_{u\in\mathbb{R}} \int_{u}^{u+1} |f(t)|\, dt < +\infty
$$

is equivalent to the conditions

$$
\sup_{u\in\mathbb{R}} \int_{u}^{u+1} |\operatorname{Re} f(t)|\, dt < +\infty \quad \text{and} \quad \sup_{u\in\mathbb{R}} \int_{u}^{u+1} |\operatorname{Im} f(t)|\, dt < +\infty.
$$

Remark 4.6. From the paper [9], we know that the convolution of a real-valued bounded Levitan almost periodic function and the function g_λ is Levitan almost periodic. Theorem 4.4 is a more general result, because every bounded LAP function satisfies condition (4.1).

The next example describes a real-valued Levitan almost periodic function, which is neither bounded below nor bounded above, and satisfies condition (4.1).

EXAMPLE 4.7. Let

$$f(x) = \begin{cases} n \sin(2\pi n x) & \text{for } x \in \left[3^n, 3^n + \dfrac{1}{n}\right] + 2 \cdot 3^{n+1}\mathbb{Z},\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is neither bounded below nor bounded above. By Lemma 2.18, we know that f is a Levitan almost periodic function. It is easy to establish that f satisfies condition (4.1). By Theorem 4.4, we know that the convolution $f * g_\lambda$ is a bounded Levitan almost periodic function.

The next theorem gives a sufficient condition which guarantees that the convolution with the function g_λ is not a Levitan almost periodic function.

THEOREM 4.8. *Let $\lambda \in \mathbb{R}$, $\lambda < 0$ and $f\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative locally integrable function which satisfies the condition*

$$\sup_{u \in \mathbb{R}} \int_u^{u+1} f(t)\, dt = +\infty. \tag{4.2}$$

If the convolution $f * g_\lambda$ exists, then it is unbounded and it is not a Levitan almost periodic function.

PROOF. From (4.2) we know that for every $\alpha > 0$

$$\sup_{k \in \mathbb{Z}} \int_{k\alpha}^{(k+1)\alpha} f(t)\, dt = +\infty. \tag{4.3}$$

Let us observe that

$$(f * g)(\tau) - (f * g)(0) = e^{\lambda\tau} \int_{-\infty}^{\tau} f(t) e^{-\lambda t}\, dt - \int_{-\infty}^{0} f(t) e^{-\lambda t}\, dt.$$

Let us assume that $f * g_\lambda$ is LAP. Then there exist numbers $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$, $\delta > 0$ such that for every number τ satisfying the inequalities $|\lambda_r \tau| < \delta \pmod{2\pi}$ for $r = 1, \ldots, p$ and $|x| < 1$, we have

$$|(f * g)(x + \tau) - (f * g)(x)| < 1.$$

In particular, for $x = 0$, we obtain

$$\left| e^{\lambda\tau} \int_{-\infty}^{\tau} f(t) e^{-\lambda t}\, dt - \int_{-\infty}^{0} f(t) e^{-\lambda t}\, dt \right| < 1. \tag{4.4}$$

Let $\omega > 0$ describe the relative density of the set

$$\{\tau \in \mathbb{R} : |\lambda_r \tau| < \delta \pmod{2\pi} \text{ for } r = 1, \ldots, p\} \quad \text{(see Lemma 2.20).}$$

For every $k \in \mathbb{Z}$ there exists $[1,1]$-a.p. $\tau_k \in ((k+1)\omega, (k+2)\omega)$. Thus

$$\begin{aligned} e^{\lambda \tau_k} \int_{-\infty}^{\tau_k} f(t) e^{-\lambda t}\, dt &\geq e^{\lambda \tau_k} \int_{k\omega}^{(k+1)\omega} f(t) e^{-\lambda t}\, dt \\ &\geq e^{\lambda \tau_k - \lambda k\omega} \int_{k\omega}^{(k+1)\omega} f(t)\, dt \geq e^{2\lambda\omega} \int_{k\omega}^{(k+1)\omega} f(t)\, dt. \end{aligned}$$

The last inequalities lead to a contradiction, because simultaneously (4.3) and (4.4) are satisfied. By the last inequalities and (4.3) we deduce that $f * g_\lambda$ is unbounded. $\square$

Remark 4.9. Let us emphasize, that in the above theorem we do not assume that f is LAP.

The next example shows that there exists an LAP function, which satisfies the conditions of Theorem 4.8.

