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PERIODIC SOLUTIONS FOR THE NON-LOCAL OPERATOR $(-\Delta + m^2)^s - m^{2s}$ WITH $m \geq 0$

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Abstract. By using variational methods, we investigate the existence of T-periodic solutions to

$$\begin{cases} [(-\Delta_x + m^2)^s - m^{2s}]u = f(x,u) & \text{in } (0,T)^N, \\ u(x+Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N,\ i = 1,\dots,N, \end{cases}$$

where $s \in (0,1)$, $N > 2s$, $T > 0$, $m \geq 0$ and f is a continuous function, T-periodic in the first variable, verifying the Ambrosetti–Rabinowitz condition, with a polynomial growth at rate $p \in (1, (N+2s)/(N-2s))$.

1. Introduction

Recently, considerable attention has been given to fractional Sobolev spaces and corresponding non-local equations, in particular to the ones driven by the fractional powers of the Laplacian. In fact, this operator naturally arises in several areas of research and finds applications in optimization, finance, the thin obstacle problem, phase transitions, anomalous diffusion, crystal dislocation, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows and water waves. For more details and applications see [4], [6], [9], [12], [15], [16], [22], [26]–[28], [30] and references therein.

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The purpose of the present paper is to study T-periodic solutions to the problem

$$\begin{cases} [(-\Delta_x + m^2)^s - m^{2s}]u = f(x,u) & \text{in } (0,T)^N, \\ u(x+Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N,\ i=1,\dots,N, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $N > 2s$, (e_i) is the canonical basis in $\mathbb{R}^N$ and $f\colon \mathbb{R}^{N+1} \to \mathbb{R}$ is a function satisfying the following hypotheses:

(f1) $f(x,t)$ is T-periodic in $x \in \mathbb{R}^N$, that is $f(x+Te_i,t) = f(x,t)$.
(f2) f is continuous in $\mathbb{R}^{N+1}$.
(f3) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^N$.
(f4) There exist $1 < p < 2^\sharp_s - 1 = 2N/(N-2s) - 1$ and $C > 0$ such that

$$|f(x,t)| \le C(1+|t|^p) \quad \text{for any } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.$$

(f5) There exist $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu F(x,t) \le t f(x,t) \quad \text{for } x \in \mathbb{R}^N \text{ and } |t| \ge r_0.$$

Here $F(x,t) = \int_0^t f(x,\tau)\,d\tau$.
(f6) $tf(x,t) \ge 0$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

We notice that (f2) and (f5) imply the existence of two constants $a, b > 0$ such that

$$F(x,t) \ge a|t|^\mu - b \quad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

Then, since $\mu > 2$, $F(x,t)$ grows at a superquadratic rate and by (f5), $f(x,t)$ grows at a superlinear rate as $|t| \to \infty$. Here, the operator $(-\Delta_x+m^2)^s$ is defined through the spectral decomposition, by using the powers of the eigenvalues of $-\Delta + m^2$ with periodic boundary conditions.

Let $u \in \mathcal{C}^\infty_T(\mathbb{R}^N)$, that is u is infinitely differentiable in $\mathbb{R}^N$ and T-periodic in each variable. Then u has a Fourier series expansion

$$u(x) = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{\imath \omega k \cdot x}}{\sqrt{T^N}}, \quad x \in \mathbb{R}^N,$$

where

$$\omega = \frac{2\pi}{T} \quad \text{and} \quad c_k = \frac{1}{\sqrt{T^N}} \int_{(0,T)^N} u(x) e^{-\imath \omega k \cdot x}\,dx, \quad k \in \mathbb{Z}^N,$$

are the Fourier coefficients of u. The operator $(-\Delta_x + m^2)^s$ is defined by setting

$$(-\Delta_x + m^2)^s u = \sum_{k \in \mathbb{Z}^N} c_k (\omega^2 |k|^2 + m^2)^s \frac{e^{\imath \omega k \cdot x}}{\sqrt{T^N}}.$$

For

$$u = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{\imath \omega k \cdot x}}{\sqrt{T^N}} \quad \text{and} \quad v = \sum_{k \in \mathbb{Z}^N} d_k \frac{e^{\imath \omega k \cdot x}}{\sqrt{T^N}},$$

we have that

$$\mathcal{Q}(u,v)=\sum_{k\in\mathbb{Z}^N}(\omega^2|k|^2+m^2)^s c_k\overline{d}_k$$

can be extended by density to a quadratic form on the Hilbert space

$$\mathbb{H}^s_{m,T}=\bigg\{u=\sum_{k\in\mathbb{Z}^N}c_k\,\frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}}\in L^2(0,T)^N:\sum_{k\in\mathbb{Z}^N}(\omega^2|k|^2+m^2)^s|c_k|^2<\infty\bigg\}$$

endowed with the norm

$$|u|^2_{\mathbb{H}^s_{m,T}}=\sum_{k\in\mathbb{Z}^N}(\omega^2|k|^2+m^2)^s|c_k|^2.$$

When $m=1$ we set $\mathbb{H}^s_T=\mathbb{H}^s_{1,T}$.

In $\mathbb{R}^N$, the physical interest of the non-local operator $(-\Delta+m^2)^s$ is manifest in the case $s=1/2$: it is the Hamiltonian for a (free) relativistic particle of mass m; see for instance [2], [19]–[22]. In particular, such operator is deeply connected with the Stochastic Process Theory: in fact it is an infinitesimal generator of a Lévy process called the α-stable process; see [4], [14] and [25].

Problems similar to (1.1) have been also studied in the local setting. The typical example is given by

$$\begin{cases} Lu=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{1.2}$$

where L is uniformly elliptic, Ω is a smooth bounded domain in $\mathbb{R}^N$ and f is a continuous function satisfying the assumptions (f3)–(f5). It is well-known that (1.2) possesses a weak solution which can be obtained as a critical point of a corresponding functional by means of minimax methods; see for instance [1], [23], [24], [29] and [31].

The aim of the following paper is to study (1.2) in the periodic setting, when we replace L by $(-\Delta+m^2)^s-m^{2s}$, $m\geq 0$ and $s\in(0,1)$. We remark that problem (1.1) with $s=1/2$ has been investigated by the same author in [3]. In this paper, we extend the results in [3] to the more general operator $(-\Delta+m^2)^s-m^{2s}$, with $s\in(0,1)$.

Our first result is the following:

THEOREM 1.1. *Let $m>0$ and $f\colon\mathbb{R}^{N+1}\to\mathbb{R}$ be a function satisfying assumptions* (f1)–(f6). *Then there exists a solution $u\in\mathbb{H}^s_{m,T}$ to* (1.1). *In particular, u belongs to $\mathcal{C}^{0,\alpha}([0,T]^N)$ for some $\alpha\in(0,1)$.*

To study problem (1.1) we will give an alternative formulation of the operator $(-\Delta+m^2)^s$ with periodic boundary conditions, which consists in realizing it as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem on the half-cylinder $(0,T)^N\times(0,\infty)$; see [3] for the case $s=1/2$. We recall that this argument is an adaptation of the idea originally introduced in [11] to study the fractional Laplacian in $\mathbb{R}^N$ (see also [7], [8]) and subsequently extended to the case of the fractional Laplacian on a bounded domain [10], [13].

As explained in more detail in Section 3, for $u \in \mathbb{H}^s_{m,T}$ one considers the problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s} v = 0 & \text{in } \mathcal{S}_T := (0,T)^N \times (0,\infty), \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial_L \mathcal{S}_T := \partial(0,T)^N \times [0,\infty), \\ v(x,0) = u(x) & \text{on } \partial^0 \mathcal{S}_T := (0,T)^N \times \{0\}, \end{cases}$$

from where the operator $(-\Delta_x + m^2)^s$ is obtained as

$$-\lim_{y\to 0} y^{1-2s} \frac{\partial v}{\partial y}(x,y) = \kappa_s(-\Delta_x + m^2)^s u(x)$$

in weak sense and $\kappa_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$. Thus, in order to study (1.1), we will exploit this fact to investigate the following problem:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s} v = 0 & \text{in } \mathcal{S}_T := (0,T)^N \times (0,\infty), \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial_L \mathcal{S}_T := \partial(0,T)^N \times [0,\infty), \\ \dfrac{\partial v}{\partial \nu^{1-2s}} = \kappa_s[m^{2s} v + f(x,v)] & \text{on } \partial^0 \mathcal{S}_T := (0,T)^N \times \{0\}, \end{cases} \tag{1.3}$$

where

$$\frac{\partial v}{\partial \nu^{1-2s}} := -\lim_{y\to 0} y^{1-2s} \frac{\partial v}{\partial y}(x,y)$$

is the conormal exterior derivative of v.

Solutions to (1.3) are obtained as critical points of the functional $\mathcal{J}_m$ associated to (1.1)

$$\mathcal{J}_m(v) = \frac{1}{2}\,||v||^2_{\mathbb{X}^s_{m,T}} - \frac{m^{2s}\kappa_s}{2}\,|v(\,\cdot\,,0)|^2_{L^2(0,T)^N} - \kappa_s \int_{\partial^0 \mathcal{S}_T} F(x,v)\,dx$$

defined on the space $\mathbb{X}^s_{m,T}$, which is the closure of the set of smooth and T-periodic (in x) functions in $\mathbb{R}^{N+1}_+$ with respect to the norm

$$||v||^2_{\mathbb{X}^s_{m,T}} := \iint_{\mathcal{S}_T} y^{1-2s}(|\nabla v|^2 + m^{2s} v^2)\,dx\,dy.$$

More precisely, we will prove that, for any fixed $m > 0$, $\mathcal{J}_m$ satisfies the hypotheses of the Linking Theorem due to Rabinowitz [24].

When m is sufficiently small, we are able to obtain estimates on critical levels α_m of the functionals $\mathcal{J}_m$ independently of m. In this way, we can pass to the limit as $m \to 0$ in (1.3) and we deduce the existence of a nontrivial solution to the problem

$$\begin{cases} (-\Delta_x)^s u = f(x,u) & \text{in } (0,T)^N, \\ u(x+Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N,\ i = 1,\dots,N. \end{cases} \tag{1.4}$$

This result can be stated as follows:

THEOREM 1.2. *Under the same assumptions on f as in Theorem* 1.1*, problem* (1.4) *admits a nontrivial solution $u \in \mathbb{H}^s_T \cap \mathcal{C}^{0,\alpha}([0,T]^N)$.*

The paper is organized as follows: in Section 2 we collect some preliminary results which we will use later to study problem (1.1); in Section 3 we show that problem (1.1) can be realized in a local manner through the nonlinear problem (1.3); in Section 4 we verify that, for any fixed $m > 0$, the functional $\mathcal{J}_m$ satisfies the linking hypotheses; in Section 5 we study the regularity of solutions of problem (1.1); in the last section we show that we can find a nontrivial Hölder continuous solution to (1.4) by passing to the limit in (1.1) as $m \to 0$.

2. Preliminaries

In this section we introduce some notation and facts which will be frequently used in the sequel of the paper. We denote the upper half-space in $\mathbb{R}^{N+1}$ by

$$\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N,\ y > 0\}.$$

Let $\mathcal{S}_T = (0,T)^N \times (0,\infty)$ be the half-cylinder in $\mathbb{R}^{N+1}_+$ with the basis $\partial^0\mathcal{S}_T = (0,T)^N \times \{0\}$ and we denote by $\partial_L\mathcal{S}_T = \partial(0,T)^N \times [0,+\infty)$ its lateral boundary. With $||v||_{L^r(\mathcal{S}_T)}$ we always denote the norm of $v \in L^r(\mathcal{S}_T)$ and with $|u|_{L^r(0,T)^N}$ the $L^r(0,T)^N$ norm of $u \in L^r(0,T)^N$.

Let $s \in (0,1)$ and $m > 0$. Let $A \subset \mathbb{R}^N$ be a domain. We denote by $L^2(A \times \mathbb{R}_+, y^{1-2s})$ the space of all measurable functions v defined on $A \times \mathbb{R}_+$ such that

$$\iint_{A\times\mathbb{R}_+} y^{1-2s} v^2\, dx\, dy < \infty.$$

We say that $v \in H^1_m(A \times \mathbb{R}_+, y^{1-2s})$ if v and its weak gradient ∇v belongs to $L^2(A \times \mathbb{R}_+, y^{1-2s})$. The norm of v in $H^1_m(A \times \mathbb{R}_+, y^{1-2s})$ is given by

$$\iint_{A\times\mathbb{R}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2)\, dx\, dy < \infty.$$

It is clear that $H^1_m(A \times \mathbb{R}_+, y^{1-2s})$ is a Hilbert space with the inner product

$$\iint_{A\times\mathbb{R}_+} y^{1-2s}(\nabla v \nabla z + m^2 vz)\, dx\, dy.$$

When $m = 1$, we set $H^1(A \times \mathbb{R}_+, y^{1-2s}) = H^1_1(A \times \mathbb{R}_+, y^{1-2s})$.

