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QUASINORMAL OPERATORS AND REFLEXIVE SUBSPACES

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ABSTRACT. In this note we will show the reflexivity of a weak*-closed subspace generated by any proper subset of $\{(T^*)^n, T^n : n \in \mathbf{N}\}$, if T is a quasinormal operator. The main tool is a direct integral of subspaces of operators. Relation between the reflexive closure of a direct integral of subspaces and a direct integral of reflexive closure of subspaces is given.

1. Introduction. All operators discussed in this note are bounded and act on a separable Hilbert space $\mathcal{H}$. We write $B(\mathcal{H})$ for the collection of such operators. By $P(\mathcal{H})$ we will denote the lattice of orthogonal projections on $\mathcal{H}$. The expression *subspace* of operators is reserved for those linear submanifolds in $B(\mathcal{H})$ which are weak*-closed. If $\mathcal{S}$ is any family of operators then $\overline{span}(\mathcal{S})$ is the subspace generated by $\mathcal{S}$ and weak*-closed and $\mathcal{W}(\mathcal{S})$ is the weak*-closed algebra generated by $\mathcal{S}$ and identity. Let $T \in B(\mathcal{H})$, denote by $\mathcal{P}_T$ the set $\{T^{(n)} : n \in \mathbf{Z}\}$, where $T^{(n)} = T^n$ if $n \geq 0$ and $T^{(n)} = (T^*)^{-n}$ if $n < 0$.

We recall that the *reflexive closure* of a subspace $\mathcal{S} \subset B(\mathcal{H})$ is the set

$$\operatorname{ref} \mathcal{S} = \{A \in B(\mathcal{H}) : Af \in \overline{\mathcal{S}f} \text{ for all } f \in \mathcal{H}\}.$$

A subspace $\mathcal{S}$ is said to be *reflexive* if $\operatorname{ref} \mathcal{S} = \mathcal{S}$. If $\mathcal{S}$ is an algebra the definition coincides with the well-known definition of a reflexive algebra.

An operator T is called *quasinormal* if T commutes with T^*T. In [**7**] Wogen proved that every quasinormal operator T is reflexive, i.e., the weak operator topology-closed algebra generated by T and the identity operator is reflexive. In [**2**] reflexive subspaces of all Toeplitz operators

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on the Hardy space H^2 were completely characterized. In particular, it was shown that the subspace $\overline{span}(\mathcal{P}_S)$, where S is a unilateral shift, is not reflexive. However, a subspace generated by any proper subset of $\mathcal{P}_S$ is reflexive. Now consider a quasinormal operator T. One cannot expect the reflexivity of $\overline{span}(\mathcal{P}_T)$. In this note we will show the reflexivity of a subspace generated by any proper subset of $\mathcal{P}_T$.

2. Direct integral of subspaces. Let us recall a definition of direct integral (see [**1**] and [**5**]). Suppose that $(\Lambda, \mathcal{M}, \mu)$ is a measure space such that Λ is a separable metric space and μ is a $\sigma-$ finite complete regular Borel measure on Λ. For $A \in L^\infty(\mu, B(\mathcal{H}))$, let M_A denote the operator on $L^2(\mu, \mathcal{H})$ defined by $(M_A f)(\lambda) = A(\lambda) f(\lambda)$. We will identify the operator A with M_A and call it *decomposable*. If $A(\lambda) = a(\lambda) I$ for all $\lambda \in \Lambda$, where $a(\lambda) \in \mathbf{C}$, then the operator A is *diagonal*. We will denote by $\mathcal{D}$ the algebra of all diagonal operators and call it the *diagonal algebra*.

Remark 1. Let us note that the results from [**1**] and [**5**] are easily adopted to weak*-closed (ultrastrongly-closed) algebras and subspaces. The easiest way to see this is to note that the strong operator and ultrastrong topologies agree on the infinite ampliation $B(\mathcal{H})^{(\infty)}$. Thus ultrastrong totality of $\{A_n\}$ in $\mathcal{S}$ is equivalent to strong totality of $\{A_n^{(\infty)}\}$ in $\mathcal{S}^{(\infty)}$.

