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A NOTE ON COMPLETENESS OF BASIC TRIGONOMETRIC SYSTEM IN $\mathcal{L}^2$

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ABSTRACT. We use the q-Lommel polynomials and the Riesz-Fisher theorem in order to give an independent proof of the completeness of the basic trigonometric system in certain weighted $\mathcal{L}^2$-spaces.

1. Introduction. In this note we present an independent proof of the completeness of the q-trigonometric system in the theory of basic Fourier series. One can look at [**2, 4, 15**–**18**] and references therein regarding the q-Fourier series. The corresponding basic exponential function on a q-quadratic grid is given by

(1.1)
$$\mathcal{E}_q(x;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \times \sum_{n=0}^{\infty} \frac{q^{n^2/4}\ \alpha^n}{(q;q)_n}\ (-i)^n \left(-iq^{(1-n)/2}e^{i\theta}, -iq^{(1-n)/2}e^{-i\theta};q\right)_n,$$

where $x=\cos\theta$ and $|\alpha|<1$. We assume that $0<q<1$ and use the standard notations [**3**] for the basic hypergeometric series and q-shifted factorials. For the analytic continuation in a larger domain and other properties see, for example, [**8, 13, 17**] and references therein. Ismail and Zhang [**8**] found the following expansion formula

(1.2)
$$\mathcal{E}_q(x;i\omega) = \frac{(q;q)_\infty\,\omega^{-\nu}}{(q^\nu;q)_\infty\,(-q\omega^2;q^2)_\infty} \times \sum_{m=0}^{\infty} i^m\left(1-q^{\nu+m}\right)\ q^{m^2/4} J^{(2)}_{\nu+m}(2\omega;q)\ C_m\left(x;q^\nu|\,q\right),$$

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where $J_{\nu+m}^{(2)}(2\omega;q)$ is Jackson's q-Bessel function defined as

$$J_{\nu}^{(2)}(x;q)=\frac{\left(q^{\nu+1};q\right)_{\infty}}{(q;q)_{\infty}}\sum_{n=0}^{\infty}q^{(\nu+n)n}\frac{(-1)^{n}\,(x/2)^{\nu+2n}}{(q;q)_{n}\,(q^{\nu+1};q)_{n}}, \tag{1.3}$$

[**5**–**6**], and the continuous q-ultraspherical polynomials $C_m(\cos\theta;\beta|q)$ are given by

$$C_m(\cos\theta;\beta|q)=\sum_{k=0}^{m}\frac{(\beta;q)_k\,(\beta;q)_{m-k}}{(q;q)_k\,(q;q)_{m-k}}\;e^{i(m-2k)\theta} \tag{1.4}$$

(see, for example, [**3**]). Relations with the q-Lommel polynomials, which are important for the further consideration, were observed for the first time in [**15**] and [**17**]; see also [**7**].

Completeness of the trigonometric system $\left\{e^{inx}\right\}_{n=-\infty}^{\infty}$ on the interval $(-\pi,\pi)$ is one of the fundamental facts in the theory of classical trigonometric series. Bustoz and Suslov [**2**] proved that the basic trigonometric system $\{\mathcal{E}_q(x;i\omega_n)\}_{n=-\infty}^{\infty}$ is complete in a weighted $\mathcal{L}^2_{\rho_1}(-1,1)$ space where $\rho_1(x)$ is the weight function for the continuous q-ultraspherical polynomials $C_m(x;\beta|q)$ with $\beta=q^{1/2}$ when $\omega_n=0,\pm\omega_1,\pm\omega_2,\pm\omega_3,\ldots$ and $\omega_0=0<\omega_1<\omega_2<\omega_3<\cdots$ are nonnegative zeros of the following basic sine function

$$S_q(\eta;\omega)=\frac{(-i\omega;q^{1/2})_\infty-(i\omega;q^{1/2})_\infty}{2i(-q\omega^2;q^2)_\infty}, \tag{1.5}$$

which can be viewed as an analog of $\sin\pi\omega$.

