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PSEUDOSPHERE ARRANGEMENTS WITH SIMPLE COMPLEMENTS

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ABSTRACT. A necessary and sufficient condition for the complements of a general kind of topological sphere arrangement (and those of all its subarrangements) to have homologically trivial components is that all intersection degeneracies be nonnegative in a certain sense. Transverse intersections are not assumed. This extends the domain of application of formulas which count these components in terms of degeneracies. In dimension 2 we show that the union of pseudocircles in an arrangement with possibly nontransverse intersections is the same as the union of an arrangement with only transverse intersections.

1. Introduction and main result. Formulas which count the regions into which $\mathbf{R}^n$ is subdivided by a collection of Euclidean hyperplanes in general position have long been known, see, e.g., [**1**], but neat formulas when the hyperplanes are not in general position are of surprisingly recent vintage [**5, 6**]. It turns out that such formulas apply more generally to certain arrangements of topological spheres. The formulas follow from additivity of the Euler characteristic once it is established that complement components of the arrangement and its subarrangements have the homology of a point. We will characterize such arrangements.

We consider indexed families of subsets of an n-sphere which intersect topologically like Euclidean spheres in a certain sense. Denote the index set $\{1,\dots,k\}$ by $[k]$.

Suppose A_i is a closed subset of a topological n-sphere $\mathcal{S}^n$ for each $i\in[k]$, and let $I\subseteq[k]$. We denote $\cap_{i\in I}A_i$ by A_I for $I\neq\varnothing$ and set $A_\varnothing:=\mathcal{S}^n$. If each A_I (and, in particular, each A_i, $i\in[k]$) is either a single point or is homeomorphic to a sphere of some dimension, we call the indexed family $\mathcal{A}=\{A_i : i\in[k]\}$ a *pseudosphere arrangement* in $\mathcal{S}^n$. (The empty set is a sphere of dimension -1.)

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If A_I is a topological sphere of dimension $r \geq -1$, we will define the corresponding *degeneracy index* d_I to be $r - (n - |I|)$, and if A_I is a point, we set $d_I = \infty$. Note that we do not assume that the spheres intersect transversally, nor do we make any additional topological assumptions, e.g., cellular decompositions, differentiability.

In our discussion, $\tilde{H}_*$ denotes reduced singular homology and $\tilde{H}^*$ reduced Čech cohomology.

Suppose $\mathcal{A}$ is a pseudosphere arrangement in $\mathcal{S}^n$ and $I, J \subseteq [k]$ with A_I not a point, and $I \cap J = \varnothing$. Set

$$\mathcal{A}_{I,J} = \{A_I \cap A_j : j \in J\}.$$

Then $\mathcal{A}_{I,J}$ is a pseudosphere arrangement in A_I which we will refer to as a subarrangement of $\mathcal{A}$. In particular, $\mathcal{A} = \mathcal{A}_{\varnothing,[k]}$.

If $\mathcal{A}$ is a pseudosphere arrangement in $\mathcal{S}^n$, we define $C(\mathcal{A}) := \mathcal{S}^n \setminus \cup_1^k A_i$. Let us call an arrangement *complement simple* if each (nonempty) component, K, of $C(\mathcal{A})$ has the homology of a point, i.e., $\tilde{H}_q(K) = 0$ for all q, and *fully complement simple* if $\mathcal{A}_{I,J}$ is complement simple for every disjoint $I, J \subseteq [k]$ with A_I not a point. Then we have our main result.

Theorem 1. *The pseudosphere arrangement $\mathcal{A}$ is fully complement simple if and only if $d_I \geq 0$ for all $I \subseteq [k]$.*

The condition that $d_I \geq 0$ for any singleton set $I = \{i\}$ implies that each A_i is either a single point, a topological sphere of dimension $n-1$ or the whole sphere $\mathcal{S}^n$, this last fact being a consequence of the Brouwer invariance of domain theorem.

Note that being fully complement simple entails more than having the arrangement induce a cell decomposition. For example, the arrangement of 1-spheres in $\mathbf{R}^2 \cup \{\infty\} \approx \mathcal{S}^2$ shown in Figure 1 is not fully complement simple. □

Some examples of pseudosphere arrangements satisfying $d_I \geq 0$ for all $I \subseteq [k]$ are the following.