We will be using very often an embedding trick, thus, for simplicity, we will be identifying subspace $L^2(\mu, \mathcal{H} \oplus \mathcal{H})$ with $L^2(\mu, \mathcal{H}) \oplus L^2(\mu, \mathcal{H})$ and elements of $L^\infty(\mu, B(\mathcal{H} \oplus \mathcal{H}))$ with operators on $L^2(\mu, \mathcal{H}) \oplus L^2(\mu, \mathcal{H})$.

Consider now a weak*-closed subspace $\mathcal{S}$ consisting of decomposable operators on $L^2(\mu, \mathcal{H})$. Assume that we can choose a countable generating set $\{A_n\} \subset L^\infty(\mu, B(\mathcal{H}))$ for $\mathcal{S}$ and fix Borel representatives $\lambda \mapsto A_n(\lambda)$ for their decomposition. For each $\lambda \in \Lambda$ we define $\mathcal{S}(\lambda)$ as $\overline{span}\{A_n(\lambda) : n \in \mathbf{N}\}$. Then the subspace $\mathcal{S}$ is *decomposable* and the *decomposition of* $\mathcal{S}(\lambda)$ *over* Λ *with respect to* $\mathcal{D}$ sometimes called *direct integral* denoted by $\int_\Lambda^\oplus \mathcal{S}(\lambda)\, d\mu(\lambda)$ is the set

$$\{M_A : A \in L^\infty(\mu, B(\mathcal{H})), A(\lambda) \in \mathcal{S}(\lambda) \text{ a.e.}\}.$$

Note that in order to see that the above definition of decomposable subspace is correct, it is necessary to show that up to a set of measure zero the subspaces $\{\mathcal{S}(\lambda)\}$ are independent of choice of generators. The definition of direct integral of subspaces is parallel to the definition in [**1**] of direct integral of algebras, where the authors showed that the definition for the algebra case does not depend on the choice of generators.

Let us choose sets $\{A_n\}, \{B_n\}$ generating $\mathcal{S}$ and fix Borel functions $\lambda \mapsto A_n(\lambda)$ and $\lambda \mapsto B_n(\lambda)$ for their decompositions. Define $\mathcal{S}(\lambda)$ as $\overline{span}\{A_n(\lambda) : n \in \mathbf{N}\}$. To show that decomposition of $\mathcal{S}$ is independent of the choice of generators it is enough to prove that $B_n(\lambda) \in \mathcal{S}(\lambda)$ for almost all $\lambda \in \Lambda$. Denote by $\mathcal{A}$ the algebra $\left\{\begin{pmatrix} aI & S \\ 0 & aI \end{pmatrix} : a \in \mathbf{C}, S \in \mathcal{S}\right\} \subset B(\mathcal{H}\oplus\mathcal{H})$. Since $\mathcal{S}$ as a subspace has a countable number of generators $\{A_n\}$, thus $\tilde{A}_n = \begin{pmatrix} 0 & A_n \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ generate $\mathcal{A}$ as an algebra. It means that $\mathcal{A}$ is decomposable (see [**1**]) and there exists $\int_\Lambda^\oplus \mathcal{A}(\lambda)\, d\mu(\lambda)$, where $\mathcal{A}(\lambda) = \left\{\begin{pmatrix} a(\lambda)I & S(\lambda) \\ 0 & a(\lambda)I \end{pmatrix} : a(\lambda) \in \mathbf{C}, S(\lambda) \in \mathcal{S}(\lambda)\right\}$. Since B_n generate $\mathcal{S}$ thus $\tilde{B}_n = \begin{pmatrix} 0 & B_n \\ 0 & 0 \end{pmatrix} \in \mathcal{A}$. Notice that $\tilde{B}_n$ is decomposable and $\tilde{B}_n(\lambda) = \begin{pmatrix} 0 & B_n(\lambda) \\ 0 & 0 \end{pmatrix}$. By [**1**, Proposition 3.3], $\tilde{B}_n(\lambda) \in \mathcal{A}(\lambda)$ for almost all $\lambda \in \Lambda$. Hence $B_n(\lambda) \in \mathcal{S}(\lambda)$ almost everywhere. Thus subspaces $\mathcal{S}(\lambda)$ do not depend on the choice of generators.