Several generalizations of this result were discussed in [**17**] and [**18**]. In particular, the following theorem has been established in [**18**], among other things, using the methods of the theory of entire functions.

Theorem 1. *Let $j_{\mu,k}(q)$ be the zeros of Jackson's q-Bessel function $J^{(2)}_{\mu}(x;q)$. The q-trigonometric system $\{\mathcal{E}_q(x;i\omega_k)\}_{k=-\infty}^{\infty}$ is complete in $\mathcal{L}^p_{\rho}(-1,1)$, where $1\le p<\infty$ and $\rho(x)$ is an integrable function, positive almost everywhere on $(-1,1)$, for $0<\nu<1$, if*

(a) $\omega_k=1/2j_{\nu-1,k}(q)$, $k=\pm1,\pm2,\ldots$;

(b) $\omega_k=1/2j_{\nu,k}(q)$, $k=0,\pm1,\pm2,\ldots$.

In the special case when $p = 2$, $0 < \nu < 1/2$ and $\rho(x)$ is the weight function for the continuous q-ultraspherical polynomials, this result had also been proven in [**17**] using a completely different method involving the Ismail and Zhang formula (1.2) and the relations with the q-Lommel polynomials. This approach does not involve the most important practical cases $\nu = 1/2$ when the corresponding systems are orthogonal. In the current paper we will be able to extend this proof to the larger interval $0 < \nu \leq 1/2$. This gives, finally, an independent proof of the completeness of the basic trigonometric systems in the orthogonal cases.

2. Proof of completeness theorem. Here we give the proof of Theorem 1 when $p = 2$, $0 < \nu \leq 1/2$ and $\rho(x)$ is the weight function for the continuous q-ultraspherical polynomials. Consider the first case: $\omega_k = 1/2 j_{\nu-1,k}(q)$, $k = \pm 1, \pm 2, \dots$. The q-Lommel polynomials introduced in [**6**] are generated by the recurrence relation

$$\begin{aligned}(2.1)\quad & q^{n\nu+n(n-1)/2} J^{(2)}_{\nu+n}(x;q) \\ &= h^{\nu}_{n}\left(\frac{1}{x};q\right) J^{(2)}_{\nu}(x;q) - h^{\nu+1}_{n-1}\left(\frac{1}{x};q\right) J^{(2)}_{\nu-1}(x;q).\end{aligned}$$

We shall use the notation $h^{\nu}_{n}(s;q) := h_{n,\nu}(s;q)$ instead of the original one in [**6**]. Letting $x = 2\omega_k = j_{\nu-1,k}(q)$ in (2.1), one gets

$$\begin{aligned}(2.2)\quad & q^{n\nu+n(n-1)/2} J^{(2)}_{\nu+n}(j_{\nu-1,k}(q);q) \\ &= h^{\nu}_{n}\left(\frac{1}{j_{\nu-1,k}(q)};q\right) J^{(2)}_{\nu}(j_{\nu-1,k}(q);q)\end{aligned}$$

and (1.2) takes a simpler form

$$\begin{aligned}(2.3)\quad & \mathcal{E}_q(x;i\omega_k)\left(-q\omega_k^2;q^2\right)_\infty \omega_k^{\nu} \\ &= \frac{(q;q)_\infty}{(q^\nu;q)_\infty} J^{(2)}_{\nu}(j_{\nu-1,k}(q);q) \sum_{n=0}^{\infty} i^n \left(1-q^{\nu+n}\right) \\ &\quad \times q^{n(2-4\nu-n)/4}\, h^{\nu}_{n}\left(\frac{1}{j_{\nu-1,k}(q)};q\right) C_n(x;q^\nu|\,q).\end{aligned}$$

The orthogonality relation for the q-Lommel polynomials established in [**6**] has the form

