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ATTRACTORS FOR SECOND ORDER NONAUTONOMOUS LATTICE SYSTEM WITH DISPERSIVE TERM

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Abstract. In this paper, we prove the existence of pullback attractor, pullback exponential attractor and uniform attractor for second order non-autonomous lattice system with dispersive term and time-dependent forces. Then we prove the existence of uniform exponential attractor for the system driven by quasi-periodic external forces.

1. Introduction

It is known that there are two important tools to study the asymptotic behavior of non-autonomous evolution equations (see [10], [6], [15]). The first one is the uniform attractor by introducing a so-called skew-product semiflows on a larger phase space, which allows one to embed a given non-autonomous system into an autonomous semiflow, and then to appeal to the general theory of autonomous semiflows. The second one is the pullback attractor (or kernel sections) directly for non-autonomous equations. The uniform attractor and pullback attractor are natural generalizations of the notion of global attractor for a non-autonomous dynamical system. However, these two attractors are usually infinite dimensional and sometimes attract orbits at a relatively slow

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speed leading to take an unexpected long time to be reached, thus the uniform exponential attractor and the pullback exponential attractor having finite fractal dimension and attracting all bounded sets exponentially were introduced, and they have become appropriate alternatives to study the asymptotic behavior of non-autonomous dynamical systems.

Recently, there are several works about the existence of uniform attractor, pullback attractor, kernel sections, pullback and uniform exponential attractor for non-autonomous lattice dynamical systems (LDSs), which arise in many applied areas, see [1]–[4], [9], [17], [21], [22], [24], and the references therein. Of those works, Zhou and Han in [21] and [22] presented some sufficient conditions for the existence of pullback and uniform exponential attractor for the continuous process on Banach space and space of infinite sequences, and applied them to prove the existence of pullback exponential attractors for the first order and partly dissipative non-autonomous LDSs and uniform exponential attractors for the non-autonomous Klein–Gordon–Schrödinger and Zakharov LDSs. Motivated by [21], [22], in this article, we consider the pullback attractor, pullback exponential attractor, uniform attractor for the following second order non-autonomous lattice dynamical system with dispersive term and time-dependent forces

$$\ddot{u}_i + \beta(A\ddot{u})_i + \alpha\dot{u}_i + \gamma(A\dot{u})_i + \lambda u_i + (Au)_i + f_i(u_i) = g_i(t), \tag{1.1}$$

and the uniform exponential attractor for second order non-autonomous lattice system driven by quasi-periodic external forces:

$$\ddot{u}_i + \beta(A\ddot{u})_i + \alpha\dot{u}_i + \gamma(A\dot{u})_i + \lambda u_i + (Au)_i + f_i(u_i) = a_i h_i(\widetilde{\sigma}(t)), \tag{1.2}$$

where $i \in \mathbb{Z}$, $t \geq \tau$, $\tau \in \mathbb{R}$, $u = (u_i)_{i\in\mathbb{Z}}$, $u_i \in \mathbb{R}$, $\beta \geq 0$, $\gamma > 0$, $\alpha > 0$, $\lambda > 0$, $a_i \in \mathbb{R}$, $g_i \in C(\mathbb{R},\mathbb{R})$, h_i, $f_i \in C^1(\mathbb{R},\mathbb{R})$, $\widetilde{\sigma}(t) \in \mathbf{T}^n$ (n-dimensional torus), A is a non-negative and self-adjoint linear operator with the decomposition $A = \overline{D}D = D\overline{D}$, and D is defined by

$$(Du)_i = \sum_{l=-m_0}^{l=m_0} d_l u_{i+l}, \quad |d_l| \leq c_0, \quad \text{for all } u = (u_i)_{i\in\mathbb{Z}},\ -m_0 \leq l \leq m_0, \tag{1.3}$$

and $\overline{D}$ is the adjoint of D. If A is defined by $(Au)_i = 2u_i - u_{i-1} - u_{i+1}$, then (1.1) can be regarded as a discrete analogue of the following continuous fourth order partial differential equation in $\mathbb{R}$:

$$u_{tt} - \beta u_{xxtt} + \alpha u_t - \gamma u_{xxt} + \lambda u - u_{xx} + f(u,x) = g(x,t), \tag{1.4}$$

which is a mathematical model for describing the spread of longitudinal strain waves in nonlinear elastic rods and weakly nonlinear ion-acoustic waves; see, e.g. [5], [11] and the references therein. The terms $-\beta u_{xxtt}$ and $-\gamma u_{xxt}$ are called the dispersive and the viscosity dissipative terms, respectively. In the autonomous case (i.e. g is independent of t) defined in a bounded domain and the stochastic equation driven by additive noise defined on the unbounded domain, the well-posedness and the existence of attractors of (1.4) have been studied by [7], [8], [13] and the references therein.

For the second order autonomous and non-autonomous lattice systems (1.1) without dispersive term (i.e. $\beta = 0$), Abdallah, Fan, Zhao and Zhou *et al.* have investigated the existence and finite-dimensionality of their global attractor and kernel sections, see [2], [3], [12], [14], [16]–[20], [23]. Here, by following the ideas of [21], [22], we consider the existence of pullback exponential attractor for system (1.1) and the existence of uniform exponential attractor for system (1.2) with $\beta \geq 0$.

The paper is organized as follows. In Section 2, we prove the existence of pullback attractor of system (1.1). In Section 3, we prove the existence of pullback exponential attractor for the system (1.1). In Section 4, we present the existence of uniform attractor for the system (1.1). In Section 5, we prove the existence of uniform exponential attractor for the system (1.2).

2. Pullback attractor

In this section, we consider the existence of pullback attractor of system (1.1) with $\beta > 0$. Note that system (1.1) with initial data can be written as a vector form

$$\begin{cases} \ddot{u} + \beta A\ddot{u} + \alpha\dot{u} + \gamma A\dot{u} + \lambda u + Au + f(u) = g(t), & t > \tau, \\ u(\tau) = (u_{i,\tau})_{i\in\mathbb{Z}} = u_\tau, \quad \dot{u}(\tau) = (u_{1i,\tau})_{i\in\mathbb{Z}} = u_{1\tau}, \end{cases} \quad \tau \in \mathbb{R}, \tag{2.1}$$

where $u = (u_i)_{i\in\mathbb{Z}}$, $A\ddot{u} = ((A\ddot{u})_i)_{i\in\mathbb{Z}}$, $A\dot{u} = ((A\dot{u})_i)_{i\in\mathbb{Z}}$, $Au = (Au_i)_{i\in\mathbb{Z}}$, $f(u) = (f_i(u_i))_{i\in\mathbb{Z}}$, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$. Let

$$l^2 = \left\{ u = (u_i)_{i\in\mathbf{Z}} : \sum_{i\in\mathbf{Z}} {u_i}^2 < \infty,\ u_i \in \mathbb{R} \right\}$$

be a Hilbert space with inner product $(u, v) = \sum_{i\in\mathbb{Z}} u_i v_i$ and norm

$$||u||^2 = (u, u) = \sum_{i\in\mathbb{Z}} u_i^2 \quad \text{for } u = (u_i)_{i\in\mathbb{Z}},\ v = (v_i)_{i\in\mathbf{Z}} \in l^2.$$

Let

$$v = \dot{u} + \varepsilon u, \tag{2.2}$$

where ε is a small positive constant such that

$$\min\{\alpha - 2\varepsilon, \gamma - \beta\varepsilon, \lambda - \alpha\varepsilon + \varepsilon^2, 1 - \gamma\varepsilon + \beta\varepsilon^2\} > 0,$$

then

$$\begin{split} (I + \beta A)\dot{v} + (\alpha - \varepsilon)v + (\lambda - \alpha\varepsilon + \varepsilon^2)u \\ + (1 - \gamma\varepsilon + \beta\varepsilon^2)Au + (\gamma - \beta\varepsilon)Av + f(u) = g(t). \end{split} \tag{2.3}$$

By the semi-positivity of operator A on l^2, we know that the operator $(I+\beta A)^{-1}$ exists, and $(I+\beta A)^{-1}$ is linear and bounded: $||(I+\beta A)^{-1}|| \leq 1$. Let $E = l^2 \times l^2$, a Hilbert space with usual inner product and norm. The system (2.1) is equivalent to the following first order evolution equation in E:

$$\dot{\varphi} + C(\varphi) = F(\varphi, t), \varphi(\tau) = \begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = \begin{pmatrix} cu_\tau \\ u_{1\tau} + \varepsilon u_\tau \end{pmatrix}, \quad t > \tau,\ \tau \in \mathbb{R}, \tag{2.4}$$

where

$$\varphi = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad F(\varphi, t) = \begin{pmatrix} 0 \\ (I+\beta A)^{-1}[-f(u) + g(t)] \end{pmatrix},$$

$$C(\varphi) = \begin{pmatrix} \varepsilon u - v \\ (I+\beta A)^{-1}[(\alpha-\varepsilon)v + (\lambda - \alpha\varepsilon + \varepsilon^2)u + (1-\gamma\varepsilon+\beta\varepsilon^2)Au + (\gamma - \beta\varepsilon)Av] \end{pmatrix}.$$

We make the following assumptions on g_i, f_i:

(H1) $g(t) = (g_i(t))_{i\in\mathbb{Z}} \in \mathbf{G}$, where

$$\mathbf{G} = \Big\{ g \in C_b(\mathbb{R}, l^2) : \text{for every } \eta > 0, \text{ there exists } I(\eta) \in \mathbb{N} \text{ such that } \sup_{t\in\mathbb{R}} \sum_{|i|>I(\eta)} g_i^2(t) < \eta \Big\},$$

and $C_b(\mathbb{R}, l^2)$ denotes the space of all continuous bounded functions from $\mathbb{R}$ into l^2, $|||g||| \doteq \sup_{t\in\mathbb{R}} ||g(t)||$;

