ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 4, Winter 2003

REFINED ARITHMETIC, GEOMETRIC AND HARMONIC MEAN INEQUALITIES

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Dedicated to Mari Mercer, in loving memory

ABSTRACT. We obtain refinements of the arithmetic, geometric, and harmonic mean inequalities. A main ingredient is Hadamard's inequality. In an application, we obtain a refined version of Ky Fan's inequality.

1. Preliminaries. For $n \geq 2$, let $x_1, x_2, \dots, x_n$ be positive numbers, and let $w_1, w_2, \dots, w_n$ be positive weights: $\sum w_j = 1$. We denote by

$$A = \sum_{j=1}^{n} w_j x_j, \quad G = \prod_{j=1}^{n} x_j^{w_j}, \quad H = \left(\sum_{j=1}^{n} \frac{w_j}{x_j}\right)^{-1},$$

the (weighted) arithmetic, geometric, and harmonic means of the x_j's.

It is well known that

$$H \leq G \leq A,$$

with the inequalities being strict unless all x_j's are equal.

In this paper we obtain various refinements, including upper and lower bounds for $A-G$, $A-H$, A/G and G/H. An important ingredient in our approach is the following.

Hadamard's inequality. *Let f be a concave function on $[a, b]$. Then*

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq f\left(\frac{a+b}{2}\right).$$

Received by the editors on July 25, 2000, and in revised form on August 15, 2001.

2. Results.

Proposition 1. *The following estimates hold, with equality occurring if and only if all x_j's are equal.*

$$\sum_{j=1}^{n} \frac{w_j(x_j - G)^2}{x_j + \max(x_j, G)} \le A - G \le \sum_{j=1}^{n} \frac{w_j(x_j - G)^2}{x_j + \min(x_j, G)}.$$

Proof. For $x > 0$, we have

$$x - 1 - \log(x) = \int_1^x \frac{t-1}{t}\, dt.$$

The integrand is concave and so Hadamard's inequality yields

$$\frac{(x-1)^2}{2x} \le x - 1 - \log(x) \le \frac{(x-1)^2}{x+1} \quad \text{for } x > 1,$$

and

$$\frac{(x-1)^2}{x+1} \le x - 1 - \log(x) \le \frac{(x-1)^2}{2x} \quad \text{for } 0 < x \le 1.$$

Equalities occur only for $x = 1$.

Substituting x_j/G for x, multiplying by w_j and summing, we obtain

$$\frac{1}{G} \sum_{x_j > G} \frac{w_j(x_j - G)^2}{2x_j} \le \sum_{x_j > G} w_j \left(\frac{x_j}{G} - 1 - \log\left(\frac{x_j}{G}\right)\right) \le \frac{1}{G} \sum_{x_j > G} \frac{w_j(x_j - G)^2}{x_j + G}$$

and

$$\frac{1}{G} \sum_{x_j \le G} \frac{w_j(x_j - G)^2}{x_j + G} \le \sum_{x_j \le G} w_j \left(\frac{x_j}{G} - 1 - \log\left(\frac{x_j}{G}\right)\right) \le \frac{1}{G} \sum_{x_j \le G} \frac{w_j(x_j - G)^2}{2x_j}$$

respectively.

Taken together, these inequalities read

$$\frac{1}{G}\sum_{j=1}^{n}\frac{w_j(x_j-G)^2}{x_j+\max(x_j,G)}\le\frac{A}{G}-1\le\frac{1}{G}\sum_{j=1}^{n}\frac{w_j(x_j-G)^2}{x_j+\min(x_j,G)},$$

as desired. □

Remarks 1.1. Observing only that the integral is nonnegative leads to a proof of the arithmetic-geometric mean inequality $0\le A-G$, cf., [**6**, Section 6.7]. Also, Proposition 1 improves

$$\frac{1}{2\max(x_j)}\sum_{j=1}^{n}w_j(x_j-G)^2\le A-G\le\frac{1}{2\min(x_j)}\sum_{j=1}^{n}w_j(x_j-G)^2,$$

which is proved in [**7**]. The lefthand inequality above is due to Alzer [**3**].

Applying the same technique, but instead substituting x_j/A and H/x_j for x respectively, we obtain the following two results.

Proposition 2. *We have*

$$\frac{1}{A}\sum_{j=1}^{n}\frac{w_j(x_j-A)^2}{x_j+\max(x_j,A)}\le\log(A)-\log(G)\le\frac{1}{A}\sum_{j=1}^{n}\frac{w_j(x_j-A)^2}{x_j+\min(x_j,A)},$$

with equality occurring if and only if all x_j's are equal.