We denote by $\mathcal{C}^\infty_T(\mathbb{R}^N)$ the space of functions $u \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that u is T-periodic in each variable, that is

$$u(x + e_iT) = u(x) \quad \text{for all } x \in \mathbb{R}^N,\ i = 1, \dots, N.$$

Let $u \in \mathcal{C}^\infty_T(\mathbb{R}^N)$. Then we know that

$$u(x) = \sum_{k\in\mathbb{Z}^N} c_k \frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}} \quad \text{for all } x \in \mathbb{R}^N,$$

where

$$\omega = \frac{2\pi}{T} \quad \text{and} \quad c_k = \frac{1}{\sqrt{T^N}} \int_{(0,T)^N} u(x) e^{-\imath k\omega\cdot x}\, dx, \quad k \in \mathbb{Z}^N,$$

are the Fourier coefficients of u. We define the fractional Sobolev space $\mathbb{H}^s_{m,T}$ as the closure of $\mathcal{C}^\infty_T(\mathbb{R}^N)$ under the norm

$$|u|^2_{\mathbb{H}^s_{m,T}} := \sum_{k\in\mathbb{Z}^N} (\omega^2|k|^2 + m^2)^s |c_k|^2.$$

When $m = 1$, we set $\mathbb{H}^s_T = \mathbb{H}^s_{1,T}$ and $|\cdot|_{\mathbb{H}^s_T} = |\cdot|_{\mathbb{H}^s_{1,T}}$. Now we introduce the functional space $\mathbb{X}^s_{m,T}$ defined as the completion of

$$\mathcal{C}^\infty_T(\overline{\mathbb{R}^{N+1}_+}) = \Big\{ v \in \mathcal{C}^\infty(\overline{\mathbb{R}^{N+1}_+}) : v(x + e_i T, y) = v(x,y) \\ \text{for every } (x,y) \in \overline{\mathbb{R}^{N+1}_+},\ i = 1,\dots,N \Big\}$$

under the $H^1_m(\mathcal{S}_T, y^{1-2s})$-norm

$$||v||^2_{\mathbb{X}^s_{m,T}} := \iint_{\mathcal{S}_T} y^{1-2s}(|\nabla v|^2 + m^2 v^2)\, dx\, dy.$$

If $m = 1$, we set $\mathbb{X}^s_T = \mathbb{X}^s_{1,T}$ and $||\cdot||_{\mathbb{X}^s_T} = ||\cdot||_{\mathbb{X}^s_{1,T}}$.

Now let us prove that it is possible to define a trace operator from $\mathbb{X}^s_{m,T}$ to the fractional space $\mathbb{H}^s_{m,T}$.

Theorem 2.1. *There exists a bounded linear operator* $\mathrm{Tr}\colon \mathbb{X}^s_{m,T} \to \mathbb{H}^s_{m,T}$ *such that*:

(a) $\mathrm{Tr}(v) = v|_{\partial^0\mathcal{S}_T}$ *for all* $v \in \mathcal{C}^\infty_T(\overline{\mathbb{R}^{N+1}_+}) \cap \mathbb{X}^s_{m,T}$.

(b) *There exists* $C = C(s) > 0$ *such that*

$$C|\mathrm{Tr}(v)|_{\mathbb{H}^s_{m,T}} \le ||v||_{\mathbb{X}^s_{m,T}} \quad \text{for every } v \in \mathbb{X}^s_{m,T}.$$

(c) Tr *is surjective.*

Proof. Let $v \in C^\infty_T(\overline{\mathbb{R}^{N+1}_+})$ be such that $\|v\|_{\mathbb{X}^s_{m,T}} < \infty$. Then v can be expressed as

$$v(x,y) = \sum_{k\in\mathbb{Z}^N} c_k(y) \frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}},$$

where

$$c_k(y) = \int_{(0,T)^N} v(x,y) \frac{e^{-\imath\omega k\cdot x}}{\sqrt{T^N}}\, dx \quad \text{and} \quad c_k \in H^1_m(\mathbb{R}_+, y^{1-2s}).$$

We notice that, by using Parseval's identity, we have

$$\|v\|^2_{\mathbb{X}^s_{m,T}} = \sum_{k\in\mathbb{Z}^N}\int_0^{+\infty} y^{1-2s}\big[(\omega^2|k|^2+m^2)|c_k(y)|^2+|c_k'(y)|^2\big]\,dy. \tag{2.1}$$

Let us show that there exists a positive constant C_s depending only on s such that

$$C_s|\mathrm{Tr}(v)|^2_{\mathbb{H}^s_{m,T}} \le \|v\|^2_{\mathbb{X}^s_{m,T}} \quad \text{for any } v\in C^\infty_T(\overline{\mathbb{R}^{N+1}_+}) \text{ such that } \|v\|_{\mathbb{X}^s_{m,T}}<+\infty,$$

or equivalently,

$$C_s\sum_{k\in\mathbb{Z}^N}(\omega^2|k|^2+m^2)^s|c_k(0)|^2 \le \sum_{k\in\mathbb{Z}^N}\int_0^{+\infty} y^{1-2s}\big[(\omega^2|k|^2+m^2)|c_k(y)|^2+|c_k'(y)|^2\big]\,dy. \tag{2.2}$$

By the Fundamental Theorem of Calculus, we have

$$|c_k(0)|\le|c_k(y)|+\left|\int_0^y c_k'(t)\,dt\right|,$$

hence, by $(|a|+|b|)^2\le 2(|a|^2+|b|^2)$,

$$|c_k(0)|^2\le 2|c_k(y)|^2+2\left|\int_0^y c_k'(t)\,dt\right|^2 \tag{2.3}$$

for any $k\in\mathbb{Z}^N$. Now, observe that, by the Hölder inequality,

$$\begin{aligned}\int_0^y |c_k'(t)|\,dt &\le \left(\int_0^y t^{1-2s}|c_k'(t)|^2\,dt\right)^{1/2}\left(\int_0^y t^{2s-1}\,dt\right)^{1/2}\\ &= \left(\int_0^y t^{1-2s}|c_k'(t)|^2\,dt\right)^{1/2}\left(\frac{y^{2s}}{2s}\right)^{1/2}.\end{aligned} \tag{2.4}$$

Putting together (2.3) and (2.4), we obtain

$$|c_k(0)|^2\le 2|c_k(y)|^2+\frac{y^{2s}}{s}\left(\int_0^{+\infty} t^{1-2s}|c_k'(t)|^2\,dt\right),$$

and multiplying both sides by y^{1-2s}, we get

$$y^{1-2s}|c_k(0)|^2\le 2y^{1-2s}|c_k(y)|^2+\frac{y}{s}\left(\int_0^{+\infty} t^{1-2s}|c_k'(t)|^2.\,dt\right). \tag{2.5}$$

Let $a_k=(\omega^2|k|^2+m^2)^{-1/2}$. Integrating (2.5) over $y\in(0,a_k)$, we deduce

$$\begin{aligned}\frac{a_k^{2-2s}}{2-2s}|c_k(0)|^2 &\le 2\int_0^{a_k} y^{1-2s}|c_k(y)|^2\,dy\\ &\quad+\left(\int_0^{a_k}\frac{y}{s}\,dy\right)\left(\int_0^{+\infty} t^{1-2s}|c_k'(t)|^2\,dt\right)\end{aligned} \tag{2.6}$$

$$\leq 2\int_0^{+\infty} y^{1-2s}|c_k(y)|^2\,dy + \frac{a_k^2}{2s}\left(\int_0^{+\infty} t^{1-2s}|c_k'(t)|^2\,dt\right)$$
$$= 2\int_0^{+\infty} t^{1-2s}|c_k(t)|^2\,dt + \frac{a_k^2}{2s}\left(\int_0^{+\infty} t^{1-2s}|c_k'(t)|^2\,dt\right).$$

Multiplying both sides of (2.6) by $a_k^{-2} = (\omega^2|k|^2 + m^2)$, we have

$$\frac{(\omega^2|k|^2+m^2)^s}{2-2s}\,|c_k(0)|^2 \leq 2(\omega^2|k|^2+m^2)\int_0^{+\infty} t^{1-2s}|c_k(t)|^2\,dt + \frac{1}{2s}\left(\int_0^{+\infty} t^{1-2s}|c_k'(t)|^2\,dt\right)$$

for any $k \in \mathbb{Z}^N$. Summing over $\mathbb{Z}^N$, we deduce

$$\begin{aligned}(2.7)\quad &\frac{1}{2-2s}\sum_{k\in\mathbb{Z}^N}(\omega^2|k|^2+m^2)^s|c_k(0)|^2\\ &\leq \sum_{k\in\mathbb{Z}^N}\bigg[2(\omega^2|k|^2+m^2)\int_0^{+\infty} t^{1-2s}|c_k(t)|^2\,dt\\ &\quad+\frac{1}{2s}\left(\int_0^{+\infty} t^{1-2s}|c_k'(t)|^2\,dt\right)\bigg]\\ &\leq \max\left\{2,\frac{1}{2s}\right\}\sum_{k\in\mathbb{Z}^N}\int_0^{+\infty} t^{1-2s}\left[(\omega^2|k|^2+m^2)|c_k(t)|^2 + |c_k'(t)|^2\right]dt.\end{aligned}$$

Taking into account (2.1) and (2.7), we get (2.2). Therefore there exists a trace operator $\mathrm{Tr}\colon \mathbb{X}^s_{m,T} \to \mathbb{H}^s_{m,T}$. Now we prove that Tr is surjective. Let

$$u = \sum_{k\in\mathbb{Z}^N} c_k\,\frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}} \in \mathbb{H}^s_{m,T}.$$

Define

$$v(x,y) = \sum_{k\in\mathbb{Z}^N} c_k\theta_k(y)\,\frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}}, \tag{2.8}$$

where $\theta_k(y) = \theta(\sqrt{\omega^2|k|^2+m^2}\,y)$ and $\theta(y) \in H^1(\mathbb{R}_+, y^{1-2s})$ solves the following ODE:

$$\begin{cases}\theta'' + \dfrac{1-2s}{y}\,\theta' - \theta = 0 & \text{in } \mathbb{R}_+,\\ \theta(0) = 1 \quad\text{and}\quad \theta(\infty) = 0.\end{cases}$$

It is known (see [17]) that $\theta(y) = (2/\Gamma(s))(y/2)^s K_s(y)$, where K_s is the Bessel function of second kind with order s, and as $K_s' = (s/y)K_s - K_{s-1}$, we get

$$\kappa_s := \int_0^\infty y^{1-2s}(|\theta'(y)|^2 + |\theta(y)|^2)\,dy = -\lim_{y\to 0} y^{1-2s}\theta'(y) = 2^{1-2s}\,\frac{\Gamma(1-s)}{\Gamma(s)}.$$

Then it is clear that v is smooth for $y > 0$, v is T-periodic in x and satisfies

$$-\operatorname{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s} v = 0 \quad \text{in } \mathcal{S}_T.$$

Now, we show that $\mathrm{Tr}(v) = u$. From standard properties of K_s, we know that $\theta(y) \to 1$ as $y \to 0$ and $0 < \theta(y) \le A_s$ for any $y \ge 0$. Then, as $u \in \mathbb{H}^s_{m,T}$, we have

$$|v(\,\cdot\,,y) - u|^2_{\mathbb{H}^s_{m,T}} = \sum_{k\in\mathbb{Z}^N} (\omega^2|k|^2 + m^2)^s |c_k|^2 |\theta_k(y) - 1|^2 \to 0 \quad \text{as } y \to 0.$$

Finally, we prove that $v \in \mathbb{X}^s_{m,T}$. By Parseval's identity, we get

$$\begin{aligned} ||v||^2_{\mathbb{X}^s_{m,T}} &= \iint_{\mathcal{S}_T} y^{1-2s}(|\nabla v|^2 + m^2 v^2)\,dx\,dy \\ &= \sum_{k\in\mathbb{Z}^N} |c_k|^2 \int_0^\infty y^{1-2s}(|\theta_k'(y)|^2 + |\theta_k(y)|^2)\,dy \\ &= \sum_{k\in\mathbb{Z}^N} |c_k|^2 \int_0^\infty y^{1-2s}(\omega^2|k|^2+m^2) \\ &\quad \cdot \big(|\theta'(\sqrt{\omega^2|k|^2+m^2}\,y)|^2 + |\theta(\sqrt{\omega^2|k|^2+m^2}\,y)|^2\big)\,dy \\ &= \sum_{k\in\mathbb{Z}^N} |c_k|^2 \frac{\sqrt{\omega^2|k|^2+m^2}}{(\omega^2|k|^2+m^2)^{(1-2s)/2}} \\ &\quad \cdot \int_0^\infty y^{1-2s}(|\theta'(y)|^2 + |\theta(y)|^2)\,dy \\ &= \kappa_s \sum_{k\in\mathbb{Z}^N} (\omega^2|k|^2+m^2)^s |c_k|^2 = \kappa_s |u|^2_{\mathbb{H}^s_{m,T}}. \end{aligned} \tag{2.9}$$

□

THEOREM 2.2. *Let $N > 2s$. Then $\mathrm{Tr}(\mathbb{X}^s_{m,T})$ is continuously embedded in $L^q(0,T)^N$ for any $1 \le q \le 2^\sharp_s$. Moreover, $\mathrm{Tr}(\mathbb{X}^s_{m,T})$ is compactly embedded in $L^q(0,T)^N$ for any $1 \le q < 2^\sharp_s$.*

PROOF. By Theorem 2.1, we know that there exists a continuous embedding from $\mathbb{X}^s_{m,T}$ to $\mathbb{H}^s_{m,T}$. Let us show that $\mathbb{H}^s_{m,T}$ is continuously embedded in $L^q(0,T)^N$ for any $q \le 2^\sharp_s$ and compactly in $L^q(0,T)^N$ for any $q < 2^\sharp_s$.