(H2) For any $i \in \mathbb{Z}$, f_i satisfies:

(H21) $f_i \in C^1(\mathbb{R};\mathbb{R})$;

(H22) let $G_i(s) = \int_0^s f_i(r)\,dr$, $f_i(s)s \geq \nu G_i(s) \geq 0$, for all $s \in \mathbb{R}$ and for some small positive constant $\nu > 0$;

(H23) there exist a function $K \in C(\mathbb{R}_+;\mathbb{R}_+)$ and a $I_0 \in \mathbb{N}$ such that

$$\sup_{|i|>I_0} \max_{s\in[-r,r]} |f_i'(s)| \leq K(r), \quad \text{for all } r \in \mathbb{R}_+.$$

Under assumptions (H1)–(H2), we see that for any $(\varphi, t) \in E \times \mathbb{R}$, $F(\varphi, t) - C(\varphi) \in E$ and $F(\varphi, t) - C(\varphi)$ is locally Lipschitz from $E \times \mathbb{R}$ into E with respect to φ for t in compact sets of $\mathbb{R}$. Thus, for any $\varphi(\tau) = (u_\tau, v_\tau)^T \in E$, there exists a unique local solution $\varphi(t,\tau) = (u(t,\tau), v(t,\tau))^T$ of (2.4) such that $\varphi(\,\cdot\,,\tau) \in C([\tau, \tau+T_0), E) \cap C^1((\tau, \tau+T_{\max}), E)$ for some $T_{\max} > 0$. Moreover, if $T_{\max} < +\infty$, then $\lim_{t\to T_{\max}} ||\varphi(t,\tau)||_E = +\infty$. It follows from Lemma 2.1 below that the solution $\varphi(t,\tau)$ of (2.4) is bounded for all $t \geq \tau$ if $\varphi(\tau)$ belongs to a bounded set, thus the local solution $\varphi(t,\tau)$ of (2.4) exists globally in $[\tau, +\infty)$, that is, $\varphi(\,\cdot\,,\tau) \in C([\tau,+\infty),E) \cap C^1((\tau,+\infty),E)$, implying that the solution maps

$$W(t,\tau)\colon \varphi(\tau) = (u_\tau, v_\tau) \to \varphi(t,\tau) = (u(t,\tau), v(t,\tau)), \quad t \geq \tau,\ \tau \in \mathbb{R},$$

generate a continuous process $\{W(t,\tau)\}_{t\geq\tau}$ on E.

First, we prove the existence of a uniform absorbing set of $\{W(t,\tau)\}_{t\geq\tau}$.

LEMMA 2.1. *The process $\{W(t,\tau)\}_{t\geq\tau}$ has a uniform absorbing set $B_0 = B_0(0,r_0) = \{\varphi \in E : ||\varphi||_E \leq r_0\} \subset E$ such that for any bounded subset $B \subset E$, there exists $T_B \geq 0$ yielding $W(t+\tau,\tau)B \subseteq B_0$ for all $t \geq T_B$, where*

$$r_0 = \sqrt{\frac{2}{\mu\sigma_0\alpha}}|||g|||, \qquad \mu = \min\{1, \lambda - \alpha\varepsilon + \varepsilon^2, \beta, 1 - \gamma\varepsilon + \beta\varepsilon^2\},$$

$$\sigma_0 = \min\left\{\alpha - 2\varepsilon, \frac{2(\gamma - \beta\varepsilon)}{\beta}, 2\varepsilon, \varepsilon\nu\right\}.$$

Particularly, $\bigcup_{t\in\mathbb{R}} W(t,t-\tau)B_0 \subseteq B_0$ for any $t \in \mathbb{R}$ and $\tau \geq T_{B_0}$.

PROOF. Let $\varphi(t,\tau) = (u(t,\tau), v(t,\tau))^T \in E$ be a solution of (2.4). By computation, we have

$$(u,v) = \frac{1}{2}\frac{d}{dt}||u||^2 + \varepsilon||u||^2, \quad (Au,v) = \frac{1}{2}\frac{d}{dt}||Du||^2 + \varepsilon||Du||^2,$$

$$(f(u),v) \geq \frac{d}{dt}\sum_{i\in\mathbb{Z}} G_i(u_i) + \varepsilon\nu\sum_{i\in\mathbb{Z}} G_i(u_i),$$

$$(g(t),v) \leq \frac{1}{2\alpha}|||g|||^2 + \frac{\alpha}{2}||v||^2.$$

Taking the inner product $(\,\cdot\,,\,\cdot\,)$ of (2.3) with $v(t,\tau)$, we have

$$\frac{d}{dt}y + \sigma_0 y \leq \frac{1}{\alpha}|||g|||^2, \quad \text{for all } t \geq \tau, \tag{2.5}$$

where

$$y = ||v||^2 + \beta||Dv||^2 + (\lambda - \alpha\varepsilon + \varepsilon^2)||u||^2 + (1 - \gamma\varepsilon + \beta\varepsilon^2)||Du||^2 + 2\sum_{i\in\mathbb{Z}} G_i(u_i).$$

Then applying Gronwall inequality to (2.5), we have

$$y(t) \leq y(\tau)e^{-\sigma_0(t-\tau)} + \frac{1}{\sigma_0\alpha}|||g|||^2, \quad \text{for all } t \geq \tau.$$

By (1.3) and (H22)–(H23),

$$||Du(\tau)||^2 = \sum_{i\in\mathbb{Z}}(Du)_i^2(\tau) \leq (2m_0+1)^2 c_0^2 ||u_\tau||^2,$$

$$\begin{aligned}\sum_{i\in\mathbb{Z}} G_i(u_i(\tau)) &\le \frac{1}{\nu}\sum_{i\in\mathbb{Z}} \max_{s\in[-||u_\tau||,||u_\tau||]} |f_i'(s)|\cdot {u_{i\tau}}^2 \\ &\le \frac{1}{\nu}\widetilde{K}(||u_\tau||)\cdot ||u_\tau||^2,\end{aligned}$$

where

$$\widetilde{K}(||u_\tau||) = \max\bigg\{\max_{|i|\le I_0, s\in[-||u_\tau||,||u_\tau||]} |f_i'(s)|, K(||u_\tau||)\bigg\}.$$

Thus,

$$\begin{aligned}y(t) \le{}& (1+\beta\widetilde{C}_0+\lambda-\alpha\varepsilon+\varepsilon^2)||v_\tau||^2 e^{-\sigma_0(t-\tau)} \\ &+ \bigg((1-\gamma\varepsilon+\beta\varepsilon^2)\widetilde{C}_0 + \frac{2}{\nu}\widetilde{K}(||u_\tau||)\bigg)||u_\tau||^2 e^{-\sigma_0(t-\tau)} + \frac{1}{\sigma_0\alpha}|||g|||^2,\end{aligned} \tag{2.6}$$

for all $t\ge\tau$, where $\widetilde{C}_0 = (2m_0+1)^2 c_0^2$. It follows that the ball $B_0 = \{\varphi\in E : ||\varphi||_E \le r_0\}$ is a uniform absorbing set for the process $\{W(t,\tau)\}_{t\ge\tau}$ on E. $\square$

Next we consider the "end" estimate of solutions of (2.4).

LEMMA 2.2. *For any $\eta>0$, there exist $T_0(\eta,B_0)>0$ and $N_0(\eta,B_0)$ such that the solution $\varphi(t+\tau,\tau) = (\varphi_i(t+\tau,\tau))_{i\in\mathbb{Z}} = (u_i(t+\tau,\tau), v_i(t+\tau,\tau))_{i\in\mathbb{Z}} \in E$ of system* (2.4) *with $\varphi(\tau)\in B_0$ satisfies*

$$\sum_{|i|>N_0(\eta,B_0)} ||\varphi_i(t+\tau,\tau)||_E^2 = \sum_{|i|>N_0(\eta,B_0)} (u_i^2(t+\tau,\tau)+v_i^2(t+\tau,\tau)) \le \eta,$$

for $t\ge T_0(\eta,B_0)$.

PROOF. Let $\tau\in\mathbb{R}$,

$$\begin{aligned}\varphi(t) &= \varphi(t+\tau,\tau) = W(t+\tau,\tau)\varphi(\tau) \\ &= (u(t+\tau,\tau), v(t+\tau,\tau)) = (u(t),v(t))\in E,\end{aligned}$$

for $t\ge0$, be a solution of (2.4) with $\varphi(\tau)\in B_0$. By (2.6),

$$\begin{aligned}&||v(t)||^2+\beta||Dv(t)||^2+(\lambda-\alpha\varepsilon+\varepsilon^2)||u(t)||^2 \\ &\qquad + (1-\gamma\varepsilon+\beta\varepsilon^2)||Du(t)||^2 \le \mu r_0^2 \le r_0^2,\end{aligned} \tag{2.7}$$

for all $t\ge T_{B_0}$. Taking the inner product $(\,\cdot\,,\,\cdot\,)$ of (2.2) and (2.3) with $\dot u(t,\tau)$, $\dot v(t,\tau)$, respectively, we have

$$||\dot u||^2 \le 2||v||^2+2\varepsilon^2||u||^2 \le 2(1+\varepsilon^2)r_0^2 \doteq 2r_1^2, \quad \text{for all } t\ge\tau+T_{B_0}. \tag{2.8}$$

and, for all $t\ge T_{B_0}$,

$$\begin{aligned}\frac{1}{2}||\dot v||^2+\frac{1}{2}\beta||D\dot v||^2 \le{}& \bigg(2(\alpha-\varepsilon+\lambda-\alpha\varepsilon+\varepsilon^2) \\ &+ \frac{(1-\gamma\varepsilon+\beta\varepsilon^2)+(\gamma-\beta\varepsilon)}{\beta} + 2\widetilde{K}^2(r_0)\bigg)r_0^2 + 2|||g|||^2 \doteq r_2^2.\end{aligned} \tag{2.9}$$