(2.4)
$$\left(1+(-1)^{m+n}\right)\sum_{k=1}^{\infty}\frac{A_k\left(\nu\right)}{j^2_{\nu-1,k}\left(q\right)}h_n^{\nu}\left(\frac{1}{j_{\nu-1,k}\left(q\right)};q\right)h_m^{\nu}\left(\frac{1}{j_{\nu-1,k}\left(q\right)};q\right) = \frac{q^{n\nu+n(n-1)/2}}{1-q^{\nu+n}}\delta_{mn}, \quad \nu>0,$$

where the jumps of the step function $A_k\left(\nu\right)$ can be found from the partial fraction decomposition

$$\sum_{k=1}^{\infty}A_k\left(\nu\right)\frac{z}{j^2_{\nu-1,k}\left(q\right)-z^2}=\frac{J_{\nu}^{(2)}\left(z;q\right)}{J_{\nu-1}^{(2)}\left(z;q\right)}. \tag{2.5}$$

We shall also write the orthogonality relation (2.4) in terms of the Lebesgue integral with respect to the corresponding purely discrete measure $\mu=\mu\left(s\right)$, $s=s_k=1/j_{\nu-1,k}\left(q\right)$ with the only cluster point of its bounded support at the origin $s=0$.

The q-trigonometric system $\left\{\mathcal{E}_q\left(x;i\omega_k\right)\right\}_{k=-\infty}^{\infty}$ is complete on $(-1,1)$ if it is closed [**1**]. Suppose that this system is not closed. Thus, there exists at least one function $\chi\left(x\right)\in\mathcal{L}^2_{\rho}\left(-1,1\right)$, not identically zero, that

$$\int_{-1}^{1}\chi\left(x\right)\mathcal{E}_q\left(x;i\omega_k\right)\rho\left(x\right)\,dx=0, \quad k=\pm1,\pm2,\dots, \tag{2.6}$$

where $\rho\left(x\right)$ is the weight function in the orthogonality relation for the continuous q-ultraspherical polynomials (1.4); see, for example, [**3**]. It can be shown that the series in (1.2) and (2.3) are uniformly convergent in x on compacts. Substituting (2.2) in (2.6) one gets

$$\sum_{n=0}^{\infty}i^n\left(1-q^{\nu+n}\right)\ q^{n(2-4\nu-n)/4}\ h_n^{\nu}\left(\frac{1}{j_{\nu-1,k}\left(q\right)};q\right) \times\int_{-1}^{1}\chi\left(x\right)\ C_n\left(x;q^{\nu}|\,q\right)\ \rho\left(x\right)\,dx=0 \tag{2.7}$$

and our goal is to show that

$$\int_{-1}^{1}\chi\left(x\right)\ C_n\left(x;q^{\nu}|\,q\right)\ \rho\left(x\right)\,dx=0 \tag{2.8}$$

for all $n = 0, 1, 2, \ldots$, which means that $\chi(x) = 0$ almost everywhere due to the completeness of the system of the continuous q-ultraspherical polynomials $C_n(x; q^\nu | q)$ in $\mathcal{L}^2_\rho(-1,1)$. In our review paper [**17**] this had been done with the help of the orthogonality relation of the q-Lommel polynomials (2.4); see also Remark 1 below. This method requires the change of the order of summation in the corresponding double series which can be justified for $0 < \nu < 1/2$ only. It does not cover the orthogonal case when $\nu = 1/2$. (See [**17**] for the details). Our main objective in the present paper is to fulfill this gap and to give an independent proof of Bustoz and Suslov's result on the completeness of the basic trigonometric system [**2**].

We propose another approach here. Let us rewrite (2.7) as

$$\sum_{n=0}^{\infty} c_n \; \varphi_n(s) = 0, \tag{2.9}$$

where

$$\varphi_n(s) = \left(1 - q^{\nu+n}\right)^{1/2} q^{-n\nu/2 - n(n-1)/4} \; h_n^\nu(s; q), \qquad s = \frac{1}{j_{\nu-1,k}(q)} \in Supp\, \mu \tag{2.10}$$

and

(2.11)
$$c_n = i^n \left(1 - q^{\nu+n}\right)^{1/2} q^{(1-2\nu)n/4} \int_{-1}^{1} \chi(x) \; C_n(x; q^\nu | q) \; \rho(x) \, dx.$$

The $\{\varphi_n(s)\}_{n=0}^{\infty}$ is an orthonormal system with respect to the purely discrete measure μ in the orthogonality relation (2.4) for the q-Lommel polynomials.