Proposition 3. *We have*

$$\sum_{j=1}^{n}\frac{w_j}{x_j}\frac{(x_j-H)^2}{H+\max(x_j,H)}\le\log(G)-\log(H)\le\sum_{j=1}^{n}\frac{w_j}{x_j}\frac{(x_j-H)^2}{H+\min(x_j,H)},$$

with equality occurring if and only if all x_j's are equal.

Again, using an argument similar to the proof of Proposition 1, but beginning with a different function, we obtain the following.

Proposition 4. *The following estimates hold, with equality occurring if and only if all x_j's are equal.*

$$\sum_{j=1}^{n} w_j (x_j - H)^2 \frac{x_j + 2H + \max(x_j, H)}{(x_j + \max(x_j, H))^2} \le A - H \le \sum_{j=1}^{n} w_j (x_j - H)^2 \frac{x_j + 2H + \min(x_j, H)}{(x_j + \min(x_j, H))^2}.$$

Proof. For $x > 0$ we have

$$x - 2 + \frac{1}{x} = \int_1^x \frac{t^2 - 1}{t^2}\, dt.$$

The integrand is concave, and Hadamard's inequality yields

$$(x-1)^2 \frac{x+1}{2x^2} \le x - 2 + \frac{1}{x} \le (x-1)^2 \frac{x+3}{(x+1)^2} \quad \text{for } x > 1,$$

and

$$(x-1)^2 \frac{x+3}{(x+1)^2} \le x - 2 + \frac{1}{x} \le (x-1)^2 \frac{x+1}{2x^2} \quad \text{for } 0 < x \le 1.$$

Equalities occur only for $x = 1$.

Now we proceed as before. Substitute x_j/H, or H/x_j, for x, multiply by w_j, and sum. □

Remark 4.1. These estimates improve

$$\frac{1}{2\max(x_j)} \sum_{j=1}^{n} w_j (x_j - H)^2 \le A - H,$$

which is obtained in [**7**].

3. An application. Here we further restrict the x_j's to be $\le 1/2$, and let $y_j = 1 - x_j$. We denote by A' $(= 1 - A)$ and G' the (weighted) arithmetic and geometric means of the y_j's. The following result is well known, e.g., [**4, 9**], and Proposition 5 below is a refinement.

Ky Fan's inequality. *We have*

$$\frac{A'}{G'} \le \frac{A}{G},$$

with equality occurring if and only if all of the x_j*'s are equal.*

Proposition 5. *If not all of the* x_j*'s are equal, then we have*

$$\frac{A'}{G'} < \left(\frac{A}{G}\right)^q,$$

where $q < 1$ *is given by*

$$q = \frac{A}{1-A}\frac{\sum_{j=1}^n w_j(x_j - A)^2/(2 - x_j - \max(x_j, A))}{\sum_{j=1}^n w_j(x_j - A)^2/(x_j + \max(x_j, A))}.$$

Proof. Applying the righthand inequality of Proposition 2 to the y_j's and the lefthand inequality to the x_j's, we obtain

$$\log(A'/G') \le \frac{1}{A'}\left(\sum_{y_j \le A'} \frac{w_j(y_j - A')^2}{2y_j} + \sum_{y_j > A'} \frac{w_j(y_j - A')^2}{y_j + A'}\right),$$

and

$$\frac{1}{A}\left(\sum_{x_j > A} \frac{w_j(x_j - A)^2}{2x_j} + \sum_{x_j \le A} \frac{w_j(x_j - A)^2}{x_j + A}\right) \le \log(A/G).$$

Taking the quotient of these estimates together with some manipulations yields

$$\frac{\log(A'/G')}{\log(A/G)} \le q,$$

as desired.

That $q < 1$ follows from $A/(1-A) < 1$, together with $x_j + \max(x_j, A) \leq 2 - x_j - \max(x_j, A)$, (with at least one of these inequalities being strict). $\square$

Remarks 5.1. The argument above clearly implies the weaker refinement

$$\left(\frac{A'}{G'}\right)^{A'} < \left(\frac{A}{G}\right)^{A}.$$

Also, using Proposition 3, one can obtain bounds for $(G'/H')/(G/H)$ in a similar way and, using Propositions 1 and 4, one can obtain bounds for $(A'-G')/(A-G)$ and $(A'-H')/(A-H)$, respectively. The interested reader may consult [**1, 2, 8, 9**] as well.

Acknowledgment. The author is grateful to Rex Mercer for valuable suggestions and encouragement.

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