Choosing a smooth increasing function $\theta \in C^1(\mathbb{R}_+, \mathbb{R})$ satisfies:

$$\begin{cases} \theta(s) = 0 & \text{for } 0 \le s \le 1, \\ 0 \le \theta(s) \le 1 & \text{for } 1 \le s \le 2, \\ \theta(s) = 1 & \text{for } s \ge 2, \\ |\theta'(s)| \le M_0 & \text{for } s \ge 0, \end{cases} \tag{2.10}$$

where $M_0 > 0$ is a constant. Let M be a fixed integer. Set $w_i = \theta(|i|/M)u_i$, $z_i = \theta(|i|/M)v_i$, for all $i \in \mathbb{Z}$, $y = (w, z) = ((w_i), (z_i))_{i\in\mathbb{Z}}$. Here we have

$$\begin{aligned} &\left|(Dz)_i - \theta\left(\frac{|i|}{M}\right)(Dv)_i\right| = \left|\sum_{l=-m_0}^{l=m_0} d_l z_{i+l} - \theta\left(\frac{|i|}{M}\right)\sum_{l=-m_0}^{l=m_0} d_l v_{i+l}\right| \\ &\qquad = \sum_{l=-m_0}^{l=m_0}\left|\left(\theta\left(\frac{|i+l|}{M}\right) - \theta\left(\frac{|i|}{M}\right)\right)d_l v_{i+l}\right| \le \frac{M_0 m_0 c_0}{M}\sum_{l=-m_0}^{l=m_0}|v_{i+l}|. \end{aligned} \tag{2.11}$$

Thus, by (2.7)–(2.9) and (2.11), for $t \ge T_{B_0}$,

$$\begin{aligned} &\left|\sum_{i\in\mathbb{Z}}(D\dot v)_i\left((Dz)_i - \theta\left(\frac{|i|}{M}\right)(Dv)_i\right)\right| \\ &\qquad \le \frac{M_0 m_0 c_0}{M}\sum_{i\in\mathbb{Z}}\left(\sum_{l=-m_0}^{l=m_0}|d_l \dot v_{i+l}|\sum_{l=-m_0}^{l=m_0}|v_{i+l}|\right) \\ &\qquad \le \frac{M_0 m_0 (2m_0+1)^2 c_0^2}{2M}\sum_{i\in\mathbb{Z}}(\dot v_i^2 + v_i^2) \le \frac{M_0 m_0 \widetilde{C}_0}{2M}(2r_2^2 + r_0^2). \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{i\in\mathbb{Z}}(Du)_i\left[(D\dot w)_i - \theta\left(\frac{|i|}{M}\right)(D\dot u)_i\right] &\le \frac{M_0 m_0 (2m_0+1)^2 c_0^2}{2M}\sum_{i\in\mathbb{Z}}(u_i^2 + \dot u_i^2) \\ &\le \frac{M_0 m_0 \widetilde{C}_0}{2M}(2r_1^2 + r_0^2), \end{aligned}$$

for $t \ge T_{B_0}$. Thus,

$$\begin{aligned} (A\dot v, z) &= \sum_{i\in\mathbb{Z}}(D\dot v)_i\left[\theta\left(\frac{|i|}{M}\right)(Dv)_i + (Dz)_i - \theta\left(\frac{|i|}{M}\right)(Dv)_i\right] \\ &\ge \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)(Dv)_i^2 - \frac{M_0 m_0 \widetilde{C}_0}{2M}(2r_2^2 + r_0^2), \\ (Av, z) &\ge \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)(Dv)_i^2 - \frac{M_0 m_0 \widetilde{C}_0}{M}r_0^2, \\ (Au, \dot w) &\ge \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)(Du)_i^2 - \frac{M_0 m_0 \widetilde{C}_0}{2M}(2r_1^2 + r_0^2), \end{aligned}$$

$$
\begin{aligned}
(Au,w) &\geq \sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)(Du)_i^2 - \frac{M_0 m_0 \widetilde{C}_0}{M} r_0^2,\\
(Au,z) &= (Au,\dot{w}+\varepsilon w)\\
&\geq \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)(Du)_i^2 + \sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)(Du)_i^2 - \frac{M_0 m_0 \widetilde{C}_0}{2M}(2r_1^2+3r_0^2),\\
(u,z) &= \sum_{i\in\mathbb{Z}} u_i \theta\bigg(\frac{|i|}{M}\bigg)(\dot{u}_i+\varepsilon u_i) = \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)u_i^2 + \varepsilon\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)u_i^2,\\
(f(u),z) &\geq \frac{d}{dt}\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)G_i(u_i) + \varepsilon\nu\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)G_i(u_i),\\
(g(t),z) &\leq \frac{\alpha}{2}\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)v_i^2 + \frac{1}{2\alpha}\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)g_i^2(t).
\end{aligned}
$$

Taking the inner product $(\,\cdot\,,\,\cdot\,)$ of (2.3) with $z(t,\tau)$, we have that, for $t \geq T_{B_0}$,

$$
\begin{aligned}
&\frac{d}{dt}\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)(v_i^2+\beta(Dv)_i^2+(\lambda-\alpha\varepsilon+\varepsilon^2)u_i^2)\\
&\quad+\frac{d}{dt}\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)((1-\gamma\varepsilon+\beta\varepsilon^2)(Du)_i^2+2G_i(u_i))\\
&\quad+\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)((\alpha-2\varepsilon)v_i^2+2(\gamma-\beta\varepsilon)(Dv)_i^2+2\varepsilon(\lambda-\alpha\varepsilon+\varepsilon^2)u_i^2)\\
&\quad+\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)(2\varepsilon(1-\gamma\varepsilon+\beta\varepsilon^2)(Du)_i^2+2\varepsilon\nu G_i(u_i))\\
&\leq \frac{1}{\alpha}\sum_{|i|\geq M} g_i^2(t) + \frac{J_0}{M},
\end{aligned}
\tag{2.12}
$$

where

$$
\begin{aligned}
J_0 = (\beta(2r_2^2+r_0^2)+4(\gamma-\beta\varepsilon)r_0^2)M_0 m_0 \widetilde{C}_0&\\
+(1-\gamma\varepsilon+\beta\varepsilon^2)(2r_1^2+r_0^2+2\varepsilon r_0^2)M_0 m_0 \widetilde{C}&.
\end{aligned}
$$

Since $g(t)=(g_i(t))_{i\in\mathbb{Z}}\in\mathbf{G}$, by the definition of $\mathbf{G}$, we have that for all $\eta>0$, there exists $N_{00}(\eta,B_0)$ such that

$$
\frac{1}{\alpha}\sup_{t\in\mathbb{R}}\sum_{|i|\geq M} g_i^2(t) + \frac{J_0}{M} \leq \frac{\mu\sigma_0}{2}\eta, \quad \text{for all } M\geq N_{00}(\eta,B_0).
$$

Write

$$
\begin{aligned}
y_M(t) = \sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)(v_i^2+\beta(Dv)_i^2+(\lambda-\alpha\varepsilon+\varepsilon^2)u_i^2)&\\
+\sum_{i\in\mathbb{Z}} \theta\bigg(\frac{|i|}{M}\bigg)((1-\gamma\varepsilon+\beta\varepsilon^2)(Du)_i^2+2G_i(u_i))&.
\end{aligned}
$$

Then by (2.12),

$$\frac{d}{dt}y_M(t) + \sigma_0 y_M(t) \le \frac{\sigma_0\mu\eta}{2}, \quad \text{for all } M \ge N_{00}(\eta, B_0),\ t \ge T_{B_0}. \tag{2.13}$$

By the Gronwall inequality to (2.13) on $[\tau + T_{B_0}, \tau + t]$ $(t \ge T_{B_0})$, we have that for $M \ge \max\{I_0, N_{00}(\eta, B_0)\}$, $t \ge T_{B_0}$,

$$\begin{aligned} y_M(t) &= \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)(v_i^2 + \beta(Dv)_i^2 + (\lambda - \alpha\varepsilon + \varepsilon^2)u_i^2) \\ &\quad + (1 - \gamma\varepsilon + \beta\varepsilon^2)\sum_{i\in\mathbf{Z}} \theta\left(\frac{|i|}{M}\right)(Du)_i^2 + 2G_i(u_i) \\ &\le y(T_{B_0})e^{-\sigma_0(t-T_{B_0})} + \frac{\mu\eta}{2} \le K_1(r_0)r_0^2 e^{-\sigma(t-T_{B_0})} + \frac{\mu\eta}{2}, \end{aligned}$$

where

$$K_1(r_0) = 1 + \beta(2m_0+1)^2 c_0^2 + (\lambda - \alpha\varepsilon + \varepsilon^2) + (1 - \gamma\varepsilon + \beta\varepsilon^2)\widetilde{C}_0 + \frac{2}{\nu}K(||r_0||).$$

Thus there exist $T_0(\eta, B_0) > T_{B_0} > 0$ and $N_0(\eta, B_0) = 2\max\{I_0, N_{00}(\eta, B_0)\} \in \mathbb{N}$ such that for $t \ge T_0(\eta, B_0)$ and $M \ge N_0(\eta, B_0)$, we have

$$\sum_{|i|>N_0(\eta,B_0)} (u_i^2(t) + v_i^2(t)) \le \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)(u_i^2(t) + v_i^2(t)) \le \eta$$

for all $t \ge T_0(\eta, B_0)$. □

As a direct consequence of Lemmas 2.1, 2.2 and Theorem 4.2 of [16], we obtain the existence of a pullback attractor for the process $\{W(t,\tau)\}_{t\ge\tau}$.