1. Let $\mathcal{A}$ be a pseudosphere arrangement in $\mathcal{S}^n$ for which A_i is a topological $(n-1)$-sphere containing the north pole N and for each

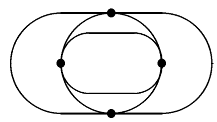


FIGURE 1.

$I \subseteq [k]$, either $A_I = N$ or A_I is a topological sphere of dimension $\geq n - |I| \geq 0$. The image of $\mathcal{A}$ in $\mathbf{R}^n$, via stereographic projection, is called an arrangement of pseudohyperplanes in [**2**]. Arrangements of Euclidean hyperplanes are a special case.

2. An arrangement of k topological $(n-1)$-spheres in *general position* in $\mathcal{S}^n$, i.e., $d_I = 0$ for $|I| \leq n$ and $d_I = -1 - n + |I|$ for $|I| > n$.

If $\mathcal{A} = \{A_i : i \in [k]\}$ is a fully complement simple sphere arrangement and $A_I = \{p\}$ for some point $p \in \mathcal{S}^n$, we will call p a *pole* of $\mathcal{A}$ of order $r = \min\{|I| : A_I = \{p\}\}$. Thus a sphere arrangement corresponding to an arrangement of pseudohyperplanes in $\mathbf{R}^n$ with no common (finite) point has a pole at N and no others. On the other hand, an arrangement of spheres in general position will have no poles. The condition $d_I \geq 0$ imposes restrictions on poles as in the following proposition.

Proposition 1. *Let $\mathcal{A} = \{A_i : i \in [k]\}$ be a fully complement simple arrangement in $\mathcal{S}^n$, and suppose p is a pole of order $r \leq n-1$. Then $p \in A_i$ for all $i \in [k]$.*

Proof. Suppose $A_I = \{p\}$ with $|I| = r$. Now suppose $l \notin I$ and $p \notin A_l$, and let $J = I \cup \{l\}$. Then $A_J = \varnothing$ and $|J| \leq n$ so $d_J = -1-(n-|J|) < 0$ and the arrangement is not fully complement simple. $\square$

It will be useful to have some conditions equivalent to our basic condition

(D) $$d_I \geq 0 \quad \text{for all} \quad I \subseteq [k].$$

In fact, (D) is equivalent to each of the following.

(C1) For all $I \subseteq [k]$ and $j \in [k]$ for which none of A_I, A_j, $A_{I\cup\{j\}}$ is a point, either $A_I \subseteq A_j$ or $\dim(A_{I\cup\{j\}}) = \dim(A_I) - 1$.

(C2) For all $I \subseteq [k]$ with A_I not a point, there exists $I' \subseteq I$ such that $A_I = A_{I'}$ and $d_{I'} = 0$.

To see this, assume (D) holds. Then remembering that $A_\varnothing = \mathcal{S}^n$, the conditions in (C1) and (C2) certainly hold for $|I| = 0$. Suppose they hold whenever $|I| < l$, and let $|I| = l$, e.g., $I = J \cup \{i\}$ for some J with $|J| = l - 1$. By induction, we can choose J' so that $A_J = A_{J'}$ and $d_{J'} = 0$. Apply (C1) to J (or J') and i: If $A_J \subseteq A_i$, then $A_I = A_J = A_{J'}$ so we can take $I' = J'$ to satisfy (C2). Otherwise, choose $I' = J' \cup \{i\}$, for then $A_{I'} = A_I$ and $0 = d_{J'} = \dim(A_{J'}) - (n - |J'|) = (\dim(A_{I'}) + 1) - (n - (|I'| - 1)) = d_{I'}$ so we see the condition in (C2) holds when $|I| = l$. To see that the condition in (C1) holds when $|I| = l$, choose I' as in (C2). If $A_I \not\subseteq A_j$, then the topological sphere $A_{I\cup\{j\}}$ is strictly contained in A_I so, by the Brouwer invariance of domain theorem $\dim(A_{I\cup\{j\}}) < \dim(A_I)$. But then, by (D) and $d_{I'} = 0$, $\dim(A_{I\cup\{j\}}) = \dim(A_{I'\cup\{j\}}) \geq n - (|I'| + 1) = \dim(A_{I'}) - 1 = \dim(A_I) - 1$. On the other hand, it is easy to verify that each of (C1) and (C2) imply (D).