Example 4.10. Let

$$f(x) = \begin{cases} n|\sin(2\pi x)| & \text{for } x \in [3^n, 3^n+1] + 2 \cdot 3^{n+1}\mathbb{Z},\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Lemma 2.18, we infer that f is a Levitan almost periodic function. Moreover, for $\lambda \in \mathbb{R}$, $\lambda < 0$, the convolution $f * g_\lambda$ exists, because we have

$$\begin{aligned} \int_{-\infty}^{0} f(t) e^{-\lambda t}\, dt &= \sum_{n=1}^{+\infty} \sum_{k=1}^{+\infty} \int_{z_{k,n}}^{z_{k,n}+1} f(t) e^{-\lambda t}\, dt \\ &\leq \sum_{n=1}^{+\infty} \sum_{k=1}^{+\infty} e^{-\lambda(z_{k,n}+1)} \int_{z_{k,n}}^{z_{k,n}+1} f(t)\, dt \\ &= \sum_{n=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{2n}{\pi} e^{-\lambda(3^n - 2k\cdot 3^{n+1}+1)} < +\infty \quad \text{(cf. Remark 4.3)} \end{aligned}$$

where $z_{k,n} = 3^n - 2k \cdot 3^{n+1}$, for $k, n \in \mathbb{N}$. Moreover,

$$\int_{3^n}^{3^n+1} f(t)\, dt = \frac{2n}{\pi} \quad \text{for } n \in \mathbb{N}.$$

By Theorem 4.8, we infer that the convolution $f * g_\lambda$ is not a Levitan almost periodic function.

Corollary 4.11. *By Theorems* 4.4 *and* 4.8*, it follows that if* $f\colon \mathbb{R} \to \mathbb{R}$ *is a nonnegative* LAP *function such that* $f * g_\lambda$ *exists* ($\lambda \in \mathbb{R}$, $\lambda < 0$)*, then the convolution* $f * g_\lambda$ *is* LAP *if and only if* f *satisfies the condition*

$$\sup_{u \in \mathbb{R}} \int_{u}^{u+1} f(t)\, dt < +\infty.$$

Corollary 4.12. *By Corollary* 4.11*, Remark* 2.14 *and linearity of convolution it follows that if* $f\colon \mathbb{R}\to\mathbb{R}$ *is a bounded below function* (*or bounded above*) *such that* $f*g_\lambda$ *exists* ($\lambda\in\mathbb{R}$, $\lambda<0$)*, then the convolution* $f*g_\lambda$ *is* LAP *if and only if* f *satisfies the condition*

$$\sup_{u\in\mathbb{R}}\int_u^{u+1}|f(t)|\,dt<+\infty.$$

Corollary 4.13. *By Theorems* 4.4 *and* 4.8 *it follows that if* $f\colon\mathbb{R}\to\mathbb{R}$ *is a nonnegative* LAP *function such that* $f*g_\lambda$ *exists* ($\lambda\in\mathbb{R}$, $\lambda<0$)*, then the convolution* $f*g_\lambda$ *is* LAP *if and only if* $f*g_\lambda$ *is bounded.*

Corollary 4.14. *By Corollary* 4.13*, Remark* 2.14 *and linearity of convolution it follows that if* $f\colon\mathbb{R}\to\mathbb{R}$ *is a bounded below function* (*or bounded above*) *such that* $f*g_\lambda$ *exists* ($\lambda\in\mathbb{R}$, $\lambda<0$)*, then the convolution* $f*g_\lambda$ *is* LAP *if and only if* $f*g_\lambda$ *is bounded.*

Now we are going to investigate the convolution of a certain classical unbounded Levitan almost periodic function with the function g_λ ($\lambda\in\mathbb{R}$, $\lambda<0$).

Example 4.15. Let

$$f(x)=\frac{1}{2+\cos x+\cos(x\sqrt{2})}\quad\text{for } x\in\mathbb{R}.$$

It was proved in the paper [10] that the function f satisfies condition (4.2) and that the convolution $f*g_\lambda$ exists ($\lambda\in\mathbb{R}$, $\lambda<0$). By Theorem 4.8, we know that $f*g_\lambda$ is not a Levitan almost periodic function.

The next example shows that convolution of an LAP function with the function g_λ does not need to exist.