By Proposition 2.1 in [5], we know that there exists a constant $C_{2^\sharp_s} > 0$ such that

$$|u|_{L^{2^\sharp_s}(0,T)^N} \le C_{2^\sharp_s} \bigg(\sum_{|k|\ge 1} \omega^{2s}|k|^{2s}|c_k|^2 \bigg)^{1/2} \tag{2.10}$$

for any $u \in \mathcal{C}^\infty_T(\mathbb{R}^N)$ such that $(1/T^N)\int_{(0,T)^N} u(x)\,dx = 0$. As a consequence, fixed $2 \le q \le 2^\sharp_s$, we have

$$|u|_{L^q(0,T)^N} \le C \bigg(\sum_{k\in\mathbb{Z}^N} |c_k|^2(\omega^2|k|^2+m^2)^s \bigg)^{1/2} \tag{2.11}$$

for any $u \in \mathbb{H}^s_{m,T}$, that is $\mathbb{H}^s_{m,T}$ is continuously embedded in $L^q(0,T)^N$ for any $2 \le q \le 2^\sharp_s$. Now, we proceed as the proof of Theorem 4 in [3] to prove that

$\mathbb{H}^s_{m,T} \Subset L^q(0,T)^N$ for any $2 \leq q < 2^\sharp_s$. Fix $q \in [2, 2^\sharp_s)$. Then, by (2.11) and the interpolation inequality, we obtain

$$|u|_{L^q(0,T)^N} \leq C|u|^\theta_{L^2(0,T)^N} \bigg(\sum_{k\in\mathbb{Z}^N} |c_k|^2(\omega^2|k|^2 + m^2)^s \bigg)^{1-\theta}, \tag{2.12}$$

for some real positive number $\theta \in (0,1)$.

Now, taking into account (2.12), it is enough to prove that $\mathbb{H}^s_{m,T} \Subset L^2(0,T)^N$ to infer that $\mathbb{H}^s_{m,T}$ is compactly embedded in $L^q(0,T)^N$ for every $q \in [2, 2^\sharp_s)$. Let us assume that $u^j \rightharpoonup 0$ in $\mathbb{H}^s_{m,T}$. Then

$$\lim_{j\to\infty} |c^j_k|^2(\omega^2|k|^2 + m^2)^s = 0 \quad \text{for all } k \in \mathbb{Z}^N, \tag{2.13}$$

$$\sum_{k\in\mathbb{Z}^N} |c^j_k|^2(\omega^2|k|^2 + m^2)^s \leq C \quad \text{for all } j \in \mathbb{N}. \tag{2.14}$$

Fix $\varepsilon > 0$. Then there exists $\nu_\varepsilon > 0$ such that $(\omega^2|k|^2 + m^2)^{-s} < \varepsilon$ for $|k| > \nu_\varepsilon$. By (2.14), we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^N} |c^j_k|^2 &= \sum_{|k|\leq\nu_\varepsilon} |c^j_k|^2 + \sum_{|k|>\nu_\varepsilon} |c^j_k|^2 \\
&= \sum_{|k|\leq\nu} |c^j_k|^2 + \sum_{|k|>\nu_\varepsilon} |c^j_k|^2(\omega^2|k|^2 + m^2)^s(\omega^2|k|^2 + m^2)^{-s} \\
&\leq \sum_{|k|\leq\nu_\varepsilon} |c^j_k|^2 + C\varepsilon.
\end{aligned}$$

Using (2.13), we deduce that $\sum_{|k|\leq\nu_\varepsilon} |c^j_k|^2 < \varepsilon$ for j large. Thus $u^j \to 0$ in $L^2(0,T)^N$. $\square$

We conclude this section with some elementary results on the nonlinearity f. More precisely, by using the assumptions (f2)–(f4), one can deduce some bounds from above and below for f and its primitive F. This part is quite standard and the proofs of the two subsequent lemmas can be found, for instance, in [1] and [24].

Lemma 2.3. *Let $f\colon [0,T]^N \times \mathbb{R} \to \mathbb{R}$ satisfy conditions* (f1)–(f3). *Then, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$|f(x,t)| \leq 2\varepsilon|t| + (p+1)C_\varepsilon|t|^p \quad \textit{for all } t \in \mathbb{R} \textit{ and all } x \in [0,T]^N, \tag{2.15}$$

$$|F(x,t)| \leq \varepsilon|t|^2 + C_\varepsilon|t|^{p+1} \quad \textit{for all } t \in \mathbb{R} \textit{ and all } x \in [0,T]^N. \tag{2.16}$$

Lemma 2.4. *Assume that $f\colon [0,T]^N \times \mathbb{R} \to \mathbb{R}$ satisfies conditions* (f1)–(f4). *Then, there exist two constants $a_3 > 0$ and $a_4 > 0$ such that*

$$F(x,t) \geq a_3|t|^\mu - a_4 \quad \textit{for all } t \in \mathbb{R} \textit{ and all } x \in [0,T]^N.$$

3. Extension problem

In this section we show that to study (1.1) it is equivalent to investigate the solutions of a problem in a half-cylinder with a Neumann nonlinear boundary condition. We start with

THEOREM 3.1. *Let $u \in \mathbb{H}^s_{m,T}$. Then there exists a unique $v \in \mathbb{X}^s_{m,T}$ such that*

$$
\begin{cases}
-\operatorname{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s} v = 0 & \text{in } \mathcal{S}_T, \\
v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial_L \mathcal{S}_T, \\
v(\,\cdot\,,0) = u & \text{on } \partial^0 \mathcal{S}_T,
\end{cases}
\tag{3.1}
$$

and

$$
-\lim_{y\to 0} y^{1-2s} \frac{\partial v}{\partial y}(x,y) = \kappa_s(-\Delta_x + m^2)^s u(x) \quad \text{in } \mathbb{H}^{-s}_{m,T}, \tag{3.2}
$$

where

$$
\mathbb{H}^{-s}_{m,T} = \bigg\{ u = \sum_{k\in\mathbb{Z}^N} c_k \frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}} : \frac{|c_k|^2}{(\omega^2|k|^2 + m^2)^s} < \infty \bigg\}
$$

is the dual of $\mathbb{H}^s_{m,T}$.

PROOF. Let $u = \sum_{k\in\mathbb{Z}^N} c_k e^{\imath\omega k\cdot x}/\sqrt{T^N} \in \mathcal{C}^\infty_T(\mathbb{R}^N)$. Consider the following problem:

$$
\min\big\{ ||v||^2_{\mathbb{X}^s_{m,T}} : v \in \mathbb{X}^s_{m,T},\ \operatorname{Tr}(v) = u \big\}. \tag{3.3}
$$

By Theorem 2.2, we can find a minimizer to (3.3). Since $||\cdot||^2_{\mathbb{X}^s_{m,T}}$ is strictly convex, such minimizer is unique and we denote it by v. As a consequence, for any $\phi \in \mathbb{X}^s_{m,T}$ such that $\operatorname{Tr}(\phi) = 0$,

$$
\iint_{\mathcal{S}_T} y^{1-2s}(\nabla v \nabla \phi + m^2 v\phi)\,dx\,dy = 0, \tag{3.4}
$$

that is v is a weak solution to (3.1). Since the function defined in (2.8) is a solution to (3.1), by the uniqueness of minimizer, we deduce that v is given by

$$
v(x,y) = \sum_{k\in\mathbb{Z}^N} c_k \theta_k(y) \frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}},
$$

where $\theta_k(y) = \theta(\sqrt{\omega^2|k|^2 + m^2}\,y)$. In particular, by (2.9), we have

$$
||v||_{\mathbb{X}^s_{m,T}} = \sqrt{\kappa_s}\,|u|_{\mathbb{H}^s_{m,T}}.
$$

Then

$$\begin{aligned}&\Big| - y^{1-2s}\,\frac{\partial v}{\partial y}(\cdot,y) - \kappa_s(-\Delta+m^2)^s u\Big|^2_{\mathbb{H}^{-s}_{m,T}}\\ &= \sum_{k\in\mathbb{Z}^N} \frac{1}{(\omega^2|k|^2+m^2)^s}\,|c_k|^2\,\Big|\sqrt{\omega^2|k|^2+m^2}\,\theta_k'(y)y^{1-2s}+\kappa_s(\omega^2|k|^2+m^2)^s\Big|^2\\ &= \sum_{k\in\mathbb{Z}^N}(\omega^2|k|^2+m^2)^s|c_k|^2\Big|\big[(\omega^2|k|^2+m^2)y\big]^{1-2s}\theta_k'(y)+\kappa_s\Big|^2\end{aligned}$$

and, by using $u\in\mathbb{H}^s_{m,T}$, $-y^{1-2s}\theta'(y)\to\kappa_s$ as $y\to 0$ and $0<-\kappa_s y^{1-2s}\theta'(y)\le B_s$ for any $y\ge 0$ (see [17]), we deduce (3.2). □

Therefore, for any given $u\in\mathbb{H}^s_{m,T}$, we can find a unique function $v=\mathrm{Ext}(u)$ in $\mathbb{X}^s_{m,T}$, which will be called the extension of u, such that

(a) v is smooth for $y>0$, T-periodic in x and v solves (3.1).
(b) $||v||_{\mathbb{X}^s_{m,T}}\le ||z||_{\mathbb{X}^s_{m,T}}$ for any $z\in\mathbb{X}^s_{m,T}$ such that $\mathrm{Tr}(z)=u$.
(c) $||v||_{\mathbb{X}^s_{m,T}}=\sqrt{\kappa_s}|u|_{\mathbb{H}^s_{m,T}}$.
(d) We have

$$\lim_{y\to 0} -y^{1-2s}\,\frac{\partial v}{\partial y}(x,y)=\kappa_s(-\Delta+m^2)^s u(x)\quad \text{in } \mathbb{H}^{-s}_{m,T}.$$

Now, modifying the proof of Lemma 2.2 in [13], we deduce

Theorem 3.2. *Let $g\in\mathbb{H}^{-s}_{m,T}$. Then, there is a unique solution to the problem:*

$$\text{find } u\in\mathbb{H}^s_{m,T} \text{ such that } (-\Delta+m^2)^s u=g.$$

Moreover, u is the trace of $v\in\mathbb{X}^s_{m,T}$, where v is the unique solution to (1.3)*, that is for every $\phi\in\mathbb{X}^s_{m,T}$ it holds*

$$\iint_{\mathcal{S}_T} y^{1-2s}(\nabla v\nabla\phi+m^2 v\phi)\,dx\,dy=\kappa_s\langle g,\mathrm{Tr}(\phi)\rangle_{\mathbb{H}^{-s}_{m,T},\mathbb{H}^s_{m,T}}.$$

Taking into account the previous results we can reformulate the non-local problem (1.1) in a local way as explained below.

Let $g\in\mathbb{H}^{-s}_{m,T}$ and consider the following two problems:

$$\begin{cases}(-\Delta_x+m^2)^s u=g & \text{in } (0,T)^N,\\ u(x+Te_i)=u(x) & \text{for } x\in\mathbb{R}^N\end{cases}\tag{3.5}$$

and

$$\begin{cases}-\operatorname{div}(y^{1-2s}\nabla v)+m^2y^{1-2s}v=0 & \text{in } \mathcal{S}_T,\\ v_{|\{x_i=0\}}=v_{|\{x_i=T\}} & \text{on } \partial_L\mathcal{S}_T,\\ \dfrac{\partial v}{\partial\nu^{1-2s}}=g(x) & \text{on } \partial^0\mathcal{S}_T.\end{cases}\tag{3.6}$$

Definition 3.3. We say that $u \in \mathbb{H}^s_{m,T}$ is a *weak solution* to (3.5) if $u = \mathrm{Tr}(v)$ and v is a weak solution to (3.6).

Remark 3.4. Later, with abuse of notation, we will denote by $v(\,\cdot\,, 0)$ the trace $\mathrm{Tr}(v)$ of a function $v \in \mathbb{X}^s_{m,T}$.