We will need the Alexander duality theorem, see [**3**]: $\tilde{H}_q(\mathcal{S}^n \setminus K) \approx \tilde{H}^{n-q-1}(K)$.

Proof of Theorem 1. First note that both conditions in the statement are inherited by subarrangements: If $\mathcal{A}$ is fully complement simple, then so is any $\mathcal{A}_{I,J}$, immediately from the definition. On the other hand, the condition (C1), equivalent to (D), holds in any $\mathcal{A}_{I,J}$ with the index set I in place of $[k]$.

Write $(n', k') \prec (n, k)$ when either $n' < n$, or $n' = n$ and $k' < k$. Equivalence of the two conditions in the theorem is clear for $n = 1$. When $k = 1$, equivalence follows from Alexander duality which implies that $\mathcal{S}^n \setminus A_1$ will have homologically trivial components if and only if either A_1 is a point, $A_1 \approx S^n$ (so $A_1 = \mathcal{S}^n$), or $A_1 \approx \mathcal{S}^{n-1}$. Assume the result has been established for all $(n', k') \prec (n, k)$ and that $n \geq 2$, $k \geq 2$.

Suppose $\mathcal{A}$ is a pseudosphere arrangement in $\mathcal{S}^n$ indexed by $[k]$ and satisfying (D). By equivalent condition (C1) with $I = \varnothing$, each A_i is either a point or a sphere with $\dim(A_i) \geq n-1$. If any $A_i = \mathcal{S}^n$ we can remove it without affecting (D), so the conclusion follows by induction. If any A_i is a point it must be contained in all the other A_j's (since if

$A_i \cap A_j = \varnothing$ for some $j, n \geq 2$ would imply $d_{\{i,j\}} = -1-(n-2) < 0$). If the A_i's are all the same point there is nothing to prove. Hence we can assume that at least one of the A_i's, say A_1 has dimension $n-1$.

All proper subarrangements are fully complement simple by the induction hypothesis and the hereditary nature of condition (D). We need only show that the components of $C(\mathcal{A})$ have the homology of a point, and for this it suffices, via Alexander duality, to show $\tilde{H}^q(\cup_1^k A_i) = 0$ when $q < n-1$.

Set

$$(1)\quad X = A_1, \quad Y = \cup_2^k A_i, \quad A'_i = A_1 \cap A_{i+1} \quad \text{for } i = 1, \dots, k-1.$$

Then $X \cap Y = \cup_1^{k-1} A'_i$ and, by induction, $\mathcal{A}_{\{1\},\{2,\dots,k\}} = \{A'_1, \dots, A'_{k-1}\}$ is fully complement simple so that $\tilde{H}^q(X \cap Y) = 0$ for $q < n-2$. Also, by induction, $\tilde{H}^q(Y) = 0$ for $q < n-1$, while $\tilde{H}^q(X) = 0$ for $q < n-1$ since $X \approx \mathcal{S}^{n-1}$. Then the exactness of the Mayer-Vietoris sequence for $\tilde{H}^*$,

$$\longrightarrow \tilde{H}^{q-1}(X \cap Y) \longrightarrow \tilde{H}^q(X \cup Y) \longrightarrow \tilde{H}^q(X) \oplus \tilde{H}^q(Y) \longrightarrow,$$

implies $\tilde{H}^q(X \cup Y) = 0$ for $q < n-1$.