THEOREM 2.3. *$\{W(t,\tau)\}_{t\ge\tau}$ possesses a pullback attractor $\{\mathcal{K}(t)\}_{t\in\mathbb{R}}$ with properties*:

(a) *for $t \in \mathbb{R}$, $\mathcal{K}(t) = \bigcap_{s\ge 0} \overline{\bigcup_{\tau\ge s} W(t, t-\tau)B_0}$ $(\subseteq B_0)$ is compact*;
(b) (invariance) *$W(t,\tau)\mathcal{K}(\tau) = \mathcal{K}(t)$ for all $-\infty < \tau \le t < \infty$*;
(c) (pullback attraction) *for any bounded set $B \subset E$ and $t \in \mathbb{R}$,*

$$\lim_{\tau\to+\infty} \mathrm{d}_E(U(t, t-\tau)B, \mathcal{K}(t)) = 0,$$

where d_E denotes the Hausdorff semi-distance between two sets of E.

3. Pullback exponential attractor

It is known from Theorem 2.3 that the process $\{W(t,\tau)\}_{t\ge\tau}$ possesses a pullback attractor $\{\mathcal{K}(t)\}_{t\in\mathbb{R}}$. Here a natural question is that whether these sets $\mathcal{K}(t)$, $t \in \mathbb{R}$ are finite dimensional or not? How are the speed of their attracting orbits? The infinite dimensionality and relatively slow attracting speed of them will make us difficult in practical applications and numerical simulations. For this reason, we consider the existence of a pullback exponential attractor for system (2.4) which has finite fractal dimension and attracts all bounded sets exponentially, moreover, includes the pullback attractor.

In this section, we assume that (H1)–(H2) hold and make a further assumption on the function K:

(H3) $K(0) = 0$, where K is defined in (H23) of Section 2.

For any $t \in \mathbb{R}$, set

$$Y(t) = \overline{\bigcup_{\tau \geq T_{B_0}} W(t, t-\tau) B_0} \subseteq B_0,$$

where B_0 is the uniform absorbing set in Lemma 2.1, then

$$W(t,\tau)Y(\tau) \subseteq Y(t) \subseteq B_0, \quad \text{for all } t \geq \tau.$$

Given any $\tau \in \mathbb{R}$ and any initial data $\varphi_\tau = (u_\tau, v_\tau)$, $\psi_\tau = (x_\tau, y_\tau) \in Y(\tau)$, let

$$\begin{aligned} \varphi(t) &= W(t,\tau)\varphi_\tau, \psi(t) = W(t,\tau)\psi_\tau, \\ \phi(t) &= \varphi(t) - \psi(t) = (\phi_i(t))_{i\in\mathbb{Z}} = (\xi_i(t), \zeta_i(t))_{i\in\mathbb{Z}}, \end{aligned}$$

then $\varphi(t)$, $\psi(t)$, $\phi(t) \in C([\tau, +\infty), E)$ satisfy:

$$||\varphi(t)|| \leq r_0, \quad ||\psi(t)|| \leq r_0, \quad ||\phi(t)|| \leq 2r_0, \quad \text{for all } t \geq \tau, \tag{3.1}$$

and

$$\begin{cases} \dot{\xi} = \zeta - \varepsilon\xi, \\ (1+\beta A)\dot{\zeta} + (\alpha - \varepsilon)\zeta + (\lambda - \alpha\varepsilon + \varepsilon^2)\xi \\ \qquad + (1 - \gamma\varepsilon + \beta\varepsilon^2)A\xi + (\gamma - \beta\varepsilon)A\xi + f(u) - f(x) = 0, \quad t \geq \tau. \end{cases} \tag{3.2}$$

First we consider the Lipschitz property of $\{W(t,\tau)\}_{t\geq\tau}$ on $Y(\tau)$, for all $\tau \in \mathbb{R}$.

Lemma 3.1. *There exists a continuous positive value function* $L\colon [0,+\infty) \to [0,+\infty)$ *such that, for every* $\tau \in \mathbb{R}$,

$$\|W(t+\tau,\tau)\varphi_\tau - W(t+\tau,\tau)\psi_\tau\|_E \leq L(t) \cdot \|\varphi_\tau - \psi_\tau\|_E,$$

for all $\varphi_\tau, \psi_\tau \in Y(\tau)$, *and*

$$\lim_{t\to\tau+0} \sup_{\varphi_\tau \in Y(\tau)} ||W(t,\tau)\varphi_\tau - \varphi_\tau||_E = 0. \tag{3.3}$$

Proof. It is easy to see that, for $t \geq \tau$,

$$(f(u) - f(x), \zeta) = \sum_{i\in\mathbb{Z}} f_i'(x_i + \vartheta_i(u_i - x_i))\xi_i\zeta_i \leq \frac{\widetilde{K}^2(r_0)}{2\alpha}||\xi||^2 + \frac{\alpha}{2}||\zeta||^2.$$

Taking the inner product $(\,\cdot\,,\,\cdot\,)$ of the second equation of (3.2) with $\zeta(t,\tau)$, we have

$$\begin{aligned}&\frac{d}{dt}(||\zeta||^2+\beta||D\zeta||^2+(\lambda-\alpha\varepsilon+\varepsilon^2)||\xi||^2+(1-\gamma\varepsilon+\beta\varepsilon^2)||D\xi||^2)\\&\quad\le(2\varepsilon-\alpha)||\zeta||^2+2(\beta\varepsilon-\gamma)||D\zeta||^2+\bigg(\frac{K^2(r_0)}{\alpha}-2\varepsilon(\lambda-\alpha\varepsilon+\varepsilon^2)\bigg)||\xi||^2\\&\qquad\qquad\qquad\qquad+2\varepsilon(\gamma\varepsilon-1-\beta\varepsilon^2)||D\xi||^2,\end{aligned}$$

for all $t\ge\tau$. Thus

$$\frac{d}{dt}Y\le\sigma_1 Y,\quad\text{for all } t\ge\tau,\tag{3.4}$$

where

$$Y=||\zeta||^2+\beta||D\zeta||^2+(\lambda-\alpha\varepsilon+\varepsilon^2)||\xi||^2+(1-\gamma\varepsilon+\beta\varepsilon^2)||D\xi||^2,$$

$$\sigma_1=\max\bigg\{2\varepsilon-\alpha,\frac{2(\beta\varepsilon-\gamma)}{\beta},\frac{K^2(r_0)/\alpha-2\varepsilon(\lambda-\alpha\varepsilon+\varepsilon^2)}{\lambda-\alpha\varepsilon+\varepsilon^2},2\varepsilon(\gamma\varepsilon-1-\beta\varepsilon^2)\bigg\}.$$

Applying Gronwall's inequality to (3.4) on $[\tau,\tau+t]$, $(t\ge 0)$, we have

$$\|Y(t+\tau)\|_E^2\le e^{\sigma_1 t}Y(\tau),\quad t\ge 0,$$

thus,

$$\begin{aligned}&\|W(t+\tau,\tau)\varphi_\tau-W(t+\tau,\tau)\psi_\tau\|_E^2\\&\qquad\qquad\le\frac{1}{\mu}C_1e^{\sigma_1 t}\cdot\|\varphi_\tau-\psi_\tau\|_E^2\doteq L(t)\cdot\|\varphi_\tau-\psi_\tau\|_E^2,\end{aligned}$$

for all $t\ge 0$, where

$$C_1=\max\{1+\beta\widetilde{C}_0,(\lambda-\alpha\varepsilon+\varepsilon^2)+(1-\gamma\varepsilon+\beta\varepsilon^2)\widetilde{C}_0\},$$

$$L(t)=\frac{1}{\mu}C_1e^{\sigma_1 t},\quad\text{for all } t\ge 0.$$

It is easy to prove (3.3). □

Then we consider the decomposition of solutions.

LEMMA 3.2. *There exist positive constants $\tau_1>0$, $\gamma_1\in[0,1/2)$ and (for $N_1\in\mathbb{Z}$) $2(2N_1+1)$-dimensional orthogonal projection $P_{N_1}\colon E\to E_{N_1}$, such that for every $\tau\in\mathbb{R}$ and φ_τ, $\psi_\tau\in Y(\tau)$,*

$$\|(I-P_{N_1})(W(\tau+\tau_1,\tau)\varphi_\tau-W(\tau+\tau_1,\tau)\psi_\tau)\|_E\le\gamma_1\|\varphi_\tau-\psi_\tau\|_E.$$