Example 4.16. Let $\operatorname{Re}\lambda<0$ and

$$f(x)=\begin{cases} e^{-2\operatorname{Re}\lambda 3^{n+1}}|\sin(2\pi x)| & \text{for } x\in[3^n,3^n+1]+2\cdot 3^{n+1}\mathbb{Z},\ n\in\mathbb{N},\\ 0 & \text{otherwise.}\end{cases}$$

By Lemma 2.18 we know that f is LAP. Moreover, we have

$$\begin{aligned}\int_{-\infty}^0|f(t)|e^{-\operatorname{Re}\lambda t}\,dt &\geq \int_{z_n}^{z_n+1}|f(t)|e^{-\operatorname{Re}\lambda t}\,dt\\ &\geq e^{-\operatorname{Re}\lambda z_n}\int_{z_n}^{z_n+1}|f(t)|\,dt=\frac{2}{\pi}\,e^{-\operatorname{Re}\lambda 3^n},\end{aligned}$$

where $z_n=3^n-2\cdot 3^{n+1}$. Therefore

$$\int_{-\infty}^0|f(t)|e^{-\operatorname{Re}\lambda t}\,dt=+\infty$$

and the convolution $f*g_\lambda$ does not exist (cf. Remark 4.3).

5. Final remarks about linear differential equations

In this section we are going to consider LAP solutions to the linear differential equation of the form

$$y'(x) = \lambda y(x) + f(x), \quad \operatorname{Re}\lambda \neq 0. \tag{5.1}$$

As follows from the lemma below, for $\operatorname{Re}\lambda < 0$, for this, it makes sense to examine the function

$$y(x) = e^{\lambda x}\int_{-\infty}^{x} f(t)e^{-\lambda t}dt = (f * g_\lambda)(x), \quad x \in \mathbb{R} \quad \text{(cf. Remark 4.3)}. \tag{5.2}$$

REMARK 5.1. It is easy to establish that if the function (5.2) is well defined, then it is a solution to equation (5.1).

REMARK 5.2. We may assume that $\operatorname{Re}\lambda < 0$, because the case when $\operatorname{Re}\lambda > 0$ can be transformed to the case $\operatorname{Re}\lambda < 0$ in the following way: if y_1 is a solution to (5.1), then $y_2(x) := -y_1(-x)$, for $x \in \mathbb{R}$, is a solution to the equation

$$y'(x) = -\lambda y(x) + \widetilde{f}(x),$$

where $\widetilde{f}(x) = f(-x)$ for $x \in \mathbb{R}$.

REMARK 5.3. The case $\operatorname{Re}\lambda = 0$ is quite different from the case $\operatorname{Re}\lambda \neq 0$. Using the variation of constants method, we obtain that the solution to equation (5.1) has the form

$$y(x) = u(x)e^{\lambda x},$$

where u is a solution to the equation

$$u'(x) = f(x)e^{-\lambda x}.$$

Since the product of LAP functions is also an LAP function (Remark 2.14), this problem concerns an antiderivative of LAP functions. Therefore each solution to equation (5.1) is LAP or each solution to equation (5.1) is not LAP. The problem of antiderivative of LAP functions we can find e.g. in [18]. Moreover, taking a continuous periodic function f given by the formula $f(x) = e^{\lambda x}$, we obtain that equation (5.1) does not have an LAP solution because the equation

$$u'(x) = 1$$

does not have an LAP solution.

REMARK 5.4. Equation (5.1) with $\operatorname{Re}\lambda < 0$ possesses at most one LAP solution. Let us assume that it possesses two different LAP solutions. Then their difference is also LAP (Remark 2.14). Since all solutions to equation (5.1) are of the shape

$$y(x) = ce^{\lambda x} + e^{\lambda x}\int_{0}^{x} f(t)e^{-\lambda t}\,dt,$$

this implies that for some $c \neq 0$ the function $g(x) = ce^{\lambda x}$ is LAP. But

$$\liminf_{x\to-\infty} |ce^{\lambda x}| = +\infty \quad \text{(cf. Remark 2.21)}.$$

Lemma 5.5. *Let us consider equation* (5.1)*, where* $\operatorname{Re}\lambda < 0$ *and* $f\colon \mathbb{R} \to \mathbb{R}$ *is constinuous. If* y_0 *is an* LAP *solution to* (5.1)*, then*

$$y_0(x) = \lim_{n\to+\infty} e^{\lambda x} \int_{\tau_n}^{x} f(t)e^{-\lambda t}\,dt,$$

for all $x \in \mathbb{R}$ *and for some sequence* (τ_n) *such that* $\tau_n \to -\infty$*, as* $n \to +\infty$*.*