Now suppose that $\mathcal{A}$ is fully complement simple, and let $I \subseteq [k]$. We show $d_I \geq 0$. If any A_i is a point, it must be contained in all the other A_j's (since $A_i \cap A_j = \varnothing$ for some $j, n \geq 2$ would imply that $\mathcal{S}^n \backslash (A_i \cup A_j)$ is not homologically trivial) and if the A_i's are all the same point there is nothing to prove. So we can assume that $I = \{1, \dots, l\}$ and that $A_1 \approx \mathcal{S}^{n-1}$. Define A'_i as in (1), replacing k by l. The arrangement $\mathcal{A}_{\{1\},\{2,\dots,l\}} = \{A'_1, \dots, A'_{l-1}\}$ is fully complement simple, so by induction, $\dim(\cap_1^{l-1} A'_i) \geq (n-1)-(l-1)$. But $\cap_1^l A_i = \cap_1^{l-1} A'_i$ so $\dim(A_I) \geq n-l$. $\square$

Following [**6, 7**] we consider the lattice of intersections of the arrangement $\mathcal{A}$. We assume here that $A_i \approx \mathcal{S}^{n-1}$ for all i. Set $L = L[\mathcal{A}] = \{A_I : I \subseteq [k]\}$. We order L by reverse inclusion, so that for $s = A_I$ and $t = A_J$, $s \leq t$ in L means $A_J \subseteq A_I$. Remembering that $A_\varnothing = \mathcal{S}^n$ by definition, we denote this element of L by 0. Let $L' = \{s \in L : s \text{ is not a point}\}$.

For $s \in L'$, let $\lambda(s) := n - \dim(s)$, so if $s = A_I$ we have $d_I = |I| - \lambda(s)$. If $r \in L'$ and s, t are in the order interval $[0, r]$ in L then $s, t \in L'$. Note that $s \vee t$ is the intersection, $s \cap t$, of the sets s, t, while $s \wedge t$ is the intersection of all members of L containing both s and t. Our condition (D) is then equivalent to condition

(M) $r \in L'$, $s, t \in [0, r] \Rightarrow \lambda(s \vee t) + \lambda(s \wedge t) \leq \lambda(s) + \lambda(t)$ (equivalently $\dim(s) + \dim(t) \leq \dim(s \vee t) + \dim(s \wedge t)$).

To see that (D) implies (M), note that this is clear for $n = 1$ and assume it is true for $n - 1$. If $\dim(s \wedge t) < n$, then $s = A_K$ for some $|K| < k$ with $\dim(s) < n$ and (D) holds as well in the subarrangement $\mathcal{A}_{K,[k]\setminus K}$. By the induction hypothesis, (M) holds in the lattice of this subarrangement, in which s and t appear. Also $s \wedge t$ in the subarrangement lattice has dimension no larger than $s \wedge t$ in the original lattice, so the conclusion of (M) follows.

Now suppose $\dim(s \wedge t) = n$. We can write $s = A_I$, $t = A_J$, with $\dim(s) = n - |I|$ and $\dim(t) = n - |J|$ and with $I \cap J = \varnothing$. Now $s \cap t = A_{I \cup J}$ so $d_{I \cup J} \geq 0$ implies $\dim(s \vee t) = \dim(s \cap t) \geq n - |I| - |J|$, which gives the conclusion of (M).

Conversely, assume (M). Let $s = A_I$ and $t = A_l$ be in L. If $s \subseteq t$, then $\dim(A_{I \cup \{l\}}) = \dim(A_I)$. Otherwise, $s \wedge t = \mathcal{S}^n$, and (M) gives $\dim(A_{I \cup \{l\}}) \geq \dim(A_I) - 1$. Thus condition (C1), equivalent to (D), holds.

Let $\mu(s, t)$ denote the Möbius function of the lattice L; see, e.g., [**4**]. Then

$$\mu(0, t) = \sum_{A_I = t} (-1)^{|I|}.$$

To see this, we need only verify the two defining conditions for $\mu(0, t)$, namely, i) $\mu(0, 0) = 1$ and ii) $\sum_{0 \leq s \leq t} \mu(0, s) = 0$. Condition (i) is obvious; we now check ii). Let $J = \{i : t \subseteq A_i\}$. Then $A_I = s \leq t$ if and only if $I \subseteq J$. Thus

$$\sum_{0 \leq s \leq t} \mu(0, s) = \sum_{0 \leq s \leq t} \sum_{A_I = s} (-1)^{|I|} = \sum_{0 \leq A_I \leq t} (-1)^{|I|} = \sum_{I \subseteq J} (-1)^{|I|} = 0,$$

the last equality following from the binomial theorem.