Let $\mathcal{S}$ be a subspace of $B(\mathcal{H})$ and $\mathcal{D}$ the diagonal algebra. Denote by $[\mathcal{S}, \mathcal{D}] = \overline{span}\{aS : a \in \mathcal{D}, S \in \mathcal{S}\}$.

Proposition 2. *Let $\mathcal{S}$ be a decomposable subspace and $\int_\Lambda^\oplus \mathcal{S}(\lambda)\, d\mu(\lambda)$ be its decomposition with respect to $\mathcal{D}$. If $A = \int_\Lambda^\oplus A(\lambda)\, d\mu(\lambda)$ is a decomposable operator, then the set of λ for which $A(\lambda) \in \mathcal{S}(\lambda)$ is measurable. Moreover, the following conditions are equivalent.*

(1) $A \in [\mathcal{S}, \mathcal{D}]$.

(2) $A(\lambda) \in \mathcal{S}(\lambda)$ *for almost all* $\lambda \in \Lambda$.

Proof. Denote by $\mathcal{A} = \left\{\begin{pmatrix} aI & S \\ 0 & aI \end{pmatrix} : a \in \mathcal{C}, S \in \mathcal{S}\right\}$ and by $\tilde{A} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$. Since $\mathcal{A}$ and $\tilde{A}$ are decomposable, then by [**1**, Proposition 3.3] the set of λ for which $\tilde{A}(\lambda) = \begin{pmatrix} 0 & A(\lambda) \\ 0 & 0 \end{pmatrix} \in \mathcal{A}(\lambda)$ is measurable. Note that $A(\lambda) \in \mathcal{S}(\lambda)$ if and only if $\tilde{A}(\lambda) \in \mathcal{A}(\lambda)$. Thus the first conclusion of the proposition is established.

Assume now that $A \in [\mathcal{S}, \mathcal{D}]$. Note that $A \in [\mathcal{S}, \mathcal{D}]$ if and only if $\tilde{A} \in \mathcal{W}(\mathcal{A}, \tilde{\mathcal{D}})$. Then [**1**, Proposition 3.3] implies that $\tilde{A}(\lambda) \in \mathcal{A}(\lambda)$ almost everywhere. Thus $A(\lambda) \in \mathcal{S}(\lambda)$ almost everywhere. Since this chain of reasoning is reversible, the proof is complete.

Let Y be a separable metric space and (Λ, μ) a complete regular σ-finite Borel measure space. Denote by 2^Y a family of closed nonempty subsets of Y. *A multifunction with closed values* is a function $F : \Lambda \to 2^Y$. Function F may be identified with a subset of $\Lambda \times Y$. We will regard a multifunction F as *measurable* if it is a measurable subset of $\Lambda \times Y$. A measurable function $f : \Lambda \to Y$ such that $f(\lambda) \in F(\lambda)$ for each $\lambda \in \Lambda$ is called a *measurable selector for* F. Denote by $C(\mathcal{H})$ the unit ball of $B(\mathcal{H})$. Note that different subspaces of $B(\mathcal{H})$ have different unit balls. Thus we may identify a family of weakly closed subspaces with a subset of $2^{C(\mathcal{H})}$. Let $\{\mathcal{S}(\lambda)\}_{\lambda \in \Lambda}$ be a family of weak*-closed subspaces. The function $\lambda \mapsto \mathcal{S}(\lambda)$ is called *attainable field of subspaces* if there exists a decomposable subspace $\mathcal{S} \subset B(L^2(\mu, \mathcal{H}))$ with decomposition $\int_\Lambda^\oplus \mathcal{S}(\lambda)\, d\mu(\lambda)$.