Proof. Since all solutions to the above equations are of the shape

$$y(x) = ce^{\lambda x} + e^{\lambda x} \int_0^x f(t)e^{-\lambda t}\,dt,$$

there exists $c_0 \in \mathbb{R}$ such that

$$y_0(x) = c_0 e^{\lambda x} + e^{\lambda x} \int_0^x f(t)e^{-\lambda t}\,dt.$$

Then, since y_0 is an LAP solution to the equation under consideration, there exists a sequence (τ_n) of $[1,1]$-almost periods such that $\tau_n \to -\infty$, as $n \to +\infty$ (see Lemma 2.20), and such that the following inequality holds:

$$\left| c_0 e^{\lambda\tau_n} + e^{\lambda\tau_n} \int_0^{\tau_n} f(t)e^{-\lambda t}\,dt - c_0 \right| < 1.$$

Therefore

$$\left| c_0 e^{\lambda\tau_n} + e^{\lambda\tau_n} \int_0^{\tau_n} f(t)e^{-\lambda t}\,dt \right| < |c_0| + 1 \leq M$$

for some constant $M > 0$. Hence

$$\left| c_0 + \int_0^{\tau_n} f(t)e^{-\lambda t}\,dt \right| < M e^{-\operatorname{Re}\lambda\tau_n} \to 0 \quad \text{as } n \to +\infty,$$

and thus

$$c_0 = \lim_{n\to+\infty} \int_{\tau_n}^{0} f(t)e^{-\lambda t}\,dt. \qquad \square$$

Remark 5.6. If equation (5.1) has an LAP solution and the convolution $f * g_\lambda$ exists, then by Remark 4.3 and Lemma 5.5, this solution is equal to $f * g_\lambda$.

Remark 5.7. If the convolution $f * g_\lambda$ exists but it is not LAP, then from Remark 4.3 and Lemma 5.5, equation (5.1) does not have an LAP solution.

Remark 5.8. If $\lambda \in \mathbb{R}$, $\lambda < 0$ and f is bounded below or bounded above and if the convolution $f * g_\lambda$ does not exist, then by Remark 4.3 and Lemma 5.5, equation (5.1) does not have a Levitan almost periodic solution. Indeed, for a function bounded below or bounded above existence of the limit

$$\lim_{n\to+\infty} \int_{\tau_n}^{0} f(t)e^{-\lambda t}\,dt$$

for some sequence (τ_n) such that $\tau_n \to -\infty$, as $n \to +\infty$, is equivalent to the condition

$$\int_{-\infty}^{0} f(t)e^{-\lambda t}\,dt < +\infty.$$

The next example shows that it may happen that the convolution $f * g_\lambda$ does not exist, but equation (5.1) possesses a uniformly almost periodic solution (in particular an LAP solution).

Example 5.9. Let $a_n = e^{2\cdot 3^{n+1}}$, $b_n = 3^{2\cdot 3^{n+1}}$ for $n \in \mathbb{N}$. Let us define

$$g(x) = \begin{cases} a_n \sin(2\pi b_n x) & \text{for } x \in [3^n, 3^n+1] + 2\cdot 3^{n+1}\mathbb{Z},\ n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G(x) = \begin{cases} \dfrac{a_n}{2\pi b_n}\,(1-\cos(2\pi b_n x)) & \text{for } x \in [3^n, 3^n+1] + 2\cdot 3^{n+1}\mathbb{Z},\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

From Lemma 2.18, we know that the functions g and G are LAP. Moreover,

$$0 \le G(x) \le \frac{a_n}{\pi b_n} \quad \text{for } x \in [3^n, 3^n+1] + 2\cdot 3^{n+1}\mathbb{Z},\ n \in \mathbb{N}.$$

Since the sets $[3^n, 3^n+1] + 2\cdot 3^{n+1}\mathbb{Z}$, for $n \in \mathbb{N}$, are pairwise disjoint (see Lemma 2.18), the function G is the limit of uniformly convergent sequence of periodic functions. This means that G is uniformly almost periodic (see Theorem 2.3). We have $G'(x) = g(x)$ for $x \in \mathbb{R}$. Let $f = G + g$. Then f is LAP (Remark 2.14). Immediately from the definition of the function f we have $G' = -G + f$, so the function G is a solution to the equation

$$y'(x) = -y(x) + f(x).$$

Moreover,

$$\int_{z_n}^{z_n+1} |g(t)|\,dt = \frac{2a_n}{\pi},$$

where $z_n = 3^n - 2\cdot 3^{n+1}$, $n \in \mathbb{N}$. The convolution $G * g_\lambda$ exists because G is bounded. A similar reasoning as in Example 4.16 (for $\lambda = -1$) establishes that $|g| * g_\lambda$ does not exist. By Remark 4.3, the convolution $g * g_\lambda$ also does not exist. Therefore the convolution $f * g_\lambda = G * g_\lambda + g * g_\lambda$ does not exist.