Since we have seen, for $s \in L'$, that the order interval $[0, s]$ is a semimodular lattice with rank function λ, [**4**, Proposition 3.10.1] implies that $\mu(0, t)$ alternates in sign.

2. Counting complement components. Let $\chi(A)$ denote the Euler characteristic of A, and let $\tilde{\chi} = \chi - 1$ be the reduced Euler characteristic. If $\mathcal{A}$ is a sphere arrangement, then additivity of χ, i.e., $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$, and likewise $\tilde{\chi}$, gives

$$\tilde{\chi}\Big(\bigcup_{i=1}^{k} A_i\Big) = \sum_{I \neq \varnothing} (-1)^{|I|+1} \tilde{\chi}(A_I). \tag{2}$$

By Alexander duality,

$$\chi(\mathcal{S}^n \setminus K) = (-1)^{n-1} \tilde{\chi}(K) + 1. \tag{3}$$

Now $\tilde{\chi}(A_I) = (-1)^r = (-1)^{d_I + n - |I|}$ when $A_I \approx \mathcal{S}^r$ and $\tilde{\chi}(A_I) = 0$ if A_I is a point. Thus, recalling that $d_\varnothing = 0$, we have $\chi(C(\mathcal{A})) = \sum_{I \neq \varnothing,\, d_I < \infty} (-1)^{d_I} + 1 = \sum_{d_I < \infty} (-1)^{d_I}$.

See [**2**] for topological details.

If $\mathcal{A}$ is a complement simple arrangement, then $\chi(C(\mathcal{A}))$ is the number of components, $n(\mathcal{A})$, of $C(\mathcal{A})$ so we have the following generalization of formulas of Winder and Zaslavsky [**5, 6**].

Corollary 1. *If $\mathcal{A}$ is a complement simple pseudosphere arrangement, then*

$$n(\mathcal{A}) = \sum_{d_I < \infty} (-1)^{d_I}.$$

Remark. If $\mathcal{A}$ is a pseudosphere arrangement for which the counting formula in Corollary 1 holds for each subarrangement, then in fact we must have $d_I \geq 0$ for every $I \subseteq [1:k]$. For suppose $d_I < 0$, where say, $I = \{1, 2, \dots, l\}$, then there is some index $j \in [2:l]$ such that $\dim(A_{[1:j]}) \leq \dim(A_{[1:j-1]}) - 2$ so A_j has codimension at least 2 in the sphere $A_{[1:j-1]}$. But $A_{[1:j-1]} \setminus A_j$ then has only one component. But the counting formula applied to the subarrangement $\mathcal{A}_{[1:j-1],\{j\}}$ would give the number of complement components as $(-1)^{d_\varnothing} + (-1)^{d_{\{j\}}} \neq 1$.

In special cases, like pseudohyperplane arrangements or spheres in general position, explicit formulas for $n(\mathcal{A})$ in terms of n and k can be found by elementary combinatorial arguments.

If a sphere arrangement is fully complement simple, we can write a formula for the number of m dimensional spherical "faces" of the arrangement. By Corollary 1,

$$n(\mathcal{A}_{I,J}) = \sum_{\substack{d_I<\infty \\ K\subseteq J}} (-1)^{d_K}(-1)^{n-\dim(A_I)},$$

so the number of m dimensional faces is given by

$$\sum_{\substack{\dim(A_I)=m \\ I\cap J=\varnothing}} n(\mathcal{A}_{I,J}) = (-1)^{n-m} \sum_{I:\dim(A_I)=m} \sum_{\substack{d_K<\infty \\ K\subseteq I}} (-1)^{d_K}.$$

From this formula one can derive Buck's formula, in [**1**], for faces of hyperplane arrangements in general affine position.

Suppose we have a fully complement simple sphere arrangement $\mathcal{A}$ in $\mathcal{S}^n$ with $A_i \approx \mathcal{S}^n$ for each i. Then, using the results and notation of the last section,

$$\begin{aligned} n(\mathcal{A}) &= \sum_{d_I<\infty} (-1)^{d_I} = \sum_{t\in L'} \sum_{A_I=t} (-1)^{|I|-\lambda(t)} \\ &= (-1)^n \sum_{t\in L'} \mu(0,t)(-1)^{\dim(t)} = \sum_{t\in L} |\mu(0,t)|. \end{aligned}$$

This last expression is Zaslavsky's formula [**6**] in our more general context.