We conclude this section giving the proof of the following sharp trace inequality:

Theorem 3.5. *For any $v \in \mathbb{X}^s_{m,T}$ we have*

$$\kappa_s |\mathrm{Tr}(v)|^2_{\mathbb{H}^s_{m,T}} \le ||v||^2_{\mathbb{X}^s_{m,T}} \tag{3.7}$$

and the equality is attained if and only if $v = \mathrm{Ext}(\mathrm{Tr}(v))$. In particular,

$$\begin{aligned} &||v||^2_{\mathbb{X}^s_{m,T}} - \kappa_s m^{2s} |\mathrm{Tr}(v)|^2_{L^2(0,T)^N} = 0 \\ &\qquad\qquad \Leftrightarrow \ v(x,y) = C\,\theta(my) \quad \textit{for some } C \in \mathbb{R}. \end{aligned} \tag{3.8}$$

Proof. By properties (b) and (c), for any $v \in \mathbb{X}^s_{m,T}$

$$\kappa_s |\mathrm{Tr}(v)|^2_{\mathbb{H}^s_{m,T}} = ||\mathrm{Ext}\,\mathrm{Tr}(v)||^2_{\mathbb{X}^s_{m,T}} \le ||v||^2_{\mathbb{X}^s_{m,T}},$$

and the equality holds if and only if $v = \mathrm{Ext}\,\mathrm{Tr}(v)$.

Now, we prove (3.8). We denote by c_k the Fourier coefficients of $\mathrm{Tr}(v)$. If $v(x,y) = C\theta_m(y) := C\theta(my)$ for some $C \in \mathbb{R}$, as $\theta(0) = 1$, we have

$$\begin{aligned} ||v||^2_{\mathbb{X}^s_{m,T}} &= C^2 T^N \int_0^{+\infty} y^{1-2s} (|\theta'_m(y)|^2 + m^2 |\theta_m(y)|^2)\, dy \\ &= C^2 T^N m^{2s} \int_0^{+\infty} y^{1-2s} (|\theta'(y)|^2 + m^2 |\theta(y)|^2)\, dy \\ &= C^2 T^N m^{2s} \kappa_s = m^{2s} \kappa_s |\mathrm{Tr}(v)|^2_{L^2(0,T)^N}. \end{aligned}$$

Now, assume that $||v||^2_{\mathbb{X}^s_{m,T}} - \kappa_s m^{2s} |\mathrm{Tr}(v)|^2_{L^2(0,T)^N} = 0$. By (b) and (c),

$$\begin{aligned} ||\mathrm{Ext}\,\mathrm{Tr}(v)||^2_{\mathbb{X}^s_{m,T}} &\le ||v||^2_{\mathbb{X}^s_{m,T}} = \kappa_s m^{2s} |\mathrm{Tr}(v)|^2_{L^2(0,T)^N} \\ &\le \kappa_s |\mathrm{Tr}(v)|^2_{\mathbb{H}^s_{m,T}} = ||\mathrm{Ext}\,\mathrm{Tr}(v)||^2_{\mathbb{X}^s_{m,T}} \end{aligned} \tag{3.9}$$

that is

$$||v||^2_{\mathbb{X}^s_{m,T}} = ||\mathrm{Ext}\,\mathrm{Tr}(v)||^2_{\mathbb{X}^s_{m,T}} = \kappa_s m^{2s} |\mathrm{Tr}(v)|^2_{L^2(0,T)^N}.$$

Let us note that $||v||^2_{\mathbb{X}^s_{m,T}} = ||\mathrm{Ext}\,\mathrm{Tr}(v)||^2_{\mathbb{X}^s_{m,T}}$ implies $v = \mathrm{Ext}(\mathrm{Tr}(v))$. In particular, from

$$\kappa_s m^{2s} |\mathrm{Tr}(v)|^2_{L^2(0,T)^N} = ||\mathrm{Ext}\,\mathrm{Tr}(v)||^2_{\mathbb{X}^s_{m,T}} = \kappa_s |\mathrm{Tr}(v)|^2_{\mathbb{H}^s_{m,T}}$$

we obtain that $c_k = 0$ for any $k \ne 0$, so we get

$$v = \mathrm{Ext}(\mathrm{Tr}(v)) = \sum_{k \in \mathbb{Z}^N} c_k \theta(\sqrt{\omega^2 |k|^2 + m^2}\, y) e^{\imath k \cdot x} = c_0 \theta(my). \qquad \square$$

4. Periodic solutions in the cylinder $\mathcal{S}_T$

In this section we prove the existence of a solution to (1.1). As shown in the previous section, we know that the study of (1.1) is equivalent to investigate the existence of weak solutions to

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s} v = 0 & \text{in } \mathcal{S}_T := (0,T)^N \times (0,\infty), \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial_L \mathcal{S}_T := \partial(0,T)^N \times [0,\infty), \\ \dfrac{\partial v}{\partial \nu^{1-2s}} = \kappa_s [m^{2s} v + f(x,v)] & \text{on } \partial^0 \mathcal{S}_T := (0,T)^N \times \{0\}. \end{cases} \tag{4.1}$$

For simplicity, we will assume that $\kappa_s = 1$. Then, we will look for the critical points of

$$\mathcal{J}_m(v) = \frac{1}{2}\,||v||^2_{\mathbb{X}^s_{m,T}} - \frac{m^{2s}}{2}\,|v(\,\cdot\,,0)|^2_{L^2(0,T)^N} - \int_{\partial^0 \mathcal{S}_T} F(x,v)\,dx$$

defined for $v \in \mathbb{X}^s_{m,T}$. More precisely, we will prove that $\mathcal{J}_m$ satisfies the assumptions of the Linking Theorem [24]:

Theorem 4.1. *Let $(X, ||\cdot||)$ be a real Banach space with $X = Y \oplus Z$, where Y is finite dimensional. Let $J \in \mathcal{C}^1(X,\mathbb{R})$ be a functional satisfying the following conditions*:

(a) *J satisfies the Palais–Smale condition.*
(b) *There exist $\eta, \rho > 0$ such that $J(v) \geq \rho$ for all $v \in Z$ such that $||v|| = \eta$.*
(c) *There exist $z \in \partial B_1 \cap Z$, $R > \eta$ and $R' > 0$ such that $J \leq 0$ on $\partial\mathcal{A}$, where*

$$\begin{aligned} \mathcal{A} &= \{v = y + rz : y \in Y,\ ||y|| \leq R' \text{ and } 0 \leq r \leq R\}, \\ \partial\mathcal{A} &= \{v = y + rz : y \in Y,\ ||y|| = R' \text{ or } r \in \{0,R\}\}. \end{aligned}$$

Then J possesses a critical value $c \geq \rho$ which can be characterized as

$$c := \inf_{\gamma\in\Gamma} \max_{v\in\mathcal{A}} J(\gamma(v)),$$

where $\Gamma := \{\gamma \in \mathcal{C}(\mathcal{A}, X) : \gamma = \mathrm{Id}$ on $\partial\mathcal{A}\}$.

Due to the assumptions on f, it is easy to prove that $\mathcal{J}_m$ is well-defined on $\mathbb{X}^s_{m,T}$ and $\mathcal{J}_m \in \mathcal{C}^1(\mathbb{X}^s_{m,T}, \mathbb{R})$. Moreover, by (3.7), we notice that the quadratic part of $\mathcal{J}_m$ is nonnegative, that is

$$||v||^2_{\mathbb{X}^s_{m,T}} - m^{2s}|v(\,\cdot\,,0)|^2_{L^2(0,T)^N} \geq 0. \tag{4.2}$$

Let us note that

$$\mathbb{X}^s_{m,T} = \langle \theta(my) \rangle \oplus \left\{ v \in \mathbb{X}^s_{m,T} : \int_{(0,T)^N} v(x,0)\,dx = 0 \right\} =: \mathbb{Y}^s_{m,T} \oplus \mathbb{Z}^s_{m,T},$$

where $\dim \mathbb{Y}^s_{m,T} < \infty$ and $\mathbb{Z}^s_{m,T}$ is the orthogonal complement of $\mathbb{Y}^s_{m,T}$ with respect to the inner product in $\mathbb{X}^s_{m,T}$. In order to prove that $\mathcal{J}_m$ verifies the linking hypotheses we need the following results.

Lemma 4.2. *$\mathcal{J}_m \leq 0$ on $\mathbb{Y}^s_{m,T}$.*

Proof. It follows directly from (3.8) and assumption (f6). □

Lemma 4.3. *There exist $\rho > 0$ and $\eta > 0$ such that*

$$\mathcal{J}_m(v) \geq \rho \quad \textit{for } v \in \mathbb{Z}^s_{m,T} \textit{ such that } ||v||_{\mathbb{X}^s_{m,T}} = \eta.$$

Proof. Firstly we show that there exists a constant $C > 0$ such that

$$||v||^2_{\mathbb{X}^s_{m,T}} - m^{2s}|v(\cdot,0)|^2_{L^2(0,T)^N} \geq C||v||^2_{\mathbb{X}^s_{m,T}} \tag{4.3}$$

for any $v \in \mathbb{Z}^s_{m,T}$. Assume, by contradiction, that there exists a sequence $(v_j) \subset \mathbb{Z}^s_{m,T}$ such that

$$||v_j||^2_{\mathbb{X}^s_{m,T}} - m^{2s}|v_j(\,\cdot\,,0)|^2_{L^2(0,T)^N} < \frac{1}{j}||v_j||^2_{\mathbb{X}^s_{m,T}}.$$

Let $z_j = v_j/||v_j||_{\mathbb{X}^s_{m,T}}$. Then $||z_j||_{\mathbb{X}^s_{m,T}} = 1$, so we can assume that $z_j \rightharpoonup z$ in $\mathbb{X}^s_{m,T}$ and $z_j(\,\cdot\,,0) \to z(\,\cdot\,,0)$ in $L^2(0,T)^N$ for some $z \in \mathbb{Z}^s_{m,T}$ ($\mathbb{Z}^s_{m,T}$ is weakly closed). Hence, for any $j \in \mathbb{N}$

$$1 - m^{2s}|z_j(\,\cdot\,,0)|^2_{L^2(0,T)^N} < \frac{1}{j},$$

so we get $|z_j(\,\cdot\,,0)|^2_{L^2(0,T)^N} \to 1/m^{2s}$ that is $|z(\,\cdot\,,0)|_{L^2(0,T)^N} = 1/m^s$.

On the other hand,

$$0 \leq ||z||^2_{\mathbb{X}^s_{m,T}} - m^{2s}|z(\,\cdot\,,0)|^2_{L^2(0,T)^N} \leq \liminf_{j\to\infty} ||z_j||^2_{\mathbb{X}^s_{m,T}} - m^{2s}|z_j(\cdot,0)|^2_{L^2(0,T)^N} = 0$$

implies that $z = c\theta(my)$ by (3.8). But $z \in \mathbb{Z}^s_{m,T}$, so $c = 0$ and this is a contradiction because of $|z(\,\cdot\,,0)|_{L^2(0,T)^N} = 1/m^s > 0$.

Taking into account (4.3), (2.16) and Theorem 2.2, we have

$$\begin{aligned}\mathcal{J}_m(v) &\geq C||v||^2_{\mathbb{X}^s_{m,T}} - \varepsilon|v(\,\cdot\,,0)|^2_{L^2(0,T)^N} - C_\varepsilon|v(\,\cdot\,,0)|^{p+1}_{L^{p+1}(0,T)^N} \\ &\geq \left(C - \frac{\varepsilon}{m}\right)||v||^2_{\mathbb{X}^s_{m,T}} - C||v||^{p+1}_{\mathbb{X}^s_{m,T}}\end{aligned}$$

for any $v \in \mathbb{Z}^s_{m,T}$. Choosing $\varepsilon \in (0, mC)$, we can find $\rho > 0$ and $\eta > 0$ such that

$$\inf\left\{\mathcal{J}_m(v) : v \in \mathbb{Z}^s_{m,T} \text{ and } ||v||_{\mathbb{X}^s_{m,T}} = \eta\right\} \geq \rho. \qquad \square$$

Lemma 4.4. *There exist $R > \eta$, $R' > 0$ and $z \in \mathbb{Z}^s_{m,T}$ such that*

$$\max_{\partial\mathbb{A}^s_{m,T}} \mathcal{J}_m(v) \leq 0 \quad \textit{and} \quad \max_{\mathbb{A}^s_{m,T}} \mathcal{J}_m(v) < \infty,$$

where $\mathbb{A}^s_{m,T} = \left\{v = y + rz : ||y||_{\mathbb{X}^s_{m,T}} \leq R' \textit{ and } r \in [0,R]\right\}$.

PROOF. From Lemma 4.2 we know that $\mathcal{J}_m \leq 0$ on $\mathbb{Y}^s_{m,T}$. Let us consider

$$w = \prod_{i=1}^{N} \sin(\omega x_i) \, \frac{1}{y+1}.$$

Note that $w \in \mathbb{Z}^s_{m,T}$ (since $\int_0^T \sin(\omega x)\, dx = 0$) and

$$\begin{aligned} ||w||^2_{\mathbb{X}^s_{m,T}} = N & \bigg(\prod_{i=1}^{N-1} \int_0^T \sin^2(\omega x)\, dx \bigg) \omega^2 \\ & \cdot \bigg(\int_0^T \cos^2(\omega x)\, dx \bigg) \bigg(\int_0^\infty y^{1-2s} \, \frac{dy}{(y+1)^2} \bigg) \\ & + \bigg(\prod_{i=1}^{N} \int_0^T \sin^2(\omega x)\, dx \bigg) \bigg(\int_0^\infty y^{1-2s} \, \frac{dy}{(y+1)^4} \bigg) \\ & + m^2 \bigg(\prod_{i=1}^{N} \int_0^T \sin^2(\omega x)\, dx \bigg) \bigg(\int_0^\infty y^{1-2s} \, \frac{dy}{(y+1)^2} \bigg). \end{aligned}$$

So there exist $C_1, C_2, C_3 > 0$ (independent of m) such that

$$C_1 \leq ||w||^2_{\mathbb{X}^s_{m,T}} \leq C_2 + m^2 C_3. \tag{4.4}$$

Set $z = w/||w||_{\mathbb{X}^s_{m,T}}$. It is clear that $z \in \mathbb{Z}^s_{m,T}$ and $||z||_{\mathbb{X}^s_{m,T}} = 1$.