Parseval's identity for the continuous q-ultraspherical polynomials,

$$\sum_{n=0}^{\infty} \frac{(q;q)_n \; (1-q^{\nu+n})}{(q^{2\nu};q)_n \; (1-q^\nu)} \left| \int_{-1}^{1} \chi(x) \; C_n(x; q^\nu | q) \; \rho(x) \, dx \right|^2 = 2\pi \frac{\left(q^\nu, q^{\nu+1}; q\right)_\infty}{\left(q, q^{2\nu}; q\right)_\infty} \int_{-1}^{1} |\chi(x)|^2 \; \rho(x) \, dx, \tag{2.12}$$

and the limit comparison tests show that

$$\sum_{n=0}^{\infty} \left| c_n \right|^2 < \infty \tag{2.13}$$

when $0 < \nu \leq 1/2$, and our goal is to prove that $c_n = 0$ for all $n = 0, 1, 2, \ldots$. Also, by the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{n=0}^{\infty} \left| c_n \right| \leq & \left(\sum_{n=0}^{\infty} \left(1 - q^{\nu+n}\right) q^{(1-2\nu)n/2} \right)^{1/2} \\ & \times \left(\sum_{n=0}^{\infty} \left| \int_{-1}^{1} \chi\left(x\right) C_n\left(x; q^{\nu} \,|\, q\right) \rho\left(x\right) \, dx \right|^2 \right)^{1/2} \end{aligned} \tag{2.14}$$

and both series on the righthand side converge when $0 < \nu < 1/2$. As a result,

$$\sum_{n=0}^{\infty} \left| c_n \right| < \infty \tag{2.15}$$

when $0 < \nu < 1/2$ and we arrive at the following lemma.

Lemma 1. *The* $\{c_n\}_{n=0}^{\infty} \in l^1$ *for* $0 < \nu < 1/2$ *and the* $\{c_n\}_{n=0}^{\infty} \in l^2$ *for* $0 < \nu \leq 1/2$.

Consider the partial sums

$$f_m\left(s\right) = \sum_{n=0}^{m} c_n \; \varphi_n\left(s\right), \tag{2.16}$$

$$g_m\left(s\right) = \sum_{n=0}^{m} \left| c_n \right| \; \left| \varphi_n\left(s\right) \right|. \tag{2.17}$$

In view of (2.9),

$$\lim_{m \to \infty} f_m\left(s\right) = 0 \tag{2.18}$$

pointwise for all $s \in E = Supp\,\mu$. Moreover, a simple estimate using the Cauchy-Schwarz inequality,

$$\begin{aligned} g_m(s) &= \sum_{n=0}^{m} |c_n| \; |\varphi_n(s)| \\ &\leq \left(\sum_{n=0}^{m} |c_n|^2 \right)^{1/2} \left(\sum_{n=0}^{m} |\varphi_n(s)|^2 \right)^{1/2} \\ &\leq \left(\sum_{n=0}^{\infty} |c_n|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} |\varphi_n(s)|^2 \right)^{1/2}, \end{aligned} \tag{2.19}$$

shows that the sequence $\{g_m(s)\}_{m=0}^{\infty}$ is bounded for all $s \in E = Supp\,\mu$. The series

$$\sum_{n=0}^{\infty} |\varphi_n(s)|^2 = \frac{1}{\mu(s)} \tag{2.20}$$

here can be summed by the dual orthogonality relation for the q-Lommel polynomials [**7**] for all $s \in E = Supp\,\mu$. The monotonicity property

$$0 \leq g_m(s) \leq g_{m+1}(s), \quad m = 0, 1, 2, \dots \tag{2.21}$$

and (2.19) show that this sequence converges

$$\lim_{m \to \infty} g_m(s) = g(s) < \infty \tag{2.22}$$

pointwise for all $s \in E = \mathrm{Supp}\,\mu$.