PROOF. For $i\in\mathbb{Z}$, let

$$\omega_i=\theta\bigg(\frac{|i|}{M}\bigg)\phi_i=\bigg(\theta\bigg(\frac{|i|}{M}\bigg)\xi_i,\theta\bigg(\frac{|i|}{M}\bigg)\zeta_i\bigg)=(\vartheta_i,\varsigma_i),\quad\omega=(\omega_i)_{i\in\mathbb{Z}},$$

where θ is as in (2.10) and $M > I_0$. We have

$$
\begin{aligned}
(A\dot\zeta,\varsigma) &\geq \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)(D\zeta)_i^2 - \frac{M_0 m_0 \widetilde{C}_0}{2M}(||\dot\zeta||^2 + ||\zeta||^2),\\
(A\zeta,\varsigma) &\geq \sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)(D\zeta)_i^2 - \frac{M_0 m_0 \widetilde{C}_0}{M}||\zeta||^2,\\
(A\xi,\varsigma) &\geq \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)(D\xi)_i^2 + \varepsilon\sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)(D\xi)_i^2\\
&\qquad - \frac{M_0 m_0 \widetilde{C}_0}{2M}(||\dot\xi||^2 + ||\xi||^2 + 2\varepsilon||\xi||^2),\\
(\dot\zeta,\varsigma) &= \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)\zeta_i^2, \qquad (\zeta,\varsigma) = \sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)\zeta_i^2,\\
(\xi,\varsigma) &= \sum_{i\in\mathbb{Z}}\xi_i\theta\bigg(\frac{|i|}{M}\bigg)(\dot\xi_i + \varepsilon\xi_i)\\
&= \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)\xi_i^2 + \varepsilon\sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)\xi_i^2,\\
2((f(u)-f(x),\varsigma) &\leq \sum_{i\in\mathbb{Z}}\xi\bigg(\frac{|i|}{M}\bigg)\bigg(\frac{1}{\alpha}|f'(x_i + \nu_i(u_i - x_i))|^2\xi_i^2 + \alpha\zeta_i^2\bigg).
\end{aligned}
$$

By (3.2), we have

$$
\begin{aligned}
||\dot\xi(t)||^2 \leq 2||\zeta(t)||^2 + 2\varepsilon^2||\xi(t)||^2 &\leq C_2(||\zeta(t)||^2 + ||\xi(t)||^2), \quad \text{for all } t \geq \tau,\\
||\dot\zeta(t)||^2 + \beta||D\dot\zeta(t)||^2 &\leq C_3(||\zeta(t)||^2 + ||\xi(t)||^2), \quad \text{for all } t \geq \tau.
\end{aligned}
$$

where $C_2 = \max\{1, 2\varepsilon^2\}$, $C_3' = \widetilde{C}_0/(2\beta)$ and

$$
C_3 = \max\{\alpha - \varepsilon, \lambda - \alpha\varepsilon + \varepsilon^2, (1 - \gamma\varepsilon + \beta\varepsilon^2)C_3', (\gamma - \beta\varepsilon)C_3', K^2(r_0)\}.
$$

Thus, taking the inner product $(\,\cdot\,,\,\cdot\,)$ of the second equation of (3.2) with $\varsigma(t,\tau)$, we have that, for all $t \geq \tau$,

$$
\begin{aligned}
\frac{d}{dt}&\sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)(\zeta_i^2 + \beta(D\zeta)_i^2 + (\lambda - \alpha\varepsilon + \varepsilon^2)\xi_i^2 + (1 - \gamma\varepsilon + \beta\varepsilon^2)(D\xi)_i^2)\\
&+ \sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)((\alpha - 2\varepsilon)\zeta_i^2 + 2(\gamma - \beta\varepsilon)(D\zeta)_i^2 + 2\varepsilon(\lambda - \alpha\varepsilon + \varepsilon^2)\xi_i^2)\\
&+ 2\varepsilon(1 - \gamma\varepsilon + \beta\varepsilon^2)\sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)(D\xi)_i^2\\
\leq &\sum_{i\in\mathbb{Z}}\xi\bigg(\frac{|i|}{M}\bigg)\frac{1}{\alpha}|f'(x_i + \vartheta_i(u_i - x_i))|^2\xi_i^2 + \frac{C_4}{M}(||\xi||^2 + ||\zeta||^2),
\end{aligned} \tag{3.5}
$$

where $\vartheta_i \in (0,1)$, C_4 is a positive constant independent of M. By the continuity of $K(s)$ and $K(0) = 0$ (see (H23)), there exists δ_0 such that

$$\Gamma^2(\delta_0) \le \alpha\varepsilon(\lambda - \alpha\varepsilon + \varepsilon^2).$$

By Lemma 2.2, for this $\delta_0 > 0$, there exist $N_2(\delta_0, B_0) \in \mathbb{N}$, $T_1(\delta_0, B_0) > 0$ such that, for $t \ge T_1(\delta_0, B_0) + \tau$,

$$\sup_{i > N_2(\delta_0, B_0)} \{|u_i(t,\tau;\varphi_\tau)|, |x_i(t,\tau;\psi_\tau)|\} \le \delta_0,$$

implying that, for $|i| > N_2(\delta_0, B_0)$, $t \ge T_1(\delta_0, B_0) + \tau$,

$$|x_i(t) + \vartheta_i(u_i(t) - x_i(t))| \le (1 - \vartheta_i)|x_i(t)| + |u_i(t)| \le \delta_0, \quad \vartheta_i \in (0,1).$$

Thus, for $t \ge T_1(\delta_0, B_0) + \tau$,

$$\sup_{|i| > N_2(\delta_0, B_0)} |f_i'(x_i(t) + \nu_i(u_i(t) - x_i(t)))|^2 \le \Gamma^2(\delta_0) \le \alpha\varepsilon(\lambda - \alpha\varepsilon + \varepsilon^2).$$

Therefore, we obtain that for all $M \ge \max\{N_2(\delta_0, B_0), I_0\}$ and $t \ge T_1(\delta_0, B_0) + \tau$,

$$\frac{d}{dt} Y_M(t) + \sigma_2 Y_M(t) \le \frac{C_4}{M}(||\xi(t)||^2 + ||\zeta(t)||^2), \tag{3.6}$$

where

$$Y_M = \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)(\zeta_i^2 + \beta(D\zeta)_i^2 + (\lambda - \alpha\varepsilon + \varepsilon^2)\xi_i^2) + (1 - \gamma\varepsilon + \beta\varepsilon^2)\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)(D\xi)_i^2,$$

$$\sigma_2 = \min\left\{\alpha - 2\varepsilon, \frac{2(\gamma - \beta\varepsilon)}{\beta}, \varepsilon\right\} > 0,$$

Applying Gronwall's inequality to (3.6) from $\tau + T_1$ ($T_1 = T_1(\delta_0, B_0)$) to $\tau + t$ ($t \ge T_1$), we have that, for $M \ge \max\{N_2(\delta_0, B_0), I_0\}$,

$$\begin{aligned}
\sum_{|i|\ge 2M} (\xi_i^2(\tau + t) + \zeta_i^2(\tau + t)) &\le \frac{1}{\mu} Y_M(\tau + t) \\
&\le \frac{1}{\mu} e^{-\sigma_2(t - T_1)} Y_M(\tau + T_1, \tau; \phi_\tau) \\
&\quad + \frac{C_4}{M\mu} \int_{\tau+T_1}^{\tau+t} e^{-\sigma_2(\tau+t-r)}(||\xi(r,\tau)||^2 + ||\zeta(r,\tau)||^2)\, dr \\
&\le \frac{1}{\mu^2}\left(C_1 e^{(\sigma_1+\sigma_2)T_1} e^{-\sigma_2 t} + \frac{1}{M}\frac{C_1 C_4}{\sigma_1 + \sigma_2} e^{\sigma_1 t}\right) \cdot \|\varphi_\tau - \psi_\tau\|_E^2,
\end{aligned} \tag{3.7}$$

for all $t \ge T_1$. Setting

$$\tau_1 = \max\left\{T_1(\delta_0, B_0), \frac{\ln(8C_1 e^{(\sigma_1+\sigma_2)T_1}/\mu^2)}{\sigma_2}\right\},$$

$$N_1 = \left\{2N_2(\delta_0, B_0) + 1, 2I_0 + 1, \frac{8C_1C_4}{\mu^2(\sigma_1 + \sigma_2)} e^{\sigma_1 \tau_1}\right\},$$

we have

$$\gamma_1 = \frac{1}{\mu}\sqrt{C_1 e^{(\sigma_1+\sigma_2)T_2} e^{-\sigma_2\tau_1} + \frac{1}{N_1}\frac{C_2C_4}{\sigma_1+\sigma_2} e^{\sigma_1\tau_1}} < \frac{1}{2}.$$

Thus, by (3.7), we have

$$\|(I - P_{N_1})(W(\tau+\tau_1,\tau)\varphi_\tau - W(\tau+\tau_1,\tau)\psi_\tau)\|_E \le \gamma_1\|\varphi_\tau - \psi_\tau\|_E. \qquad \square$$

From Lemmas 3.1, 3.2 and Theorem 2 of [21], we obtain the following result of existence of a pullback exponential attractor.

THEOREM 3.3. *Let* (H1)–(H3) *hold. Then the process* $\{W(t,\tau)\}_{t\ge\tau}$ *associated with* (2.4) *possesses a pullback exponential attractor* $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ *with the following properties*:

(a) *for any* $t \in \mathbb{R}$, $\mathcal{K}(t) \subseteq \mathcal{A}(t) \subseteq Y(t) \subseteq B_0$;
(b) *the fractal dimension of* $\mathcal{A}(t)$, $\dim_f \mathcal{A}(t) \le \ln N_\varepsilon / -\ln a_\varepsilon$;
(c) *for any bounded set* $B \subset E$,

$$\mathrm{d}_E(W(t,\tau)B, \mathcal{A}(t)) \le \frac{L(\tau_1)r_0 e^{\omega_1 T_B}}{(a_\varepsilon)^2} e^{-\omega_1(t-\tau)},$$

for $-\infty < \tau + T_B < t \le +\infty$;
(d) $\lim_{t\to s} \mathrm{d}_E(\mathcal{A}(t),\mathcal{A}(s)) = 0$, $-\infty < s < \infty$, *where*

$$a_\varepsilon = 2(\gamma_1 + \varepsilon L(\tau_1)), \quad \varepsilon = \frac{1-2\gamma_1}{4L(\tau_1)}, \quad \omega_1 = \frac{-\ln a_\varepsilon}{\tau_1}$$

and N_ε *is the minimal number of closed balls of* E *with radius* θ *covering the closed unit ball* $B_{N_1}(0,1)$ *of* $E_{N_1} = l^2_{N_1} \times l^2_{N_1}$ *centered at* 0, *here* $l^2_{N_1} = \{u = (u_i)_{i\in\mathbb{Z}} \in l^2 : u_i = 0$ *for* $|i| > N_1\}$.