We now consider, as in [**6, 7**], the number of bounded components of the complement in $\mathbf{R}^n$ of $\cup A_i$ where A_i are pseudohyperplanes as we defined them earlier. Thus we suppose $\mathcal{A}$ is a fully complement simple sphere arrangement and that there is a point $N \in \mathcal{S}^n$ with $N \in A_i$ for each i. Call a component C of $\mathcal{S}^n \setminus \cup A_i$ *bounded* if N is not a boundary point of C, and let B denote $\cup\{\text{closure}(C) : C \text{bounded}\}$, so $N \notin B$. Suppose also that the following condition holds:

(T1) There is a contractible neighborhood $U \subset \mathcal{S}^n$ of N, disjoint from B, such that $\chi(A_I \setminus U) = 1$ for each I satisfying $d_I < \infty$.

This condition at N is clearly satisfied in the Euclidean case. Let $B_i = A_i \setminus U$, so also $B_I = A_I \setminus U$. It follows that $\chi(B_I) = 1$ for all I

with $d_I < \infty$, and $\chi(B_I) = 0$ otherwise. Applying (2) with χ in place of $\tilde{\chi}$ to the B_is, along with (3), gives

$$\chi(\mathcal{S}^n \setminus \cup B_i) = (-1)^{n-1} \sum_{\substack{I \neq \varnothing \\ d_I < \infty}} (-1)^{|I|+1} + 1 + (-1)^n.$$

Now the components of $\mathcal{S}^n \setminus \cup B_i$ consist of the bounded components of $\mathcal{S}^n \setminus \cup A_i$ together with the connected open set $V = \cup\{C : C \text{ unbounded}\} \cup U$. Then the Euler characteristic of the union of the bounded components is

$$\chi(\mathcal{S}^n \setminus \cup B_i) - \chi(V) = (-1)^n \sum_{d_I < \infty} (-1)^{|I|} + 1 - \chi(V). \tag{4}$$

Now suppose additionally

(T2) For each unbounded component, C, $C \cup U$ is homologically trivial.

This rules out (in the Euclidean case) what are called in [**6**] *relatively bounded* components (which arise from parallelism). If we have an *essential* arrangement of Euclidean hyperplanes, that is, one in which some collection of hyperplanes intersect in a (finite) point, we will show that (T2) holds.

As pointed out in [**6**], there is some topological subtlety in proving the formula for the number of bounded components even in the Euclidean case.

With (T2) it follows from additivity of χ that $\chi(V) = 1$, so with homological triviality of the bounded components, (4) yields the number of bounded components:

$$n_b = \Big| \sum_{d_I < \infty} (-1)^{|I|} \Big| = \Big| \sum_{t \in L} \mu(0, t) \Big|.$$

The middle term is the bounded analog of Winder's formula; the last is Zaslavsky's formula.

Here is an argument that $C \cup U$ is homologically trivial when C is an unbounded component of an essential affine Euclidean arrangement in $\mathbf{R}^n$.

(i) First, the assumption that $\mathcal{A}$ is essential together with elementary linear algebra shows that no line is contained in C.

(ii) Next we show that $C \subset \mathbf{R}^n$ contains a half-line. If not, take a point x in the open set C and a sphere S within C centered at x. For each $y \in S$, extend the line from x to y maximally in closure (C). By assumption, this line meets the boundary of C at a point $f(y)$. Then $g(y) := \|x - f(y)\|$ is continuous, so bounded on S. C is star-shaped with respect to x so it follows that C is bounded, a contradiction.

(iii) Now let $u \in \mathbf{R}^n$ be such that there is a half-line in C parallel to u. Then C is a union of half-lines parallel to u. For suppose $x, y \in C$ and l_x, l_y are lines through x, y parallel to u. These lines do not lie entirely in any A_i and l_x meets A_i if and only if l_y does. Thus $l_x \cap C$ is bounded if and only if $l_y \cap C$ is bounded, so by (ii) all $l_x \cap C$ are unbounded, and by (i) they are all half-lines.