Our next step is to prove that this limiting function $g(s)$ is integrable in the sense $g(s) \in \mathcal{L}^1(\mu)$ for $0 < \nu \leq 1/2$. Indeed, when $0 < \nu < 1/2$ one can write

$$\begin{aligned} \int_E g_m(s)\; d\mu &= \sum_{n=0}^{m} |c_n| \int_E |\varphi_n(s)|\; d\mu \\ &\leq \sum_{n=0}^{m} |c_n| \left(\int_E |\varphi_n(s)|^2\; d\mu \right)^{1/2} \left(\int_E d\mu \right)^{1/2} \\ &\leq \sum_{n=0}^{\infty} |c_n| < \infty \end{aligned} \tag{2.23}$$

(see (2.15)), and by the Lebesgue monotone convergence theorem [**10, 12**]

$$\begin{aligned}\int_E g(s)\, d\mu &= \int_E \lim_{m\to\infty} g_m(s)\, d\mu \\ &= \lim_{m\to\infty} \int_E g_m(s)\, d\mu < \infty. \end{aligned} \tag{2.24}$$

We could also use Beppo Levi's theorem [**9, 11**] here.

On the other hand, inequality (2.19) together with (2.20) give

$$\int_E g_m(s)\, d\mu \le \left(\sum_{n=0}^{\infty} |c_n|^2\right)^{1/2} \int_E \frac{d\mu}{\mu^{1/2}(s)}. \tag{2.25}$$

In general it does not look easy to give an estimate for the last integral, but luckily in the orthogonal case $\nu = 1/2$ the measure of the corresponding q-Lommel polynomials $h_n^{1/2}(s;q)$ can be found explicitly [**15**] as

$$\begin{aligned}\int_E d\mu &= \left(1 - q^{1/2}\right) \sum_{k=-\infty}^{\infty} \frac{1}{\kappa(\omega_k)\ \omega_k^2} \\ &= 2\left(1 - q^{1/2}\right) \sum_{k=1}^{\infty} \frac{1}{\kappa(\omega_k)\ \omega_k^2} = 1, \end{aligned} \tag{2.26}$$

where

$$\kappa(\omega) = \sum_{n=0}^{\infty} \frac{q^{n/2}}{1 + \omega^2 q^n}$$

and, therefore,

$$\int_E \frac{d\mu}{\mu^{1/2}(s)} = 2\left(1 - q^{1/2}\right)^{1/2} \sum_{k=1}^{\infty} \frac{1}{\kappa^{1/2}(\omega_k)\ |\omega_k|} < \infty \tag{2.27}$$

in view of the asymptotic relations $|\omega_k|\,\kappa(\omega_k) \sim$ *constant* and $|\omega_k| \sim q^{3/4-|k|}$ as $|k| \to \infty$; (see, for example, [**15, 16**]). Thus, (2.24) holds also when $\nu = 1/2$, and we have proven the following result.

Lemma 2. *The* $g(s) \in \mathcal{L}^1(\mu)$ *for* $0 < \nu \leq 1/2$.

By the Riesz-Fisher theorem [**9, 11, 12**], the series (2.16) converges in the mean square to some element $f \in \mathcal{L}^2(\mu)$,

$$\lim_{m \to \infty} \int_E |f_m(s) - f(s)|^2 \, d\mu = 0 \tag{2.28}$$

and

$$c_n = \int_E f(s) \ \varphi_n(s) \, d\mu, \tag{2.29}$$

$$\sum_{n=0}^{\infty} |c_n|^2 = \int_E |f(s)|^2 \, d\mu \tag{2.30}$$

where $E = \mathit{Supp}\, \mu$. In the case of the finite measure under consideration, the convergence in the mean square in (2.28) implies convergence in the mean,