Our considerations in the previous section lead to the following results.

Theorem 5.10. *Suppose that* $\operatorname{Re}\lambda < 0$ *and* $f\colon \mathbb{R} \to \mathbb{C}$ *is an* LAP *function. Then one of the following cases holds*:

(a) *the convolution* $f * g_\lambda$ *is a solution to equation* (5.1) *and it is its unique* LAP *solution*;

(b) *the convolution* $f * g_\lambda$ *is a solution to equation* (5.1), *but equation* (5.1) *does not have an* LAP *solution*;

(c) *the convolution* $f * g_\lambda$ *does not exist and equation* (5.1) *does not have an* LAP *solution*;

(d) *the convolution* $f * g_\lambda$ *does not exist, but equation* (5.1) *has one* LAP *solution.*

Proof. The case (a) follows from Theorem 4.4, Remarks 5.1 and 5.4. The case (b) follows from Theorem 4.8, Example 4.10 and Remark 5.7. The case (c) follows from Example 4.16 and Remark 5.8, while the case (d) follows from Example 5.9 and Remark 5.4. □

Theorem 5.11. *Suppose that* $\lambda \in \mathbb{R}$, $\lambda < 0$ *and* $f\colon \mathbb{R} \to \mathbb{R}$ *is a bounded below* (*or bounded above*) LAP *function. Then the equation* (5.1) *possesses an* LAP *solution if and only if* f *satisfies the condition*

$$\sup_{u\in\mathbb{R}} \int_u^{u+1} |f(t)|\,dt < +\infty.$$

Then the convolution $f * g_\lambda$ *exists and it is an* LAP *solution of this equation.*

Theorem 5.12. *Suppose that* $\lambda \in \mathbb{R}$, $\lambda < 0$ *and* $f\colon \mathbb{R} \to \mathbb{R}$ *is a bounded below* (*or bounded above*) LAP *function. Then equation* (5.1) *possesses an* LAP *solution if and only if this equation has a bounded solution. Then the convolution* $f * g_\lambda$ *exists and it is equal to the bounded solution to equation* (5.1).

The last example shows that sometimes equation (5.1) with $\lambda \in \mathbb{R}$, $\lambda < 0$ and $f\colon \mathbb{R} \to \mathbb{R}$ possesses an unbounded LAP solution given by the convolution.

Example 5.13. Let

$$g(x) = \begin{cases} n\sin(2\pi x) & \text{for } x \in [3^n, 3^n+1] + 2\cdot 3^{n+1}\mathbb{Z},\ n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G(x) = \begin{cases} \dfrac{n}{2\pi}\,(1-\cos(2\pi x)) & \text{for } x \in [3^n, 3^n+1] + 2\cdot 3^{n+1}\mathbb{Z},\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

From Lemma 2.18, we know that the functions g and G are LAP. Moreover, $G'(x) = g(x)$ for $x \in \mathbb{R}$. Let $f = -\lambda G + g$ ($\lambda \in \mathbb{R}$, $\lambda < 0$). Then f is LAP (Remark 2.14) and $G' = \lambda G + f$, so the function G is a solution to the equation

$$y'(x) = \lambda y(x) + f(x).$$

The function G is not bounded because for $n \in \mathbb{N}$ we have

$$G\biggl(3^n + \frac{1}{2}\biggr) = \frac{n}{\pi}.$$

Moreover,

$$\int_{z_{k,n}}^{z_{k,n}+1} |g(t)|\,dt = \frac{2n}{\pi} \quad \text{and} \quad \int_{z_{k,n}}^{z_{k,n}+1} |G(t)|\,dt = \frac{n}{2\pi},$$

where $z_{k,n} = 3^n - 2k\cdot 3^{n+1}$, for $k,n \in \mathbb{N}$. A similar reasoning as in Example 4.10 shows that $|g| * g_\lambda$ and $|G| * g_\lambda$ exist. Therefore, from Remark 4.3, $g * g_\lambda$ and $G * g_\lambda$ exist and also $f * g_\lambda = -\lambda G * g_\lambda + g * g_\lambda$ exists. From Remark 5.6, $G = f * g_\lambda$.