By the Hölder inequality, we can observe that if $v = y + rz \in \mathbb{Y}^s_{m,T} \oplus \mathbb{R}_+ z$

$$\begin{aligned} |v(\,\cdot\,,0)|^\mu_{L^\mu(0,T)^N} & \geq C |v(\,\cdot\,,0)|^\mu_{L^2(0,T)^N} \\ & = C \bigg(\int_{(0,T)^N} (c + rz)^2\, dx \bigg)^{\mu/2} \geq C'(m^{2s} c^2 T^N + r^2)^{\mu/2}. \end{aligned}$$

Then, for any $v = y + rz \in \mathbb{Y}^s_{m,T} \oplus \mathbb{R}_+ z$,

$$\begin{aligned} \mathcal{J}_m(v) & = \frac{1}{2}\, ||z||^2_{\mathbb{X}^s_{m,T}} - \frac{m^{2s}}{2}\, |z(\,\cdot\,,0)|^2_{L^2(0,T)^N} - \int_{\partial^0 \mathcal{S}_T} F(x,v)\, dx \\ & \leq \frac{r^2}{2} - A|v(\,\cdot\,,0)|^\mu_{L^\mu(0,T)^N} + BT^N \\ & \leq \frac{r^2}{2} - C''(m^{2s} c^2 T^N + r^2)^{\mu/2} + BT^N \qquad (4.5) \\ & \leq (m^{2s} c^2 T^N + r^2) - C''(m^{2s} c^2 T^N + r^2)^{\mu/2} + BT^N \qquad (4.6) \\ & = ||v||^2_{\mathbb{X}^s_{m,T}} - E||v||^\mu_{\mathbb{X}^s_{m,T}} + F. \qquad (4.7) \end{aligned}$$

Recall that $\mu > 2$. By (4.5), there exists $R > 0$ such that

$$\mathcal{J}_m(y + rz) \leq 0 \quad \text{for any } r \geq R \text{ and } y \in \mathbb{Y}^s_{m,T}.$$

Let $r \in [0, R]$. By (4.6), we can find $R' > 0$ such that $\mathcal{J}_m(y + rz) \leq 0$ for $||y||_{\mathbb{X}^s_{m,T}} \geq R'$. By (4.7), we deduce that there exists a constant $\delta > 0$ such that $\mathcal{J}_m(v) \leq \delta$ for any $v \in \mathbb{A}^s_{m,T}$. □

Finally, we show that $\mathcal{J}_m$ satisfies the Palais–Smale condition:

LEMMA 4.5. *Let $c\in\mathbb{R}$. Let $(v_j)\subset\mathbb{X}^s_{m,T}$ be a sequence such that*

$$\mathcal{J}_m(v_j)\to c \quad \text{and} \quad \mathcal{J}'_m(v_j)\to 0. \tag{4.8}$$

Then there exist a subsequence $(v_{j_h})\subset(v_j)$ and $v\in\mathbb{X}^s_{m,T}$ such that $v_{j_h}\to v$ in $\mathbb{X}^s_{m,T}$.

PROOF. We start proving that (v_j) is bounded in $\mathbb{X}^s_{m,T}$. Fix $\beta\in(1/\mu,1/2)$. By Lemma 2.3 with $\varepsilon=1$, we get

$$\begin{aligned}\bigg|\int_{\partial^0\mathcal{S}_T\cap\{|v_j|\le r_0\}}(\beta f(x,v_j)v_j-F(x,v_j))\,dx\bigg| \\ \le((2\beta+1)r_0^2+C_1(p+2)r_0^{p+1})T^N=\iota_1\end{aligned} \tag{4.9}$$

and

$$\bigg|\int_{\partial^0\mathcal{S}_T\cap\{|v_j|\le r_0\}}F(x,v_j)\,dx\bigg|\le\big(r_0^2+C_1r_0^{p+1}\big)T^N=\iota_2. \tag{4.10}$$

Taking into account Lemma 2.4, (f5), (4.2), (4.8)–(4.10), we have for j large enough

$$\begin{aligned}c+1\,+||v_j||_{\mathbb{X}^s_{m,T}} &\ge\mathcal{J}_m(v_j)-\beta\langle\mathcal{J}'(v_j),v_j\rangle\\
&=\Big(\frac12-\beta\Big)\big[||v_j||^2_{\mathbb{X}^s_{m,T}}-m^{2s}|v_j(\cdot,0)|^2_{L^2(0,T)^N}\big]\\
&\quad+\int_{\partial^0\mathcal{S}_T}[\beta f(x,v_j)v_j-F(x,v_j)]\,dx\\
&\ge\int_{\partial^0\mathcal{S}_T}[\beta f(x,v_j)v_j-F(x,v_j)]\,dx\\
&=\int_{\partial^0\mathcal{S}_T\cap\{|v_n|\ge r_0\}}[\beta f(x,v_j)v_j-F(x,v_j)]\,dx\\
&\quad+\int_{\partial^0\mathcal{S}_T\cap\{|v_n|\le r_0\}}[\beta f(x,v_j)v_j-F(x,v_j)]\,dx\\
&\ge(\mu\beta-1)\int_{\partial^0\mathcal{S}_T\cap\{|v_j|\ge r_0\}}F(x,v_j)\,dx-\iota_1\\
&\quad\ge(\mu\beta-1)\int_{\partial^0\mathcal{S}_T}F(x,v_j)\,dx-(\mu\beta-1)\iota_2-\iota_1\\
&=(\mu\beta-1)\int_{\partial^0\mathcal{S}_T}F(x,v_j)\,dx-\iota \qquad (4.11)\\
&\ge(\mu\beta-1)\big[a_3|v_j(\,\cdot\,,0)|^\mu_{L^\mu(0,T)^N}-a_4T^N\big]-\iota\\
&\ge(\mu\beta-1)\big[a_3|v_j(\,\cdot\,,0)|^\mu_{L^2(0,T)^N}T^{-N(\mu-2)/2}-a_4T^N\big]-\iota. \qquad (4.12)\end{aligned}$$

Hence, by (4.11) and (4.12), we deduce that

$$\begin{aligned}||v_j||^2_{\mathbb{X}^s_{m,T}} &= 2\mathcal{J}_m(v_j) + m^{2s}|v_j(\,\cdot\,,0)|^2_{L^2(0,T)^N} + 2\int_{\partial^0\mathcal{S}_T} F(x,v_j)\,dx \\ &\leq C_1 + C_2\big(C_3 + 1 + ||v_j||_{\mathbb{X}^s_{m,T}}\big)^{2/\mu} + C_4\big(C_5 + 1 + ||v_j||_{\mathbb{X}^s_{m,T}}\big) \\ &\leq C_6 + C_7||v_j||_{\mathbb{X}^s_{m,T}}\end{aligned}$$

that is (v_j) is bounded in $\mathbb{X}^s_{m,T}$.

By Theorem 2.1, we can assume, up to a subsequence, that

$$\begin{aligned} v_j &\rightharpoonup v && \text{in } \mathbb{X}^s_{m,T}, \\ v_j(\,\cdot\,,0) &\to v(\,\cdot\,,0) && \text{in } L^{p+1}(0,T)^N, \\ v_j(\,\cdot\,,0) &\to v(\,\cdot\,,0) && \text{a.e. in } (0,T)^N \end{aligned} \tag{4.13}$$

as $j \to \infty$ and there exists $h \in L^{p+1}(0,T)^N$ such that

$$|v_j(x,0)| \leq h(x) \quad \text{a.e. in } x \in (0,T)^N, \text{ for all } j \in \mathbb{N}. \tag{4.14}$$

Taking into account (f2), (f4), (4.13), (4.14) and the Dominated Convergence Theorem, we get

$$\int_{\partial^0\mathcal{S}_T} f(x,v_j)v_j\,dx \to \int_{\partial^0\mathcal{S}_T} f(x,v)v\,dx \tag{4.15}$$

and

$$\int_{\partial^0\mathcal{S}_T} f(x,v_j)v\,dx \to \int_{\partial^0\mathcal{S}_T} f(x,v)v\,dx \tag{4.16}$$

as $j \to \infty$. Due to (4.8) and boundedness of $(v_j)_{j\in\mathbb{N}}$ in $\mathbb{X}^s_{m,T}$, we deduce that $\langle \mathcal{J}'_m(v_j), v_j\rangle \to 0$, that is

$$||v_j||^2_{\mathbb{X}^s_{m,T}} - m^{2s}|v_j(\,\cdot\,,0)|^2_{L^2(0,T)^N} - \int_{\partial^0\mathcal{S}_T} f(x,v_j)v_j\,dx \to 0 \tag{4.17}$$

as $j \to \infty$. By (4.13), (4.15) and (4.17) we have

$$||v_j||^2_{\mathbb{X}^s_{m,T}} \to m^{2s}|v(\,\cdot\,,0)|^2_{L^2(0,T)^N} - \int_{\partial^0\mathcal{S}_T} f(x,v)v\,dx. \tag{4.18}$$

Moreover, by (4.8) and $v \in \mathbb{X}^s_{m,T}$, we have $\langle \mathcal{J}'_m(v_j), v\rangle \to 0$ as $j \to \infty$, that is

$$\langle v_j, v\rangle_{\mathbb{X}^s_{m,T}} - m^{2s}\langle v_j, v\rangle_{L^2(0,T)^N} - \int_{\partial^0\mathcal{S}_T} f(x,v_j)v\,dx \to 0. \tag{4.19}$$

Taking into account (4.13), (4.14), (4.16) and (4.19), we obtain

$$||v||^2_{\mathbb{X}^s_{m,T}} = m^{2s}|v(\,\cdot\,,0)|^2_{L^2(0,T)^N} - \int_{\partial^0\mathcal{S}_T} f(x,v)v\,dx. \tag{4.20}$$

Thus, (4.18) and (4.20) imply that

$$||v_j||^2_{\mathbb{X}^s_{m,T}} \to ||v||^2_{\mathbb{X}^s_{m,T}} \quad \text{as } j \to \infty. \tag{4.21}$$

Since $\mathbb{X}^s_{m,T}$ is a Hilbert space, we have

$$||v_j - v||^2_{\mathbb{X}^s_{m,T}} = ||v_j||^2_{\mathbb{X}^s_{m,T}} + ||v||^2_{\mathbb{X}^s_{m,T}} - 2\langle v_j, v\rangle_{\mathbb{X}^s_{m,T}}$$

and, due to $v_j \rightharpoonup v$ in $\mathbb{X}^s_{m,T}$ and (4.21), we can conclude that $v_j \to v$ in $\mathbb{X}^s_{m,T}$, as $j \to \infty$. □

PROOF OF THEOREM 1.1. Taking into account Lemmas 4.2–4.5, by Theorem 4.1, we deduce that for any fixed $m > 0$, there exists of a function $v_m \in \mathbb{X}^s_{m,T}$ such that $\mathcal{J}_m(v_m) = \alpha_m$, $\mathcal{J}'_m(v_m) = 0$, where

$$\alpha_m = \inf_{\gamma\in\Gamma_m} \max_{v\in\mathbb{A}^s_{m,T}} \mathcal{J}_m(\gamma(v)) \tag{4.22}$$

and $\Gamma_m = \{\gamma \in \mathcal{C}(\mathbb{A}^s_{m,T}, \mathbb{X}^s_{m,T}) : \gamma = \mathrm{Id} \text{ on } \partial\mathbb{A}^s_{m,T}\}$. □

REMARK 4.6. Let us observe that an easy consequence of Theorem 1.1 is the existence of infinitely many distinct T-periodic solutions to (1.1). To prove it, one can proceed as in the proof of [24, Corollary 6.44].

5. Regularity of solutions to (1.1)

In this section we study the regularity of weak solutions to problem (1.1).

LEMMA 5.1. *Let $v \in \mathbb{X}^s_{m,T}$ be a weak solution to*

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s} v = 0 & \text{in } \mathcal{S}_T, \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial_L\mathcal{S}_T, \\ \dfrac{\partial v}{\partial \nu^{1-2s}} = m^{2s} v + f(x,v) & \text{on } \partial^0\mathcal{S}_T. \end{cases} \tag{5.1}$$

Then $v(\,\cdot\,,0) \in L^q(0,T)^N$ for all $q < \infty$.