$$\lim_{m \to \infty} \int_E |f_m(s) - f(s)| \, d\mu = 0, \tag{2.31}$$

and our final step is to prove that $f(s) \equiv 0$ for all $s \in E = \mathit{Supp}\, \mu$. In principle, this can be done using the fact that convergence in the mean implies convergence in measure, which, in turn, implies the existence of a convergent subsequence [**9, 10**]

$$f_{m_k}(s) \longrightarrow f(s), \tag{2.32}$$

which converges to zero due to (2.9), but these arguments involve convergence almost everywhere, and in our degenerate case of a purely discrete measure this condition should be replaced by convergence everywhere on the support of the measure.

On second thought, using the facts established above, we can give a direct proof interchanging the limit and integral in (2.31). The following inequalities hold

$$\begin{aligned} |f_m(s) - f(s)| &\leq |f_m(s)| + |f(s)| \\ &\leq g_m(s) + |f(s)| \leq g(s) + |f(s)| \end{aligned} \tag{2.33}$$

for all $m = 0, 1, 2, \dots$ due to the monotonicity of $g_m(s)$. Since $g(s) \in \mathcal{L}^1(\mu)$, $f(s) \in \mathcal{L}^2(\mu)$ (and, therefore, $f(s) \in \mathcal{L}^1(\mu)$ in the case of the finite measure under consideration), the hypotheses of the Lebesque dominated convergence theorem [**9, 10, 12**] are satisfied and

$$\begin{aligned} 0 &= \lim_{m\to\infty} \int_E |f_m(s) - f(s)|\ d\mu = \int_E \lim_{m\to\infty} |f_m(s) - f(s)|\ d\mu \\ &= \int_E \left| \lim_{m\to\infty} f_m(s) - f(s) \right| d\mu = \int_E |f(s)|\ d\mu. \end{aligned}$$

But the measure $\mu(s)$ is purely discrete. Therefore $f(s) \equiv 0$ for all $s \in E = Supp\,\mu$ and $c_n = 0$ for all $n = 0, 1, 2, \cdots$ in view of (2.29) or (2.30). This completes the proof in the first case.

The second case, when $\omega_k = 1/2 j_{\nu,k}(q)$, $k = 0, \pm 1, \pm 2, \dots$, can be considered in a similar manner. The analog of (2.9) is

$$\sum_{n=0}^{\infty} c_n\ \phi_{n-1}(s) = 0, \tag{2.34}$$

where

$$\phi_{n-1}(s) = \varphi_n(s)\big|_{n\to n-1, \nu\to\nu+1} \tag{2.35}$$

and c_n are given by (2.11). For $0 < \nu \leq 1/2$ the same arguments show that $c_n = 0$, for all $n = 1, 2, 3, \dots$. Moreover, $c_0 = 0$ because condition (2.6) holds for $\omega_0 = 0$ in this case. The measure of the q-Lommel polynomials $h_n^{3/2}(s; q)$ for the corresponding orthogonal case $\nu = 1/2$ has been found explicitly in [**15**]. This completes the proof. □

Remark 1. In view of (2.9) and (2.16) one can also write formally

$$0 = \int_E \left(\lim_{m\to\infty} f_m(s) \right) \varphi_n(s)\ d\mu = \lim_{m\to\infty} \int_E f_m(s)\ \varphi_n(s)\ d\mu = c_n.$$

Interchanging of the limit and the integral here can be justified by the Lebesgue dominated convergence theorem when $0 < \nu < 1/2$ due to

$$\begin{aligned} \int_E |f_m(s)\ \varphi_n(s)|\ d\mu &\leq \int_E g_m(s)\ |\varphi_n(s)|\ d\mu \\ &= \sum_{k=0}^{m} |c_k| \int_E |\varphi_k(s)\ \varphi_n(s)|\ d\mu \\ &\leq \sum_{k=0}^{m} |c_k| \leq \sum_{k=0}^{\infty} |c_k| < \infty \end{aligned}$$

(see (2.15)). This consideration is equivalent to justification of interchanging of the order of summation in [**17**].