4. Uniform attractor

Another important concept describing the asymptotic behavior of non-autonomous dynamical systems is uniform attractor. In this section, we consider the existence of uniform attractor for the system (2.4).

Given a fixed function $g_0 \in \mathbf{G} \subset C_b(\mathbb{R}; l^2)$, take

$$\mathcal{H}(g_0) = \{g_0(\,\cdot\,+h) : h \in \mathbb{R}\} \subset C_b(\mathbb{R}; l^2)$$

as a symbol space, then $T(h)\mathcal{H}(g_0) = \mathcal{H}(g_0)$, where $T(h)g(\,\cdot\,) = g(\,\cdot\,+h)$, for all $g \in \mathcal{H}(g_0)$, for all $h \in \mathbb{R}_+$. From Section 2, for any $g \in \mathcal{H}(g_0)$ in (3.2), the solution $\varphi(t) = (\varphi_i(t))_{i\in\mathbb{Z}}$ of (2.4) exists globally in $[\tau,+\infty)$, and maps of solutions

$$W_g(t,\tau)\colon \varphi_\tau \mapsto \varphi(t) = W_g(t,\tau)\varphi_\tau \in E, \quad \text{for all } t \ge \tau,\ g \in \mathcal{H}(g_0),$$

generate a family of continuous processes $\{W_g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$ on E which satisfying

$$W_g(t+h,\tau+h) = W_{T(h)g}(t,\tau), \quad \text{for all } g \in \mathcal{H}(g_0),$$

for all $t \geq \tau$, $\tau \in \mathbb{R}$ and all $h \in \mathbb{R}_+$.

We have the following theorem for the existence of a compact uniform (w.r.t. $g \in \mathcal{H}(g_0)$) attractor of the family of processes $\{W_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ in E.

Theorem 4.1. *Assume that* (H1)–(H2) *hold and* $g_0 \in \mathbf{G}$, *then*

(a) $\{W_g(t,\tau)\}_{g\in\mathcal{H}(g_0), t\geq\tau}$ *possesses a uniform closed bounded absorbing ball*

$$\widetilde{B}_0 = \widetilde{B}_0(0, r_0) = \{\varphi \in E : ||\varphi||_E \leq r_0\} \subset E,$$

where $\widetilde{r}_0 = \sqrt{2/(\mu\sigma_0\alpha)}|||g_0|||$, *such that, for any* $\tau \in \mathbb{R}$ *and each bounded set* $B \subset E$, *there exists* $T_B \geq 0$ *such that* $\bigcup_{g\in\mathcal{H}(g_0)} W_g(t,\tau)B \subseteq \widetilde{B}_0$ *for all* $t \geq T_B + \tau$. *In particular, there exists a time* $T_{\widetilde{B}_0} > 0$ *such that*

$$\bigcup_{g\in\mathcal{H}(g_0)} W_g(t,\tau)\widetilde{B}_0 \subseteq \widetilde{B}_0, \quad \textit{for all } t \geq T_{\widetilde{B}_0} + \tau, \ \tau \in \mathbb{R};$$

(b) *Set* $\widetilde{B} = \bigcup_{g\in\mathcal{H}(g_0)} \overline{\bigcup_{t\geq T_{\widetilde{B}_0}} W_g(t,0)\widetilde{B}_0} \subseteq \widetilde{B}_0$. *For any* $\tau \in \mathbb{R}$ *and any* $\eta > 0$, *there exist* $\widetilde{N}_1(\eta, \widetilde{B}_0) \in \mathbb{N}$ *and* $\widetilde{T}_1(\eta, \widetilde{B}_0) \geq 0$ *such that for any* $\varphi(\tau) \in \widetilde{B}$, *the solution* $\varphi(t) = ((u_i(t), v_i(t)))_{i\in\mathbb{Z}} = W_g(t,\tau)\varphi(\tau)$ *of* (2.4) *satisfies*

$$\sup_{g\in\mathcal{H}(g_0)} \sum_{|i|>\widetilde{N}_1(\eta,\widetilde{B}_0)} (u_i^2(t) + v_i^2(t)) \leq \eta, \quad \textit{for all } t \geq \widetilde{T}_1(\eta, \widetilde{B}_0) + \tau.$$

(c) $\{W_g(t,\tau)\}_{g\in\mathcal{H}(g_0), t\geq\tau}$ *possesses a uniform attractor* $\mathcal{A} \subset \widetilde{B} \subset E$*with properties*:

(c1) $\mathcal{A}$ *is closed*;

(c2) $\lim_{t\to+\infty} \sup_{g\in\mathcal{H}(g_0)} \mathrm{d}_E(W_g(t,\tau)B, \mathcal{A}) = 0$ *for any fixed* $\tau \in \mathbb{R}$ *and any bounded set* B *of* E;

(c3) $\mathcal{A}$ *is the minimal set* (*for inclusion relation*) *among those satisfying* (c2).

Proof. Noticing that for any $g \in \mathcal{H}(g_0)$, $|||g||| \leq |||g_0|||$, thus (c1) and (c2) are similar to the proof of Lemmas 2.1 and 2.2. The proof of (c3) is obtained by Theorem 3.1 in [24]. □

5. Uniform exponential attractor

It is known from Theorem 3.1 in [24] that the uniform attractor $\mathcal{A}$ with respect to $g \in \mathcal{H}(g_0)$ for the family of processes $\{W_g(t,\tau)\}_{g\in\mathcal{H}(g_0), t\geq\tau}$ on the state space E is just the projection of the global attractor $\Theta \subset E \times \mathcal{H}(g_0)$ on E of semigroup $\{S(t)\}_{t\geq 0}$ (skew-product semiflow) on a extended phase space $E \times \mathcal{H}(g_0)$:

$$S(t)\colon E \times \mathcal{H}(g_0) \to E \times \mathcal{H}(g_0), \qquad (\varphi, g) \to (W_g(t,0)\varphi, T(t)g),$$

for all $t \geq 0$. Thus the dimension of $\mathcal{A}$ and Θ depend on the dimension of the symbol space $\mathcal{H}(g_0)$. In fact, generally, the dimension of $\mathcal{A}$ and Θ is infinite dimensional when $\mathcal{H}(g_0)$ is infinite dimensional. So it is necessary to consider the existence of a uniform exponential attractor for system (2.4) which includes the uniform attractor and has finite fractal dimension and attracts all bounded sets exponentially. However, it is difficult to prove the existence of such uniform exponential attractor for system (2.4) with an infinite dimensional symbol space $\mathcal{H}(g_0)$. In this section, we consider the existence of a uniform exponential attractor for system (1.2) with quasi-periodic external force.

Let $\mathbf{T}^n$ be the n-dimensional torus:

$$\mathbf{T}^n = \{\rho = (\rho_1, \ldots, \rho_n) : \rho_j \in [-\pi, \pi], \text{for all } j = 1, \ldots, n\}$$

with the identification

$$(\rho_1, \ldots, \rho_{j-1}, -\pi, \rho_{j+1}, \ldots, \rho_n) \sim (\rho_1, \ldots, \rho_{j-1}, \pi, \rho_{j+1}, \ldots, \rho_n),$$

for all $j = 1, \ldots, n$, and the topology and metric induced from the norm given by

$$\|\rho\|_{\mathbf{T}^n} = \bigg(\sum_{j=1}^{n} \rho_j^2\bigg)^{1/2}, \quad \text{for all } \rho = (\rho_1, \ldots, \rho_n) \in \mathbf{T}^n.$$

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be a fixed vector such that $x_1, \ldots, x_n$ are rationally independent, i.e. if there exist integers $k_1, \ldots, k_n$ such that $\sum_{j=1}^{n} k_j x_j = 0$, then $k_j = 0$ for $j = 1, \ldots, n$. For $s \in \mathbb{R}$, define

$$\widetilde{T}(s)\rho = (\mathbf{x}s + \rho) \operatorname{mod}(\mathbf{T}^n), \quad \rho \in \mathbf{T}^n,$$

then $\{\widetilde{T}(s)\}_{s\in\mathbb{R}}$ is a translation group on $\mathbf{T}^n$ with

$$T(s)\mathbf{T}^n = \mathbf{T}^n, \quad \text{for all } s \in \mathbb{R}.$$

Consider the second order lattice system (1.2) with quasi-periodic external forces and initial data, which is equivalent to the following vector form:

$$\begin{cases} \ddot{u} + \beta(A\ddot{u}) + \alpha\dot{u} + \gamma(A\dot{u}) + \lambda u + Au + f(u) = ah(\widetilde{\rho}(t)), & t > \tau, \\ u(\tau) = (u_{i,\tau})_{i\in\mathbb{Z}}, \quad \dot{u}(\tau) = (\dot{u}_{i,\tau})_{i\in\mathbb{Z}}, & \tau \in \mathbb{R}, \end{cases} \tag{5.1}$$

where β, α, γ, λ, A, f are as in (2.1) of Section 2; $ah(\widetilde{\rho}(t)) = (a_i h_i(\widetilde{\rho}(t)))_{i\in\mathbb{Z}}$, $\widetilde{\rho}(t) = \widetilde{T}(t)\rho = (\mathbf{x}t + \rho)\operatorname{mod}(\mathbf{T}^n) \in \mathbf{T}^n$, $\rho \in \mathbf{T}^n$.