PROOF. We proceed as in the proof of Lemma 7 in [3]. Since v is a critical point for $\mathcal{J}_m$, we know that

$$\iint_{\mathcal{S}_T} y^{1-2s}(\nabla v \nabla \eta + m^2 v\eta)\,dx,dy = \int_{\partial^0(\mathcal{S}_T} m^{2s} v\eta + f(x,v)\eta)\,dx \tag{5.2}$$

for all $\eta \in \mathbb{X}^m_T$. Let $w = v v_K^{2\beta} \in \mathbb{X}^s_{m,T}$, where $v_K = \min\{|v|, K\}$, $K > 1$ and $\beta \geq 0$. Taking $\eta = w$ in (5.2), we deduce that

$$\begin{aligned} \iint_{\mathcal{S}_T} y^{1-2s} v_K^{2\beta}(|\nabla v|^2 + m^2 v^2)\,dx\,dy + \iint_{D_{K,T}} 2\beta y^{1-2s} v_K^{2\beta}|\nabla v|^2\,dx\,dy \\ = m^{2s}\int_{\partial^0\mathcal{S}_T} v^2 v_K^{2\beta}\,dx + \int_{\partial^0\mathcal{S}_T} f(x,v) v v_K^{2\beta}\,dx, \end{aligned} \tag{5.3}$$

where $D_{K,T} = \{(x,y) \in \mathcal{S}_T : |v(x,y)| \leq K\}$.

It is easy to see that

$$\iint_{\mathcal{S}_T} y^{1-2s}|\nabla(vv_K^\beta)|^2\,dx\,dy = \iint_{\mathcal{S}_T} y^{1-2s}v_K^{2\beta}|\nabla v|^2\,dx\,dy + \iint_{D_{K,T}} (2\beta+\beta^2)y^{1-2s}v_K^{2\beta}|\nabla v|^2\,dx\,dy. \tag{5.4}$$

Then, putting together (5.3) and (5.4), we get

$$\begin{aligned} ||vv_K^\beta||^2_{\mathbb{X}^s_{m,T}} &= \iint_{\mathcal{S}_T} y^{1-2s}\big[|\nabla(vv_K^\beta)|^2 + m^2v^2v_K^{2\beta}\big]\,dx\,dy \\ &= \iint_{\mathcal{S}_T} y^{1-2s}v_K^{2\beta}\big[|\nabla v|^2 + m^2v^2\big]\,dx\,dy \\ &\quad + \iint_{D_{K,T}} 2\beta\left(1+\frac{\beta}{2}\right)y^{1-2s}v_K^{2\beta}|\nabla v|^2\,dx\,dy \\ &\le c_\beta\bigg[\iint_{\mathcal{S}_T} y^{1-2s}v_K^{2\beta}\big[|\nabla v|^2+m^2v^2\big]\,dx\,dy \\ &\quad + \iint_{D_{K,T}} 2\beta y^{1-2s}v_K^{2\beta}|\nabla v|^2\,dx\,dy\bigg] \\ &= c_\beta\int_{\partial^0\mathcal{S}_T}\big(m^{2s}v^2v_K^{2\beta} + f(x,v)vv_K^{2\beta}\big)\,dx, \end{aligned} \tag{5.5}$$

where $c_\beta = 1+\beta/2$. By Lemma 2.3 with $\varepsilon = 1$, we deduce that

$$m^{2s}v^2v_K^{2\beta} + f(x,v)vv_K^{2\beta} \le (m^{2s}+2)v^2v_K^{2\beta} + (p+1)C_1|v|^{p-1}v^2v_K^{2\beta}$$

on $\partial^0\mathcal{S}_T$. Now, we prove that $|v|^{p-1} \le 1+h$ on $\partial^0\mathcal{S}_T$ for some $h \in L^{N/2s}(0,T)^N$. Firstly, we observe that

$$|v|^{p-1} = \chi_{\{|v|\le 1\}}|v|^{p-1} + \chi_{\{|v|>1\}}|v|^{p-1} \le 1 + \chi_{\{|v|>1\}}|v|^{p-1} \quad \text{on } \partial^0\mathcal{S}_T.$$

If $(p-1)N < 4s$ then

$$\int_{\partial^0\mathcal{S}_T}\chi_{\{|v|>1\}}|v|^{N(p-1)/(2s)}\,dx \le \int_{\partial^0\mathcal{S}_T}\chi_{\{|v|>1\}}|v|^2\,dx < \infty$$

while if $4s \le (p-1)N$ we have that $(p-1)N/(2s) \in [2, 2N/(N-2s)]$. Therefore, there exist a constant $c = m^{2s}+2+(p+1)C_1$ and a function $h \in L^{N/2s}(0,T)^N$, $h \ge 0$ and independent of K and β, such that

$$m^{2s}v^2v_K^{2\beta} + f(x,v)vv_K^{2\beta} \le (c+h)v^2v_K^{2\beta} \quad \text{on } \partial^0\mathcal{S}_T. \tag{5.6}$$

Taking into account (5.5) and (5.6), we have

$$||vv_K^\beta||^2_{\mathbb{X}^s_{m,T}} \le c_\beta\int_{\partial^0\mathcal{S}_T}(c+h)v^2v_K^{2\beta}\,dx,$$

and, by the Monotone Convergence Theorem (v_K is increasing with respect to K), we have as $K \to \infty$

$$(5.7)\qquad |||v|^{\beta+1}||^2_{\mathbb{X}^s_{m,T}} \le cc_\beta \int_{\partial^0\mathcal{S}_T} |v|^{2(\beta+1)}\,dx + c_\beta \int_{\partial^0\mathcal{S}_T} h|v|^{2(\beta+1)}\,dx.$$

Fix $M > 0$ and let $A_1 = \{h \le M\}$ and $A_2 = \{h > M\}$. Then

$$(5.8)\qquad \int_{\partial^0\mathcal{S}_T} h|v(\,\cdot\,,0)|^{2(\beta+1)}\,dx \le M||v(\,\cdot\,,0)|^{\beta+1}|^2_{L^2(0,T)^N} + \varepsilon(M)||v(\,\cdot\,,0)|^{\beta+1}|^2_{L^{2^\sharp}(0,T)^N},$$

where

$$\varepsilon(M) = \bigg(\int_{A_2} h^{N/2s}\,dx\bigg)^{2s/N} \to 0 \quad \text{as } M \to \infty.$$

Taking into account (5.7), (5.8), we get

$$(5.9)\qquad |||v|^{\beta+1}||^2_{\mathbb{X}^s_{m,T}} \le c_\beta(c+M)||v(\,\cdot\,,0)|^{\beta+1}|^2_{L^2(0,T)^N} + c_\beta\varepsilon(M)||v(\,\cdot\,,0)|^{\beta+1}|^2_{L^{2^\sharp_s}(0,T)^N}.$$

By Theorem 2.2, we know that there exists a constant $C^2_{2^\sharp_s,m} > 0$ such that

$$(5.10)\qquad ||v(\,\cdot\,,0)|^{\beta+1}|^2_{L^{2^\sharp_s}(0,T)^N} \le C^2_{2^\sharp_s,m}|||v|^{\beta+1}||^2_{\mathbb{X}^s_{m,T}}.$$

Then, choosing M large enough so that $\varepsilon(M)c_\beta C^2_{2^\sharp,m} < 1/2$, by (5.9) and (5.10), we obtain

$$(5.11)\qquad ||v(\,\cdot\,,0)|^{\beta+1}|^2_{L^{2^\sharp_s}(0,T)^N} \le 2C^2_{2^\sharp_s,m}c_\beta(c+M)||v(\,\cdot\,,0)|^{\beta+1}|^2_{L^2(0,T)^N}.$$

Then we can start a bootstrap argument: since $v(\,\cdot\,,0) \in L^{2N/(N-2s)}$ we can apply (5.11) with $\beta_1 + 1 = N/(N-2s)$ to deduce that

$$v(\,\cdot\,,0) \in L^{(\beta_1+1)2N/(N-2s)}(0,T)^N = L^{2N^2/(N-2s)^2}(0,T)^N.$$

Applying (5.11) again, after k iterations we find $v(\,\cdot\,,0) \in L^{2N^k/(N-2s)^k}(0,T)^N$, and so $v(\,\cdot\,,0) \in L^q(0,T)^N$ for all $q \in [2,\infty)$. □

THEOREM 5.2. *Let $v \in \mathbb{X}^s_{m,T}$ be a weak solution to*

$$(5.12)\qquad \begin{cases} -\operatorname{div}(y^{1-2s}\nabla v) + m^2y^{1-2s}v = 0 & \text{in } \mathcal{S}_T,\\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial_L\mathcal{S}_T,\\ \dfrac{\partial v}{\partial \nu^{1-2s}} = \kappa_s[m^{2s}v + f(x,v)] & \text{on } \partial^0\mathcal{S}_T. \end{cases}$$

Let us assume that v is extended by periodicity to the whole $\mathbb{R}^{N+1}_+$. Then $v(\,\cdot\,,0) \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.

PROOF. It is clear that $v \in H^1_m(A \times \mathbb{R}_+, y^{1-2s})$ for any bounded domain $A \subset \mathbb{R}^N$. By Lemma 5.1 here and Proposition 3.5 in [18], the statement follows. □

6. Passage to the limit as $m \to 0$

In this last section, we give the proof of Theorem 1.2. We verify that it is possible to take the limit in (1.3) as $m \to 0$ so that we deduce the existence of a nontrivial weak solution to (1.4). In particular, we will prove that such solution is Hölder continuous. We remark that in Section 4 we proved that for any $m > 0$ there exists $v_m \in \mathbb{X}^s_{m,T}$ such that

$$\mathcal{J}_m(v_m) = \alpha_m \quad \text{and} \quad \mathcal{J}'_m(v_m) = 0, \tag{6.1}$$

where α_m is defined in (4.22). In order to attain our aim, we estimate from above and below the critical levels of the functional $\mathcal{J}_m$ independently of m.

Let us assume that $0 < m < m_0 := 1/2C^2_{2^\sharp_s}$, where $C_{2^\sharp_s}$ is the Sobolev constant which appears in (2.10). We start proving that there exists a positive constant δ independent of m such that

$$\alpha_m \le \delta \quad \text{for all } 0 < m < m_0. \tag{6.2}$$

Due to (4.4) and $m < m_0$, we know that

$$C_1 \le ||z||^2_{\mathbb{X}^s_{m,T}} \le C_2 + m_0^2 C_3.$$

Moreover (see Lemma 4.4), we have for any $v = y + rz \in \mathbb{Y}^s_{m,T} \oplus \mathbb{R}_+ z$

$$\begin{aligned} |v(\,\cdot\,,0)|^\mu_{L^\mu(0,T)^N} &\ge T^{-N(\mu-2)/2} |v(\,\cdot\,,0)|^\mu_{L^2(0,T)^N} \\ &= T^{-N(\mu-2)/2} \bigg(\int_{(0,T)^N} (c+rz)^2\,dx \bigg)^{\mu/2} \\ &\ge T^{-N(\mu-2)/2} \bigg(c^2 T^N + \Big(\frac{T}{2}\Big)^N \frac{r^2}{||z||^2_{\mathbb{X}^s_{m,T}}} \bigg)^{\mu/2} \\ &\ge T^{-N(\mu-2)/2} \min\bigg\{ \frac{1}{m_0^{2s}}, \frac{(T/2)^N}{C_2 + m_0^2 C_3} \bigg\} (m^{2s} c^2 T^N + r^2)^{\mu/2} \\ &= C ||v||^\mu_{\mathbb{X}^s_{m,T}} \end{aligned} \tag{6.3}$$

for some $C = C(m_0, T, N, s) > 0$. Then, for any $v = y + rz \in \mathbb{Y}^s_{m,T} \oplus \mathbb{R}_+ z$ and $0 < m < m_0$ we get

$$\begin{aligned} \mathcal{J}_m(v) &= \frac{1}{2} ||v||^2_{\mathbb{X}^s_{m,T}} - \frac{m^{2s}}{2} |v(\,\cdot\,,0)|^2_{L^2(0,T)^N} - \int_{\partial^0 \mathcal{S}_T} F(x,v)\,dx \\ &\le \frac{1}{2} ||v||^2_{\mathbb{X}^s_{m,T}} - A |v(\,\cdot\,,0)|^\mu_{L^\mu(0,T)^N} + B\,T^N \\ &= ||v||^2_{\mathbb{X}^s_{m,T}} - C ||v||^\mu_{\mathbb{X}^s_{m,T}} + D \le \delta, \end{aligned} \tag{6.4}$$

where $A, B, C, D, \delta > 0$ are independent of m.