Finally, let $U := \{x \in \mathbf{R}^n : \|x\| > M\}$ together with the north pole N, where M is chosen so that U only meets unbounded components. We will give a deformation retract of $C \cup U$ to $U \cup (\partial U \cap C)$ which is contractible, showing that $C \cup U$ is homologically trivial. Let u be as in (iii). Each $x \in C \setminus U$ lies on a line $l_x \| u$ which meets ∂U in a point $F(x)$. Let $d(x) = \|x - F(x)\|$ and parametrize $C \setminus U$ by $x \sim (d(x), F(x))$. Then $H(x, r) = ((1-r)d(x), F(x))$ for $x \in C \setminus U$ and $r \in [0,1]$ gives the required deformation retract.

3. Nontransverse intersections. Suppose $\mathcal{A}$ is a sphere arrangement in $\mathcal{S}^n$ with $\dim(A_i) = n-1$ for all i. $A_1, A_2 \in \mathcal{A}$ will be said to intersect *transversally* if A_1 meets both components of the complement of A_2 (and consequently A_2 meets both components of A_1^c). If $A_1 \cap A_2$ is a point, then the intersection must be nontransverse, but if this intersection is a sphere of dimension ≥ 0, then it may be either transverse or nontransverse. If whenever two spheres in $\mathcal{A}$ or in any of its subarrangements intersect in a sphere of dimension ≥ 0 that intersection is transverse, we will call $\mathcal{A}$ a *transverse* arrangement.

Theorem 2. *If $\mathcal{A}$ is an arrangement of k topological circles in $\mathcal{S}^2$, then there is a transverse arrangement $\mathcal{B}$ of k topological circles in $\mathcal{S}^2$ such that $\cup_{A \in \mathcal{A}} A = \cup_{B \in \mathcal{B}} B$.*

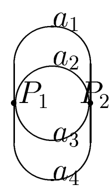


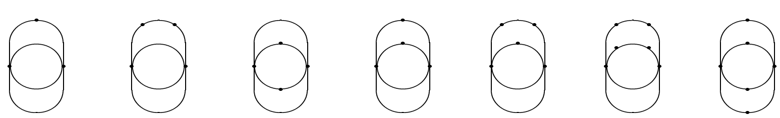
FIGURE 2.

Proof. Suppose $A_1, A_2 \in \mathcal{A}$ intersect nontransversally in two points. We can represent them schematically, with no loss of generality, as in Figure 2.

Here A_1 consists of the (open) arcs a_1, a_4 and A_2 consists of arcs a_2, a_3, each together with points $\{P_1, P_2\}$. Let B_1 be the topological circle consisting of arcs a_1, a_3 and let B_2 consist of a_2, a_4, each together with $\{P_1, P_2\}$. Then B_1, B_2 intersect transversally. We now show that, if A_1, A_2 are replaced by B_1, B_2, the resulting collection is still a sphere arrangement, i.e., for any $A_3 \in \mathcal{A}$, $A_3 \cap B_1$ and $A_3 \cap B_2$ are sets with 0, 1 or 2 points, and moreover, the number of nontransitive intersections among $A_3 \cap B_1$ and $A_3 \cap B_2$ is the same as the number of nontransverse intersections among $A_3 \cap A_1$ and $A_3 \cap A_2$. We can then repeat this procedure until there are no nontransverse intersections.

We need to consider several cases. First suppose that A_3 does not contain P_1 or P_2. We will indicate the possible intersections with A_3 as follows. Let x_i be the number (0,1 or 2) of intersections of A_3 with the arc a_i (so that $x_1 + x_4 \leq 2$, $x_2 + x_3 \leq 2$) and denote the corresponding case by $x_1x_2x_3x_4$, as illustrated in Figure 3.

Note that $y_1y_2y_3y_4$ and $y_2y_1y_4y_3$ describe essentially the same case since this permutation amounts to representing A_2 by a_1a_4 and A_1



1000 2000 0110 1100 2100 2200 1111

FIGURE 3.

by a_2a_3 and this is topologically equivalent, on the 2-sphere, to the representation of A_1 by a_1a_4 and A_2 by a_2a_3. Likewise, $y_1y_2y_3y_4$ is equivalent to $y_4y_3y_2y_1$. In each of the cases in the figure above, as well as the case 0000, it is easy to check that A_3 has acceptable intersections with B_1 and B_2. Notice that in the case 0110, A_3 intersects A_2 nontransversally in an $\mathcal{S}^0$ but the intersections of B_1 and B_2 with A_3 are both points, i.e., the replacement introduces poles.