We make the following assumptions on h_i, a_i, $i \in \mathbb{Z}$ in (5.1):

(H4) for all $i \in \mathbb{Z}$, $h_i(0_{\mathbf{T}^n}) = 0$ and there exists $k_0 > 0$ such that

$$|h_i(\widetilde{\rho}_1) - h_i(\widetilde{\rho}_2)| \leq k_0 \|\rho_1 - \rho_2\|_{\mathbf{T}^n}, \quad \rho_1, \rho_2 \in \mathbf{T}^n.$$

(H5) $a = (a_i)_{i\in\mathbb{Z}} \in l^2$.

The problem (5.1) can be written as

$$\begin{cases} \dot{u} = \varepsilon u - v, \\ (1+\beta A)\dot{v} + (\alpha - \varepsilon)v + (\lambda - \alpha\varepsilon + \varepsilon^2)u \\ \qquad +(1-\gamma\varepsilon + \beta\varepsilon^2)Au + (\gamma - \beta\varepsilon)Av + f(u) = ah(\widetilde{\rho}(t)), \\ u(\tau) = (u_{i,\tau})_{i\in\mathbb{Z}}, \quad \dot{u}(\tau) = (\dot{u}_{i,\tau})_{i\in\mathbb{Z}}, \end{cases}$$

that is,

$$\dot{\varphi} + C(\varphi) = \widetilde{F}(\varphi, t),\ \varphi(\tau) = \begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = \begin{pmatrix} u_\tau \\ u_{1\tau} + \varepsilon u_\tau \end{pmatrix}, \quad t > \tau,\ \tau \in \mathbb{R}, \tag{5.2}$$

where

$$\widetilde{F}(\varphi, t) = \begin{pmatrix} 0 \\ (1+\beta A)^{-1}[-f(u) + ah(\widetilde{\rho}(t))] \end{pmatrix}.$$

In the following, we assume (H2)–(H5) hold. By replacing $g(t)$ in Section 4 by $ah(\widetilde{\rho}(t))$, and by

$$|||ah(\widetilde{\rho}(t))|||^2 = \sum_{i\in\mathbb{Z}} a_i^2 h_i^2(\widetilde{\rho}(t)) \le \sum_{i\in\mathbb{Z}} a_i^2 k_0^2 ||\rho||^2_{\mathbf{T}^n} \le n\pi^2 k_0^2 \|a\|^2,$$

similar to Theorem 4.1, we have the following theorem.

THEOREM 5.1. (a) *For any $\tau \in \mathbb{R}$, any initial data $\varphi(\tau) = (u(\tau), v(\tau)) \in E$ and $\rho \in \mathbf{T}^n$, there exists a unique solution of* (5.2):

$$\varphi(t) = (u(t), v(t)) \in C([\tau, \infty), E) \cap C^1((\tau, \infty), H), \quad t \ge \tau,$$

which generate a family of continuous processes $\{U^\rho(t,\tau)\}_{\rho\in\mathbf{T}^n, t\ge\tau}$ on E :

$$U^\rho(t,\tau)\colon \varphi(\tau) \mapsto \varphi(t), \quad t > \tau,\ \tau \in \mathbb{R},\ \rho \in \mathbf{T}^n.$$

(b) *The family of processes $\{U^\rho(t,\tau)\}_{\rho\in\mathbf{T}^n, t\ge\tau}$ possesses a uniform closed bounded absorbing ball $\mathcal{B}_0 = \mathcal{B}_0(0, R_0) \subset H$ centered at 0 with radius $R_0 = \sqrt{2n/(\mu\sigma_0\alpha)}\pi k_0\|a\|$.*

(c) *Set $\widetilde{\mathcal{B}}_0 = \bigcup_{\rho\in\mathbf{T}^n} \overline{\bigcup_{t\ge T_{\mathcal{B}_0}} U^\rho(t,0)\mathcal{B}_0} \subseteq \mathcal{B}_0$. For any $\tau \in \mathbb{R}$ and any $\eta > 0$, there exist $N_3(\eta, R_0) \in \mathbb{N}$ and $T_3(\eta, R_0) \ge 0$ such that for any $\varphi(\tau) \in \widetilde{\mathcal{B}}_0$, the solution $\varphi(t) = ((u_i(t), v_i(t)))_{i\in\mathbb{Z}} = U^\rho(t,\tau)\varphi(\tau)$ of* (5.2) *satisfies*

$$\sup_{\rho\in\mathbf{T}^n} \sum_{|i|>N_3(\eta,R_0)} (u_i^2(t) + v_i^2(t)) \le \eta, \quad \text{for all } t \ge T_3(\eta, R_0) + \tau.$$

(d) *The family of continuous processes $\{U^\rho(t,\tau)\}_{\rho\in\mathbf{T}^n, t\ge\tau}$ possesses a uniform attractor $\widetilde{\mathcal{A}} \subset \widetilde{\mathcal{B}} \subset H$.*

Now we first verify the Lipschitz continuity of $\{U^\rho(t,\tau)\}_{\rho\in\mathbf{T}^n, t\ge\tau}$ and provide an estimation of the tail of the difference between two solutions of (5.2). Then we obtain the existence of a uniform exponential attractor of (5.2).

For $j = 1, 2$, $\varphi^{(j0)} \in \widetilde{\mathcal{B}}_0$, $\rho_j \in \mathbf{T}^n$ and $t \geq 0$, let $\varphi^{(j)}(t) = U^{\rho_j}(t,0)\varphi^{(j0)} = (u^{(j)}(t), v^{(j)}(t))(\in \widetilde{\mathcal{B}}_0)$ be the solutions of (5.2). Set

$$\Phi(t) = \varphi^{(1)}(t) - \varphi^{(2)}(t) = U^{\rho_1}(t,0)\varphi^{(10)} - U^{\rho_2}(t,0)\varphi^{(20)} = (\xi(t), \zeta(t)),$$

for all $t \geq 0$, we have

$$||\varphi^{(j)}(t)|| \leq R_0, \quad ||\Phi(t)|| \leq 2R_0, \quad \text{for all } t \geq 0,\ j = 1, 2,$$

and

$$\begin{cases} \dot{\xi} = \zeta - \varepsilon\xi, \\ (1+\beta A)\dot{\zeta} + (\alpha - \varepsilon)\zeta + (\lambda - \alpha\varepsilon + \varepsilon^2)\xi + (1 - \gamma\varepsilon + \beta\varepsilon^2)A\xi \\ \qquad +(\gamma - \beta\varepsilon)A\xi + f(u) - f(x) = ah(\widetilde{\rho}_1(t)) - ah(\widetilde{\rho}_2(t)), \qquad t > 0. \end{cases} \tag{5.3}$$

LEMMA 5.2. *There exists a positive valued continuous function $L_1 \colon [0, +\infty) \to [0, +\infty)$ such that, for $t \geq 0$,*

$$\begin{aligned} &\|U^{\rho_1}(t,0)\varphi^{(10)} - U^{\rho_2}(t,0)\varphi^{(20)}\|_E \\ &\qquad\qquad \leq L_1(t)(\|\varphi^{(10)} - \varphi^{(20)}\|_H^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2)^{1/2}. \end{aligned} \tag{5.4}$$

PROOF. Note that $\|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2$ is independent of t, and for any $t \geq 0$,

$$\begin{aligned} (f(u) - f(x), \zeta) &= \sum_{i\in\mathbb{Z}} f_i'(x_i + \vartheta_i(u_i - x_i))\xi_i\zeta_i \\ &\leq \frac{K^2(R_0)}{\alpha}||\xi||^2 + \frac{\alpha}{4}||\zeta||^2, \\ (ah(\widetilde{\rho}_1(t)) - ah(\widetilde{\rho}_2(t)), \zeta) &= \sum_{i\in\mathbb{Z}} a_i(h_i(\widetilde{\rho}_1(t)) - h_i(\widetilde{\rho}_2(t)))\zeta_i \\ &\leq \frac{k_0^2||a||^2}{\alpha}\|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2 + \frac{\alpha}{4}||\zeta||^2. \end{aligned}$$

Taking the inner product $(\,\cdot\,,\,\cdot\,)$ of the second equation of (5.3) with $\zeta(t,\tau)$, we have

$$\frac{d}{dt}Z(t) \leq \sigma_3 Z(t), \quad \text{for all } t \geq 0, \tag{5.5}$$

where

$$\begin{aligned} Z = ||\zeta||^2 &+ \beta||D\zeta||^2 + (\lambda - \alpha\varepsilon + \varepsilon^2)||\xi||^2 \\ &+ (1 - \gamma\varepsilon + \beta\varepsilon^2)||D\xi||^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2, \end{aligned}$$

$$\sigma_3 = \max\left\{2\varepsilon - \alpha, \frac{2(\beta\varepsilon - \gamma)}{\beta}, \frac{2K^2(r_0)/\alpha - 2\varepsilon(\lambda - \alpha\varepsilon + \varepsilon^2)}{\lambda - \alpha\varepsilon + \varepsilon^2}, \frac{k_0^2||a||^2}{\alpha}\right\}.$$

Applying Gronwall's inequality to (5.5) on $[0,t]$ $(t \geq 0)$, we have

$$\begin{aligned} ||\zeta(t)||^2 + ||\xi(t)||^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2 &\leq \frac{1}{\mu}\|Z(t)\|_E^2 \\ &\leq \frac{1}{\mu}C_5 e^{\sigma_3 t} \cdot (\|\Phi(0)\|_E^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2) \doteq L_1(t) \cdot \|\varphi_\tau - \psi_\tau\|_E^2, \end{aligned}$$

for all $t \geq 0$, where

$$C_5 = \max\{1 + \beta\widetilde{C}_0, (\lambda - \alpha\varepsilon + \varepsilon^2) + (1 - \gamma\varepsilon + \beta\varepsilon^2)\widetilde{C}_0, 1\},$$

$$L_1(t) = \frac{1}{\mu} C_5 e^{\sigma_3 t}, \quad \text{for all } t \geq 0. \qquad \square$$