Now we prove that there exists $\lambda > 0$ independent of m such that

$$\alpha_m \ge \lambda \quad \text{for all } 0 < m < m_0. \tag{6.5}$$

Let $v \in \mathbb{Z}^s_{m,T}$ and $\varepsilon > 0$. We denote by c_k the Fourier coefficients of the trace of v. By (2.10) and (3.7) (with $\kappa_s = 1$),

$$|v|_{L^q(0,T)^N} \le C_{2^\sharp_s}\Big(\sum_{|k|\ge 1}\omega^{2s}|k|^{2s}|c_k|^2\Big)^{1/2} \le C_{2^\sharp_s}|v|_{\mathbb{H}^s_{m,T}} \le C_{2^\sharp_s}||v||_{\mathbb{X}^s_{m,T}} \tag{6.6}$$

for any $q \in [2, 2^\sharp_s]$. By Lemma 2.3 and (6.6), we can see that for every $0 < m < m_0$

$$\begin{aligned}\mathcal{J}_m(v) &= \frac{1}{2}\iint_{\mathcal{S}_T} y^{1-2s}(|\nabla v|^2 + m^2v^2)\,dx\,dy \\ &\quad - \frac{m^{2s}}{2}\int_{\partial^0\mathcal{S}_T}|v|^2\,dx - \int_{\partial^0\mathcal{S}_T}F(x,v)\,dx \\ &\ge \frac{1}{2}||v||^2_{\mathbb{X}^s_{m,T}} - \Big(\frac{m}{2}+\varepsilon\Big)|v(\,\cdot\,,0)|^2_{L^2(0,T)^N} - C_\varepsilon|v(\,\cdot\,,0)|^{p+1}_{L^{p+1}(0,T)^N} \\ &\ge \Big[\frac{1}{2} - C^2_{2^\sharp_s}\Big(\frac{m}{2}+\varepsilon\Big)\Big]||v||^2_{\mathbb{X}^m_T} - C_\varepsilon C^{p+1}_{2^\sharp_s}||v||^{p+1}_{\mathbb{X}^s_{m,T}} \\ &\ge \Big(\frac{1}{4} - C^2_{2^\sharp_s}\varepsilon\Big)||v||^2_{\mathbb{X}^s_{m,T}} - C'_\varepsilon||v||^{p+1}_{\mathbb{X}^s_{m,T}}.\end{aligned}$$

Choosing $0 < \varepsilon < 1/(4C^2_{2^\sharp_s})$, we have that $b := 1/4 - C^2_{2^\sharp_s}\varepsilon > 0$. Let $\rho := (b/(2C'_b))^{1/(p-1)}$. Then, for every $v \in \mathbb{Z}^s_{m,T}$ such that $||v||_{\mathbb{X}^s_{m,T}} = \rho$,

$$\mathcal{J}_m(v) \ge b\rho^2 - C'_b\,\rho^{p+1} = \frac{b}{2}\Big(\frac{b}{2C'_b}\Big)^{2/(p-1)} =: \lambda.$$

Therefore, taking into account (6.2) and (6.5), we deduce that

$$\lambda \le \alpha_m \le \delta \quad \text{for every } 0 < m < m_0. \tag{6.7}$$

Now, we estimate the $H^1_{\rm loc}(\mathcal{S}_T, y^{1-2s})$-norm of v_m in order to pass to the limit in (1.3) as $m \to 0$. Fix $\beta \in (1/\mu, 1/2)$. By (6.1) and (6.7), we have for any $m \in (0, m_0)$

$$\begin{aligned}\delta &\ge \mathcal{J}_m(v_m) - \beta\langle\mathcal{J}'_m(v_m), v_m\rangle \\ &= \Big(\frac{1}{2}-\beta\Big)\big[||v_m||^2_{\mathbb{X}^s_{m,T}} - m^{2s}|v_m(\,\cdot\,,0)|^2_{L^2(0,T)^N}\big] \qquad (6.8)\\ &\quad + \int_{\partial^0\mathcal{S}_T}[\beta f(x,v_m)v_m - F(x,v_m)]\,dx \\ &\ge \int_{\partial^0\mathcal{S}_T}[\beta f(x,v_m)v_m - F(x,v_m)]\,dx \\ &\ge (\mu\beta-1)\int_{\partial^0\mathcal{S}_T}F(x,v_m)\,dx - \widetilde{\kappa} \qquad (6.9)\\ &\ge (\mu\beta-1)\big[a_3|v_m(\,\cdot\,,0)|^\mu_{L^\mu(0,T)^N} - a_4T^N\big] - \widetilde{\kappa} \\ &\ge (\mu\beta-1)\big[a_3|v_m(\,\cdot\,,0)|^\mu_{L^2(0,T)^N}T^{-N(\mu-2)/2} - a_4T^N\big] - \widetilde{\kappa}. \qquad (6.10)\end{aligned}$$

By (6.10), we deduce that the trace of v_m is bounded in $L^2(0,T)^N$

$$|v_m(\,\cdot\,,0)|_{L^2(0,T)^N} \leq K(\delta) \quad \text{for every } m \in (0,m_0). \tag{6.11}$$

Taking into account (6.1), (6.7), (6.9) and (6.11), we deduce

$$\begin{aligned} ||\nabla v_m||^2_{L^2(\mathcal{S}_T,y^{1-2s})} &\leq ||v_m||^2_{\mathbb{X}^s_{m,T}} \\ &= 2J_m(v_m) + m|v_m(\cdot,0)|^2_{L^2(0,T)^N} + 2\int_{\partial^0\mathcal{S}_T} F(x,v_m)\,dx \\ &\leq 2\delta + \frac{\omega}{2}\,K(\delta) + C(\delta) =: K'(\delta). \end{aligned} \tag{6.12}$$

Now, let c_k^m be the Fourier coefficients of the trace of v_m. By (3.7), we can see that

$$K'(\delta) \geq ||v_m||^2_{\mathbb{X}^s_{m,T}} \geq |v_m(\,\cdot\,,0)|^2_{\mathbb{H}^s_{m,T}} \geq \sum_{k\in\mathbb{Z}^N} \omega^{2s}|k|^{2s}|c_k^m|^2, \tag{6.13}$$

which, together with (6.11), implies that

$$|v_m(\,\cdot\,,0)|_{\mathbb{H}^s_T} \leq K''(\delta) \quad \text{for every } m \in (0,m_0), \tag{6.14}$$

that is $\mathrm{Tr}(v_m)$ is bounded in $\mathbb{H}^s_T$.

Finally, we estimate the $L^2_{\mathrm{loc}}(\mathcal{S}_T,y^{1-2s})$-norm of v_m uniformly in m. Fix $\alpha > 0$ and let $v \in C^\infty_T(\overline{\mathbb{R}^{N+1}_+})$ be such that $||v_m||_{\mathbb{X}^s_{m,T}} < \infty$. For any $x \in [0,T]^N$ and $y \in [0,\alpha]$, we have

$$v(x,y) = v(x,0) + \int_0^y \partial_y v(x,t)\,dt.$$

Due to $(a+b)^2 \leq 2a^2 + 2b^2$ for all $a,b \geq 0$, we obtain

$$|v(x,y)|^2 \leq 2|v(x,0)|^2 + 2\left(\int_0^y |\partial_y v(x,t)|\,dt\right)^2,$$

and, applying the Hölder inequality, we deduce

$$|v(x,y)|^2 \leq 2\left[|v(x,0)|^2 + \left(\int_0^y t^{1-2s}|\partial_y v(x,t)|^2\,dt\right)\frac{y^{2s}}{2s}\right]. \tag{6.15}$$

Multiplying both sides by y^{1-2s}, we have

$$y^{1-2s}|v(x,y)|^2 \leq 2\left[y^{1-2s}|v(x,0)|^2 + \left(\int_0^y t^{1-2s}|\partial_y v(x,t)|^2\,dt\right)\frac{y}{2s}\right]. \tag{6.16}$$

Integrating (6.16) over $(0,T)^N \times (0,\alpha)$, we have

$$\begin{aligned} &||v||^2_{L^2((0,T)^N\times(0,\alpha),y^{1-2s})} \\ &\qquad \leq \frac{\alpha^{2-2s}}{1-s}\,|v(\,\cdot\,,0)|^2_{L^2(0,T)^N} + \frac{\alpha^2}{2s}\,||\partial_y v||^2_{L^2(\mathcal{S}_T,y^{1-2s})}. \end{aligned} \tag{6.17}$$

By density, the above inequality holds for any $v \in \mathbb{X}^s_{m,T}$. Then, by (6.17), (6.11) and (6.12), for any $0 < m < m_0$, we have

$$\begin{aligned}||v_m||^2_{L^2((0,T)^N\times(0,\alpha),y^{1-2s})} &\leq \frac{\alpha^{2-2s}}{1-s}\,|v_m(\,\cdot\,,0)|^2_{L^2(0,T)^N} + \frac{\alpha^2}{2s}\,||\partial_y v_m||^2_{L^2(\mathcal{S}_T,y^{1-2s})} \\ &\leq C(\alpha,s)K(\delta)^2 + C'(\alpha,s)K'(\delta).\end{aligned}$$

As a consequence, we can extract a subsequence, that for simplicity we will denote again with (v_m), and a function v such that

- $v \in L^2_{\mathrm{loc}}(\mathcal{S}_T, y^{1-2s})$ and $\nabla v \in L^2(\mathcal{S}_T, y^{1-2s})$;
- $v_m \rightharpoonup v$ in $L^2_{\mathrm{loc}}(\mathcal{S}_T, y^{1-2s})$ as $m \to 0$;
- $\nabla v_m \rightharpoonup \nabla v$ in $L^2(\mathcal{S}_T, y^{1-2s})$ as $m \to 0$;
- $v_m(\,\cdot\,,0) \rightharpoonup v(\,\cdot\,,0)$ in $\mathbb{H}^s_T$ and $v_m(\,\cdot\,,0) \to v(\,\cdot\,,0)$ in $L^q(0,T)^N$ as $m \to 0$, for any $q \in [2, 2N/(N-2s))$.

Now we prove that v is a weak solution to

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathcal{S}_T := (0,T)^N \times (0,\infty), \\ v_{|\{x_i=0\}} = v_{|\{x_i=T\}} & \text{on } \partial_L\mathcal{S}_T := \partial(0,T)^N \times [0,\infty), \\ \dfrac{\partial v}{\partial \nu^{1-2s}} = f(x,v) & \text{on } \partial^0\mathcal{S}_T := (0,T)^N \times \{0\}. \end{cases} \tag{6.18}$$

Fix $\varphi \in \mathbb{X}^s_T$. We know that v_m satisfies

$$\iint_{\mathcal{S}_T} y^{1-2s}(\nabla v_m \nabla \eta + m^2 v_m \eta)\,dx\,dy = \int_{\partial^0\mathcal{S}_T} [m^{2s}v_m + f(x,v_m)]\eta\,dx \tag{6.19}$$

for every $\eta \in \mathbb{X}^s_{m,T}$. Now, we consider $\xi \in \mathcal{C}^\infty([0,\infty))$ defined as follows:

$$\begin{cases} \xi = 1 & \text{if } 0 \leq y \leq 1, \\ 0 \leq \xi \leq 1 & \text{if } 1 \leq y \leq 2, \\ \xi = 0 & \text{if } y \geq 2. \end{cases} \tag{6.20}$$

We set $\xi_R(y) = \xi(y/R)$ for $R > 1$. Then choosing $\eta = \varphi\xi_R \in \mathbb{X}^s_{m,T}$ in (6.19) and taking the limit as $m \to 0$, we have

$$\iint_{\mathcal{S}_T} y^{1-2s}\nabla v \nabla(\varphi\xi_R)\,dx\,dy = \int_{\partial^0\mathcal{S}_T} f(x,v)\varphi\,dx. \tag{6.21}$$

Taking the limit as $R \to \infty$, we deduce that v verifies

$$\iint_{\mathcal{S}_T} y^{1-2s}\nabla v \nabla \varphi\,dx\,dy - \int_{\partial^0\mathcal{S}_T} f(x,v)\varphi\,dx = 0 \quad \text{for all } \varphi \in \mathbb{X}^s_T.$$

Now let us prove that $v \not\equiv 0$. Let $\xi \in \mathcal{C}^\infty([0,\infty))$ as in (6.20), note that $\xi v \in \mathbb{X}^s_{m,T}$. Then

$$0 = \langle \mathcal{J}'_m(v_m), \xi v\rangle = \iint_{\mathcal{S}_T} y^{1-2s}(\nabla v_m \nabla(\xi v) + m^2 v_m \xi v)\,dx\,dy \\ - m^{2s}\int_{\partial^0\mathcal{S}_T} v_m v\,dx - \int_{\partial^0\mathcal{S}_T} f(x,v_m)v\,dx$$

and, taking the limit as $m \to 0$, we get

$$0 = \iint_{\mathcal{S}_T} y^{1-2s}\nabla v \nabla(\xi v)\,dx\,dy - \int_{\partial^0\mathcal{S}_T} f(x,v)v\,dx. \tag{6.22}$$

Due to (6.1), (6.7), $\langle \mathcal{J}'_m(v_m), v_m\rangle = 0$ and $F \geq 0$, we have

$$\begin{aligned} 2\lambda &\leq 2\mathcal{J}_m(v_m) + m^{2s}|v_m(\,\cdot\,,0)|^2_{L^2(0,T)^N} + 2\int_{\partial^0\mathcal{S}_T} F(x,v_m)\,dx \\ &= \|v_m\|^2_{\mathbb{X}^m_T} = m^{2s}|v_m(\,\cdot\,,0)|^2_{L^2(0,T)^N} + \int_{\partial^0\mathcal{S}_T} f(x,v_m)v_m\,dx. \end{aligned} \tag{6.23}$$

Taking the limit in (6.23) as $m \to 0$, we obtain

$$2\lambda \leq \int_{\partial^0\mathcal{S}_T} f(x,v)v\,dx. \tag{6.24}$$

Hence, (6.22) and (6.24) give

$$0 < 2\lambda \leq \int_{\partial^0\mathcal{S}_T} f(x,v)v\,dx = \iint_{\mathcal{S}_T} y^{1-2s}\nabla v \nabla(\xi v)\,dx\,dy,$$

that is v is not a trivial solution to (6.18).