Remark 2. The q-Lommel polynomials $\{h_n^{\nu}(s;q)\}_{n=0}^{\infty}$ form a complete set in $\mathcal{L}^2(\mu)$ when $\nu > 0$. Our method shows that the corresponding dual system $\{h_n^{\nu}(s_k;q)\}_{k=-\infty}^{\infty}$, where $s_k = 1/j_{\nu-1,k}(q) \in Supp\,\mu$, is closed in l^1 for $\nu > 0$. Moreover, in view of (2.25) this system is closed for the dual l^2 if

$$\int_E \frac{d\mu}{\mu^{1/2}(s)} < \infty.$$

We were able to verify the last condition for $\nu = 1/2, 3/2$ when the corresponding measures are known explicitly. This result can be generalized to other systems of discrete orthogonal polynomials.

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REFERENCES

1. N.I. Akhiezer, *Theory of approximation*, Frederick Ungar Publ. Co., New York, 1956.

2. J. Bustoz and S.K. Suslov, *Basic analog of Fourier series on a q-quadratic grid*, Methods Appl. Anal. **5** (1998), 1–38.

3. G. Gasper and M. Rahman, *Basic hypergeometric series*, Cambridge Univ. Press, Cambridge, 1990.

4. R.W. Gosper, Jr. and S.K. Suslov, *Numerical investigation of basic Fourier series*, in *q-Series from a contemporary perspective* (M.E.H. Ismail, D.R. Stanton, eds.), Contemp. Math., vol. 254, Amer. Math. Soc., Providence, RI, 2000, pp. 199–227.

5. M.E.H. Ismail, *The basic Bessel functions and polynomials*, SIAM J. Math. Anal. **12** (1981), 454–468.

6. ———, *The zeros of basic Bessel functions, the functions $J_{\nu+ax}(x)$, and associated orthogonal polynomials*, J. Math. Anal. Appl. **86** (1982), 1–19.

7. ———, *Orthogonality and completeness of q-Fourier type systems*, preprint.

8. M.E.H. Ismail and R. Zhang, *Diagonalization of certain integral operators*, Adv. Math. **108** (1994), 1–33.

9. A.N. Kolmogorov and S.V. Fomin, *Introductory real analysis*, Dover, New York, 1970.

10. J.N. McDonald and N.A. Weiss, *A course in real analysis*, Academic Press, New York, 1999.

11. F. Riesz and B. Sz.-Nagy, *Functional analysis*, Dover, New York, 1990.

12. W. Rudin, *Principles of mathematical analysis*, 3rd ed., McGraw-Hill, New York, 1976.

13. S.K. Suslov, *"Addition" theorems for some q-exponential and q-trigonometric functions*, Methods Appl. Anal. **4** (1997), 11–32.

14. ———, *Another addition theorem for the q-exponential function*, J. Phys. A: Math. Gen. **33** (2000), L375–L380.

15. ———, *Some expansions in basic Fourier series and related topics*, J. Approx. Theory, **115** (2002), 289–353.

16. ———, *Asymptotics of zeros of basic sine and cosine functions*, J. Approx. Theory **121** (2003), 292–335.

17. ———, *Basic exponential functions on a q-quadratic grid*, in *"Special functions 2000: Current perspective and future directions"* (J. Bustoz, M.E.H. Ismail and S.K. Suslov, eds.), NATO Sci. Ser. II: Math., Phys., Chem., vol. 30, Kluwer Acad. Publ., Dordrecht, 2001, pp. 411–456.

18. ———, *Completeness of basic trigonometric system in $\mathcal{L}^p$*, in *"q-Series with applications to combinatorics, number theory, and physics"* (B.C. Berndt and K. Ono, eds.), Contemp. Math., vol. 291, Amer. Math. Soc., Providence, RI, 2001, pp. 229–241.

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