References

[1] J. Andres, A.M. Bersani and R.F. Grande, *Hierarchy of almost-periodic function spaces*, Rend. Mat. Appl. **26** (7) (2006), 121–188.

[2] B. Basit and H. Günzler, *Difference property for perturbations of vector-valued Levitan almost periodic functions and their analogs*, Russ. J. Math. Phys. **12** (4) (2005), 424–438.

[3] A.S. Besicovitch, *On generalized almost periodic functions*, Proc. London Math. Soc. **25** (2) (1926), 495–512.

[4] H. Bohr, *Zur Theorie der fastperiodischen Funktionen*, I. *Teil Eine Verallgemeinerung der Theorie der Fourierreihen*, Acta Math. **45** (1925), 29–127.

[5] H. Bohr, *Zur Theorie der fastperiodischen Funktionen*, II. *Teil Zusammenhang der fastperiodischen Functionen mit Funktionen von unendkich vielen Variablen; gleichmassige Approximation durch trigonometrische Summen*, Acta Math. **46** (1925), 101–214.

[6] H. Bohr, *Zur Theorie der fastperiodischen Funktionen*, III. *Teil Dirichletentwicklung analytischer Functionen*, Acta Math. **47** (1926), 237–281.

[7] G. Bruno and A. Pankov, *On convolution operators in the spaces of almost periodic functions*, Z. Anal. Anwendungen **19** (2) (2000), 359–367.

[8] D. Bugajewski and T. Diagana, *Almost automorphy of the convolution operator and applications to differential and functional differential equations*, Nonlinear Studies **13**(2) (2006), 129–140.

[9] D. Bugajewski, X. Gan and P. Kasprzak, *Mappings of higher order and nonlinear equations in some spaces of almost periodic functions*, Nonlinear Anal. **75** (2012), 5294–5310.

[10] D. Bugajewski and A. Nawrocki, *Some remarks on almost periodic functions in view of the Lebesgue measure with applications to linear differential equations*, Ann. Acad. Sci. Fenn. Math. (in press).

[11] T. Caraballo and D. Cheban, *Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition* I, J. Differential Equations **246** (2009), 108–128.

[12] C. Corduneau, *Almost Periodic Functions*, AMS/Chelsea Publication Series, Chelsea Publishing Company, New York, 1989.

[13] T. Diagana, *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*, Springer, New York, 2013.

[14] J. Favard, *Sur les equations differentielles a coefficients presque-periodiques*, Acta Math. **51** (1927), 31–81.

[15] G. N'Guerekata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic/Plenum Publishers, New York, 2001.

[16] G. N'Guerekata, *Topics in Almost Automorphy*, Springer, New York, 2005.

[17] B.M. Levitan, *Almost Periodic Functions*, Gosudarstv. Izdat. Tekhn.-Teor. Lit., Moscow, 1953 (in Russian).

[18] B.M. Levitan and V.V. Zhikov, *Almost Periodic Functions and Differenial Equations*, Cambridge University Press, Cambridge, 1982.

[19] B.M. Levitan and V.V. Zhikov, *Favard theory*, Russian Math. Surveys **32** (1977), 129–180 (in Russian).

[20] M.G. Lyubarskiĭ, *An extension of Favard's theory to the case of a system of linear differential equations with unbounded Levitan almost periodic coefficients*, Dokl. Akad. Nauk SSSR **206** (1972), 808–810 (in Russian).

[21] W. Marzantowicz and J. Signerska, *Firing map of an almost periodic input function*, Discret. Contin. Dyn. Syst. (2011), 1032–1041.

[22] Y. Meyer, *Quasicrystals, almost periodic patterns, mean-periodic functions and irregular sampling*, Afr. Diaspora J. Math. (N.S.) **13** (2012), 1–45.

[23] V.V. Stepanov, *Über einige Verallgemeinerungen der fastperiodischen Funktionen*, Ann. Math. **95** (1926), 437–498.

[24] S. Stoiński, *Almost periodic function in the Lebesgue measure*, Comment. Math. Prace Mat. **34** (1994), 189–198.

[25] S. Stoiński, *Almost Periodic Functions*, Scientific Publisher UAM, Poznań, 2008 (in Polish).

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