Finally, we show that $v \in \mathcal{C}^{0,\alpha}([0,T]^N)$, for some $\alpha \in (0,1)$. We start proving that $v(\,\cdot\,,0) \in L^q(0,T)^N$ for any $q < \infty$. We proceed as in the proof of Lemma 5.1 and we use estimate (6.14). Let $w_m = v_m v^{2\beta}_{m,K}$, where $v_{m,K} = \min\{|v_m|, K\}$, $K > 1$ and $\beta \geq 0$. Then, replacing $vv^{2\beta}_K$ by $v_m v^{2\beta}_{m,K}$ in (5.5), we can see that

$$||v_m v^{\beta}_{m,K}||^2_{\mathbb{X}^s_{m,T}} \leq c_\beta \int_{(0,T)^N} \big[m^{2s} v^2_m v^{2\beta}_{m,K} + f(x,v_m)v_m v^{2\beta}_{m,K}\big]\,dx, \tag{6.25}$$

where $c_\beta = 1 + \beta/2 \geq 1$. Using Lemma 2.3 with $\varepsilon = 1$, we get

$$m^{2s} v^2_m v^{2\beta}_{m,K} + f(x,v_m)v_m v^{2\beta}_{m,K} \leq (m^{2s}+2) v^2_m v^{2\beta}_{m,K} + (p+1)C_1|v_m|^{p-1} v^2 v^{2\beta}_{m,K}.$$

Since v_m converges strongly in $L^{N(p-1)/(2s)}(0,T)^N$ (because of $N(p-1)/(2s) < 2^\sharp_s$), we can assume that, up to subsequences, there exists a function z in $L^{N(p-1)/(2s)}(0,T)^N$ such that $|v_m(x,0)| \leq z(x)$ in $(0,T)^N$ for every $m < m_0$. Therefore, there exist a constant $c = m_0^{2s} + 2 + (p+1)C_1$ and a function $h := 1 + z^{p-1} \in L^{N/(2s)}(0,T)^N$, $h \geq 0$ and independent of K, m and β such that

$$m^{2s} v^2_m v^{2\beta}_{m,K} + f(x,v_m)v_m v^{2\beta}_{m,K} \leq (c+h) v^2_m v^{2\beta}_{m,K} \quad \text{on } \partial^0\mathcal{S}_T. \tag{6.26}$$

As a consequence

$$\|v_m v_{m,K}^{\beta}\|_{\mathbb{X}^s_{m,T}}^2 \leq c_\beta \int_{(0,T)^N} (c+h) v_m^2 v_{m,K}^{2\beta}\, dx. \tag{6.27}$$

Taking the limit as $K \to \infty$ ($v_{m,K}$ is increasing with respect to K), we get

$$\||v_m|^{\beta+1}\|_{\mathbb{X}^s_{m,T}}^2 \leq c c_\beta \int_{(0,T)^N} |v_m|^{2(\beta+1)}\, dx + c_\beta \int_{(0,T)^N} h |v_m|^{2(\beta+1)}\, dx. \tag{6.28}$$

For any $M>0$, let $A_1=\{h\leq M\}$ and $A_2=\{h>M\}$. Then

$$\begin{aligned}\int_{(0,T)^N} h|v_m(\,\cdot\,,0)|^{2(\beta+1)}\, dx \\ \leq M\||v_m(\,\cdot\,,0)|^{\beta+1}|_{L^2(0,T)^N}^2 + \varepsilon(M)\||v_m(\,\cdot\,,0)|^{\beta+1}|^2_{L^{2^\sharp_s}(0,T)^N},\end{aligned} \tag{6.29}$$

where

$$\varepsilon(M) = \left(\int_{A_2} h^{N/(2s)}\, dx\right)^{2s/N} \to 0 \quad \text{as } M\to\infty.$$

Taking into account (6.28), (6.29), we have

$$\begin{aligned}&\||v_m|^{\beta+1}\|_{\mathbb{X}^s_{m,T}}^2 \\ &\leq c_\beta(c+M)\||v_m(\cdot,0)|^{\beta+1}|_{L^2(0,T)^N}^2 + c_\beta\varepsilon(M)\||v_m(\,\cdot\,,0)|^{\beta+1}|^2_{L^{2^\sharp}(0,T)^N}.\end{aligned} \tag{6.30}$$

Now, by (2.10), we know that for every $w\in\mathcal{C}_T^\infty(\mathbb{R}^N)$ with mean zero, there exists $\mu_0 := C_{2^\sharp_s} > 0$ such that

$$|w|_{L^{2^\sharp_s}(0,T)^N} \leq \mu_0 \bigg(\sum_{|k|\neq 0} \omega^{2s}|k|^{2s}|b_k|^2\bigg)^{1/2}, \tag{6.31}$$

where b_k are the Fourier coefficients of w. Therefore, if $w\in\mathcal{C}_T^\infty(\mathbb{R}^N)$ and

$$\overline{w} := \frac{1}{T^N}\int_{(0,T)^N} w(x)\, dx,$$

by the Hölder inequality, we have

$$\begin{aligned}|w|_{L^{2^\sharp_s}(0,T)^N} &\leq |w-\overline{w}|_{L^{2^\sharp_s}(0,T)^N} + |\overline{w}|_{L^{2^\sharp_s}(0,T)^N} \\ &\leq \mu_0\bigg(\sum_{|k|\neq 0}\omega^{2s}|k|^{2s}|b_k|^2\bigg)^{1/2} + |\overline{w}|_{L^{2^\sharp_s}(0,T)^N} \\ &\leq \mu_0\bigg(\sum_{|k|\neq 0}\omega^{2s}|k|^{2s}|b_k|^2\bigg)^{1/2} + \mu_1|w|^2_{L^2(0,T)^N} \\ &\leq \mu_0|w|^2_{\mathbb{H}^s_{m,T}} + \mu_1|w|^2_{L^2(0,T)^N},\end{aligned} \tag{6.32}$$

where $\mu_1 = T^{(N-2)/2} > 0$. Taking into account (6.30), (6.32) and (3.7), we deduce that

$$
\begin{aligned}
||v_m(\,\cdot\,,0)|^{\beta+1}|^2_{L^{2^\sharp_s}(0,T)^N} &- \mu_1||v_m(\,\cdot\,,0)|^{\beta+1}|^2_{L^2(0,T)^N} \\
\leq \mu_0||v_m(\,\cdot\,,0)|^{\beta+1}|^2_{\mathbb{H}^s_{m,T}} &\leq \mu_0|||v_m|^{\beta+1}||^2_{\mathbb{X}^s_{m,T}} \\
\leq \mu_0\big[c_\beta(c+M)||v_m(\,\cdot\,,0)|^{\beta+1}|^2_{L^2(0,T)^N} & \\
+ c_\beta\varepsilon(M)||v_m(\,\cdot\,,0)|^{\beta+1}|^2_{L^{2^\sharp_s}(0,T)^N}\big]. &
\end{aligned}
\tag{6.33}
$$

Choosing M large enough so that $c_\beta\mu_0\varepsilon(M) < 1/2$, by (6.33), we obtain

$$
||v_m(\,\cdot\,,0)|^{\beta+1}|^2_{L^{2^\sharp_s}(0,T)^N} \leq 2\big[\mu_0 c_\beta(c+M)+\mu_1\big]||v_m(\,\cdot\,,0)|^{\beta+1}|^2_{L^2(0,T)^N}. \tag{6.34}
$$

Let us notice that, by (6.14) and $\mathbb{H}^s_T \subset L^{2^\sharp_s}(0,T)^N$, we get

$$
|v_m(\,\cdot\,,0)|_{L^{2^\sharp_s}(0,T)^N} \leq K'''(\delta), \tag{6.35}
$$

for any $m < m_0$. By applying (6.34) with $\beta+1 = N/(N-2s)$ (that is $\beta = 2s/(N-2s)$) and by using (6.35), we have that

$$
||v_m|^{N/(N-2s)}|^2_{L^{2^\sharp_s}(0,T)^N} \leq 2\big[c_{2s/(N-2s)}\mu_0(c+M)+\mu_1\big]K'''(\delta)^{2N/(N-2s)},
$$

and taking the limit as $m \to 0$, we deduce $v(\,\cdot\,,0) \in L^{2N^2/(N-2s)^2}(0,T)^N$.

By (6.34), we find, after k iterations, that $v(\,\cdot\,,0)$ in $L^{2N^k/(N-2s)^k}(0,T)^N$ for all $k \in \mathbb{N}$. Then $v(\,\cdot\,,0) \in L^q(0,T)^N$ for all $q \in [2,\infty)$, and by invoking Proposition 3.5 in [18], we conclude that $v \in \mathcal{C}^{0,\alpha}([0,T]^N)$, for some $\alpha \in (0,1)$.

References

[1] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[2] V. Ambrosio, *Existence of heteroclinic solutions for a pseudo-relativistic Allen–Cahn type equation*, Adv. Nonlinear Stud. **15** (2015), 395–414.

[3] ———, *Periodic solutions for a pseudo-relativistic Schrödinger equation*, Nonlinear Anal. **120** (2015), 262–284.

[4] D. Applebaum, *Lévy processes and stochastic calculus*, Cambridge Stud. Adv. Math. **93** (2004).

[5] A. Bènyi and T. Oh, *The Sobolev inequality on the torus revisited*, Publ. Math. Debrecen **83** (2013), no. 3, 359–374.

[6] P. Biler, G. Karch and W.A. Woyczynski, *Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws*, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), 613–637.

[7] X. Cabré and Y. Sire, *Nonlinear equations for fractional Laplacians* I: *regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), 23–53.

[8] ———, *Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions*, Trans. Amer. Math. Soc. **367** (2015), 911–941.

[9] X. Cabré and J. Solà-Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. **58** (2005), 1678–1732.

[10] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. **224** (2010), 2052–2093.

[11] L.A. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260.

[12] L. Caffarelli and A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. **2** 171 (2010), no. 3, 1903–1930.

[13] A. Capella, J. Dávila, L. Dupaigne and Y. Sire, *Regularity of radial extremals solutions for some non-local semilinear equation*, Comm. Partial Differential Equations **36** (2011), 1353–1384.

[14] R. Carmona, W.C. Masters and B. Simon, *Relativistic Schrödinger operators; Asymptotic behaviour of the eigenfunctions*, J. Funct. Anal. **91** (1990), 117–142.

[15] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman and Hall/CRC Financ. Math. Ser., Chapman and Hall/CRC, Boca Raton, FL (2004).

[16] G. Duvaut and J.L. Lions, *Inequalities in Mechanics and Physics*, Grundlehren Math. Wiss., vol. 219, Springer–Verlag, Berlin, 1976. Transl. from French by C.W. John.

[17] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Higher Trascendental Functions*, vol. 1,2 McGraw-Hill, New York (1953).

[18] M.M. Fall and V. Felli, *Unique continuation properties for relativistic Schrödinger operators with a singular potential*, DCDS A. http://arxiv.org/abs/1312.6516 (to appear).

[19] J. Fröhlich, B.L.G. Jonsson and E. Lenzmann, *Boson stars as solitary waves*, Comm. Math. Phys. **274** (1), 1–30.

[20] E.H. Lieb and M. Loss, *Analysis*, 2nd Edition. Vol. 14 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI. 33.

[21] E.H. Lieb and H.T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Comm. Math. Phys. **112** (1987), 147–174.

[22] ———, *The stability and instability of relativistic matter*, Comm. Math. Phys. 118 (2), (1988) 177–213.

[23] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, Springer–Verlag, New York **74** (1989).

[24] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics **65** (1986).

[25] M. Ryznar, *Estimate of Green function for relativistic α-stable processes*, Potential Analysis **17** (2002), 1–23.

[26] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2006), 67–112.

[27] Y. Sire and E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. **256** (2009), 1842–1864.

[28] J.J. Stoker, *Water Waves: The Mathematical Theory with Applications*, Pure Appl. Math., vol. IV, Interscience Publishers, Inc., New York, (1957).

[29] M. Struwe, *Variational methods: Application to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer–Verlag, Berlin, (1990).

[30] J.F. Toland, *The Peierls–Nabarro and Benjamin–Ono equations*, J. Funct. Anal. **145** (1997), 136–150.

[31] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications **24** (1996).

[32] A. ZYGMUND, *Trigonometric Series*, Vol. 1, 2, Cambridge University Press, Cambridge (2002).

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