LEMMA 5.3. *There exist $T^* > 0$ and $M^* \in \mathbb{N}$ such that*

$$\sum_{|i|>M^*} |(U^{\rho_1}(T^*,0)\varphi^{(10)} - U^{\rho_2}(T^*,0)\varphi^{(20)})_i|_E^2 = \sum_{|i|>M^*} [\xi_i^2(t) + \zeta_i^2(t))] \\ \leq \frac{1}{128}(\|\varphi^{(10)} - \varphi^{(20)}\|_E^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2).$$

PROOF. For $i \in \mathbb{Z}$, let

$$\omega_i = \theta\bigg(\frac{|i|}{M}\bigg)\phi_i = \bigg(\theta\bigg(\frac{|i|}{M}\bigg)\xi_i, \theta\bigg(\frac{|i|}{M}\bigg)\zeta_i\bigg) = (\vartheta_i, \varsigma_i),$$

$\omega = (\omega_i)_{i\in\mathbb{Z}}$, θ is as in (2.10) and $M > I_0$. Here

$$(ah(\widetilde{\rho}_1(t)) - ah(\widetilde{\rho}_2(t)), \varsigma) \leq \sum_{i\in\mathbb{Z}} \xi\bigg(\frac{|i|}{M}\bigg)\bigg(\frac{k_0^2 a_i^2}{\alpha}\|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2 + \frac{\alpha\zeta_i^2}{4}\bigg).$$

and

$$||\dot{\zeta}(t)||^2 + \beta||D\dot{\zeta}(t)||^2 \leq C_3(||\zeta(t)||^2 + ||\xi(t)||^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2),$$

for all $t \geq 0$. Similar to (3.5), taking the inner product $(\,\cdot\,,\,\cdot\,)$ of the second equation of (5.3) with $\varsigma(t,\tau)$, we have that for all $t \geq 0$,

$$\begin{aligned}
&\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)(\zeta_i^2 + \beta(D\zeta)_i^2 + (\lambda - \alpha\varepsilon + \varepsilon^2)\xi_i^2 + (1 - \gamma\varepsilon + \beta\varepsilon^2)(D\xi)_i^2) \\
&\quad + \sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)((\alpha - 2\varepsilon)\zeta_i^2 + 2(\gamma - \beta\varepsilon)(D\zeta)_i^2 + 2\varepsilon(\lambda - \alpha\varepsilon + \varepsilon^2)\xi_i^2) \\
&\quad + 2\varepsilon(1 - \gamma\varepsilon + \beta\varepsilon^2)\sum_{i\in\mathbb{Z}}\theta\bigg(\frac{|i|}{M}\bigg)(D\xi)_i^2 \\
&\leq \sum_{i\in\mathbb{Z}}\xi\bigg(\frac{|i|}{M}\bigg)\frac{1}{\alpha}|f'(x_i + \vartheta_i(u_i - x_i))|^2\xi_i^2 \\
&\quad + \frac{C_6}{M}(||\xi||^2 + ||\zeta||^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2) + \frac{2k_0^2}{\alpha}\|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2\sum_{i\in\mathbb{Z}}\xi\bigg(\frac{|i|}{M}\bigg)a_i^2,
\end{aligned}$$

where C_6 is a constant independent of M. By the continuity of $K(s)$, $K(0) = 0$ and Lemma 5.2, there exist $\delta_1 > 0$, and $N_4(\delta_1, R_0) \in \mathbb{N}$, $T_4(\delta_1, R_0) > 0$ such that

$$\sup_{|i|>N_2(\delta_1,R_0)} |f_i'(x_i + \vartheta_i(u_i - x_i))|^2 \leq \Gamma^2(\delta_1) \leq \alpha\varepsilon(\lambda - \alpha\varepsilon + \varepsilon^2), t \geq T_4(\delta_1, R_0).$$

Therefore, we obtain that for all $M \geq \max\{N_4(\delta_1, R_0), I_0\}$ and $t \geq T_4(\delta_1, R_0)$,

$$\begin{aligned} \frac{d}{dt} Y_M + \sigma_4 Y_M \leq{}& \frac{C_6 C_5}{\mu M} e^{\sigma_3 t} \cdot (\|\Phi(0)\|_E^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2) \\ &+ \frac{2k_0^2}{\alpha} \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2 \sum_{i \in \mathbb{Z}} \xi\left(\frac{|i|}{M}\right) a_i^2, \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} Y_M ={}& \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{M}\right) (\zeta_i^2 + \beta (D\zeta)_i^2 + (\lambda - \alpha\varepsilon + \varepsilon^2)\xi_i^2) \\ &+ (1 - \gamma\varepsilon + \beta\varepsilon^2) \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{M}\right) (D\xi)_i^2, \\ \sigma_4 ={}& \min\left\{\alpha - 2\varepsilon, \frac{2(\gamma - \beta\varepsilon)}{\beta}, \varepsilon\right\} > 0. \end{aligned}$$

Integrating (5.6) from $T_4 = T_4(\delta_1, R_0)$ to t $(> T_4)$, we have

$$\begin{aligned} \sum_{|i| \geq 2M} (\xi_i^2(t) + \zeta_i^2(t)) \leq{}& \frac{1}{\mu} Y_M(t) \\ \leq{}& \frac{1}{\mu}\left(e^{-\sigma_4(t - T_4)} \cdot Y_M(T_4) + \frac{1}{M} \frac{C_6 C_5}{\mu(\sigma_4 + \sigma_3)} e^{\sigma_3 t} \cdot (\|\Phi(0)\|_H^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2)\right) \\ &+ \frac{2k_0^2}{\mu\alpha\sigma_4} \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2 \sum_{|i| \geq M} a_i^2 \\ \leq{}& \frac{1}{\mu^2}\left(C_5 e^{(\sigma_3 + \sigma_4)T_4} e^{-\sigma_4 t} + \frac{1}{M} \frac{C_6 C_5}{(\sigma_4 + \sigma_3)} e^{\sigma_3 t}\right)(\|\Phi(0)\|_H^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2) \\ &+ \frac{2k_0^2}{\mu\alpha\sigma_4} \sum_{|i| \geq M} a_i^2 \cdot \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2. \end{aligned}$$

By the condition (H5), there exists $M_5 \in \mathbb{N}$ such that

$$\frac{2k_0^2}{\mu\alpha\sigma_4} \sum_{|i| \geq M} a_i^2 \leq \frac{1}{512}, \quad M \geq M_5.$$

Letting

$$T^* = \max\left\{\frac{(\sigma_3 + \sigma_4)T_4 + 2\ln(1024 C_5/\mu^2)}{\sigma_4}, T_4\right\}, \tag{5.7}$$

$$M^* = \max\left\{M_4, M_5, I_0, \frac{1024 C_6 C_5}{\mu^2(\sigma_4 + \sigma_3)} e^{\sigma_3 T^*}\right\}, \tag{5.8}$$

we then have

$$\begin{aligned} \frac{1}{\mu^2}\left(C_5 e^{(\sigma_3 + \sigma_4)T_2} e^{-\sigma_2 T^*} + \frac{1}{M^*} \frac{C_6 C_5}{(\sigma_4 + \sigma_3)} e^{\sigma_3 T^*}\right) &\leq \frac{1}{512}, \\ \sum_{|i| > 2M^*} (\xi_i^2(T^*) + \zeta_i^2(T^*)) &\leq \frac{1}{128}(\|\Phi(0)\|_H^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2), \end{aligned}$$

that is,

$$\sum_{|i|>2M^*} |(U^{\rho_1}(T^*,0)\varphi^{(10)} - U^{\rho_2}(T^*,0)\varphi^{(20)})_i|^2$$
$$\leq \frac{1}{128}(\|\varphi^{(10)} - \varphi^{(20)}\|_H^2 + \|\rho_1 - \rho_2\|_{\mathbf{T}^n}^2). \quad \square$$

By Lemmas 5.1–5.3 and Theorem 2.2 of [22], our main result of this section is as follows.

Theorem 5.4. *If conditions* (H2)–(H5) *are satisfied, then the family of processes* $\{U^\rho(t,\tau)\}_{\rho\in\mathbf{T}^n, t\geq\tau}$, *possesses a uniform exponential attractor* $\mathcal{M} \subset \widetilde{\mathcal{B}}$ *with the following properties*:

(a) $\mathcal{M}$ *is compact*;
(b) $\mathcal{A} \subset \mathcal{M} \subset \mathcal{B}$, *where* $\mathcal{A}$ *is the uniform attractor*;
(c) $\mathcal{M}$ *has a finite fractal dimension*

$$\dim_f(\mathcal{M}) \leq K_0(n + 2(4M^* + 1)) \ln \sqrt{L_1(T^*) + 1} + 1,$$

where K_0 *is a constant,* T^* *and* M^* *are as in* (5.7)–(5.8);
(d) *there exist two positive constants* k_1 *and* k_2 *such that*

$$\sup_{\rho\in\mathbf{T}^n} \mathrm{d}_E(U^\rho(t,\tau)\mathcal{B}, \mathcal{M}) \leq k_1 e^{-k_2(t-\tau)}, \quad \textit{for all } t \geq \tau,\ \tau \in \mathbb{R}.$$

Remark 5.5. Similar results in this article are still valid for non-autonomous lattice systems (1.1) and (1.2) defined on $\mathbb{Z}^l$ with $l \geq 2$ and $\beta \geq 0$.

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