Now these cases and those equivalent to them are the only ones possible. The other $x_1x_2x_3x_4$ satisfying $x_1 + x_4 \leq 2$, $x_2 + x_3 \leq 2$, namely 1010, 2010, 1101, 1201, 2020 and their equivalents, cannot occur since in each of these cases, there is an arc a_i not meeting A_3 which intervenes between two arcs containing points of A_3.

Now suppose that A_3 contains exactly one of the points P_i, say P_1. The other points in which A_3 meets $A_1 \cup A_2$ are then on the arcs a_1, a_2, a_3, a_4, and we can indicate the (equivalence classes) of possibilities for these additional intersections by $x_1x_2x_3x_4$ where now $x_1 + x_4 \leq 1$ and $x_2 + x_3 \leq 1$. The cases 0000,1000, and 1100 are easily checked to result in acceptable intersections of A_3 with B_1 and B_2, while the other, 1010, is not possible topologically.

Finally, if A_3 contains both P_1 and P_2, it is clear that A_3 has acceptable intersections with B_1, B_2. $\square$

It is clear from the proof that the transverse arrangement $\mathcal{B}$ constructed above is unique. From consideration of the 0110 case we see that the lattice structure of the arrangement $\mathcal{A}$ is generally not isomorphic to that of $\mathcal{B}$ and in fact the replacement of A_1, A_2 by B_1, B_2 can introduce new poles even if the arrangement is fully complement simple. If we have an arrangement of pseudolines the configuration 0110 cannot occur but the lattice of the transverse rearrangement $\mathcal{B}$ may still not be isomorphic to that of $\mathcal{A}$. For example, suppose A_1, A_2 are as in Figure 2 with the pole N identified with P_1. Let A_3 be a pseudocircle through P_1 having intersection pattern 1000 as above, and let A_4 be a pseudocircle through P_1 having intersection pattern 0001. In the original arrangement, the two-fold intersections which equal $\{P_1\}$ involve the circle pairs $\{2,3\}, \{3,4\}$ and $\{1,4\}$ while, in the transverse rearrangement, the circle pair intersections equaling $\{P_1\}$ are $\{2,3\}, \{3,4\}$ and $\{2,4\}$ in which only three circles appear.

The analog of Theorem 2 does not hold for $n = 3$. Consider the following three topological $\mathcal{S}^2$s, A_1, A_2, A_3 in $\mathbf{R}^3 \subseteq \mathcal{S}^3$: a plane, $A_1 = \{(x, y, z) : z = 0\}$, a truncated elliptical cylinder, $A_2 = \{(x, y, z) : x^2/4 + y^2 = 1, -2 < z < 3\} \cup \{(x, y, z) : x^2/4 + y^2 \leq 1, z \in \{-2, 3\}\}$ and a cone, $A_3 = \{(x, y, z) : x^2 + y^2 = (z+1)^2, -1 \leq z < 4\} \cup \{(x, y, z) : x^2 + y^2 \leq 25, z = 4\}$. Then A_1, A_2, A_3 intersect transversally in topological $\mathcal{S}^1$s, A_{12}, A_{23}, A_{13} and the threefold intersection, A_{123}, is an $\mathcal{S}^0$. But the arrangement induced by A_2, A_3 on A_1 has nontransverse intersections. In fact, all the induced arrangements are nontransverse. Moreover, by taking one of the two points in A_{123} as $N \in \mathcal{S}^3$ we can interpret this example as showing that there is a transverse arrangement of three pseudoplanes in $\mathbf{R}^3$ such that every induced subarrangement of pseudolines is nontransverse. Consideration of the $\mathcal{S}^1$s in this arrangement shows that $\cup A_i$ cannot be written as a union of three topological $\mathcal{S}^2$s in a transverse arrangement.

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