Before we start the lemma, let us notice that for any $\mathcal{S} \subset B(\mathcal{H})$ an operator $T \in \operatorname{ref} \mathcal{S}$ if and only if for any projections $P, Q \in P(\mathcal{H})$ we have $Q^\perp T P = 0$ whenever $QSP = SP$ for all $S \in \mathcal{S}$.

Lemma 3. *Let $F : \lambda \mapsto \mathcal{S}(\lambda)$ be an attainable field of subspaces. Then ref$F : \lambda \mapsto$ ref$\mathcal{S}(\lambda)$ is measurable.*

Proof. Let for every $\lambda \in \Lambda$ function $\lambda \mapsto A_n(\lambda)$ be Borel and $\{A_n(\lambda)\}$ generate $\mathcal{S}(\lambda)$. Denote by $\mathcal{F}$ the set $\{(\lambda, (P, Q)) \in \Lambda \times P(\mathcal{H}) \times P(\mathcal{H}) : QA_n(\lambda)P = A_n(\lambda)P$ for all $n\}$. Since composition is weak*-continuous on the unit ball of $B(\mathcal{H})$ then $(\lambda, (P, Q)) \mapsto QA_n(\lambda)P - A_n(\lambda)P$ is measurable, thus $\mathcal{F}$ is the measurable subset of $\Lambda \times P(\mathcal{H}) \times P(\mathcal{H})$. By [**1**, Proposition 5.1] there exists $\{(P_k, Q_k)\}_{k=1}^\infty$ a countable dense set of measurable selectors for $\mathcal{F}$. Hence $\operatorname{ref} F = \{(\lambda, B) \in \Lambda \times B(\mathcal{H}) : Q_k^\perp(\lambda) B P_k(\lambda) = 0$ for all $k\}$. Let us consider functions $\omega_k : \Lambda \times C(\mathcal{H}) \ni (\lambda, B) \mapsto Q_k^\perp(\lambda) B P_k(\lambda)$. Since P_k, Q_k are measurable, so is $Q_k^\perp$.

Moreover P_k, Q_k are bounded so the composition is measurable, too. Hence ω_k is measurable. Thus $\operatorname{ref} F = \cap_{k=1}^{\infty} \omega_k^{-1}(\{0\})$ is measurable as well. According to the identification we have made, the lemma is proved. $\square$

Theorem 4. *Let $\mathcal{S}$ be a decomposable subspace and $\int_\Lambda^\oplus \mathcal{S}(\lambda)\, d\mu(\lambda)$ be the decomposition of $\mathcal{S}$ with respect to $\mathcal{D}$. Then $\operatorname{ref}[\mathcal{S},\mathcal{D}] = \int_\Lambda^\oplus \operatorname{ref}\mathcal{S}(\lambda)\, d\mu(\lambda)$. In particular, reflexivity of $[\mathcal{S},\mathcal{D}]$ is equivalent to the reflexivity of almost all $\mathcal{S}(\lambda)$.*

Proof. Since $\mathcal{S}$ is decomposable, the multifunction $\lambda \mapsto \mathcal{S}(\lambda)$ is the attainable field of subspaces. Note now that [**1**, Proposition 5.2], which says that each measurable field of algebras is attainable, remains true with the same proof if we change subalgebras to subspaces. Lemma 3 shows that $\lambda \mapsto \operatorname{ref}\mathcal{S}(\lambda)$ is measurable and hence by [**1**, Proposition 5.2] it is attainable. Therefore there exists a unique subspace $\mathcal{B} = \int_\Lambda^\oplus \operatorname{ref}\mathcal{S}(\lambda)\, d\mu(\lambda)$.

Suppose $B \in \mathcal{B}$. Then $B(\lambda) \in \operatorname{ref}.\mathcal{S}(\lambda)$ almost everywhere. Denote $\left\{\begin{pmatrix} aI & S \\ 0 & aI\end{pmatrix} : a \in \mathbf{C}, S \in \mathcal{S}\right\}$ by $\mathcal{A}$. The next part of the proof is based on the simple fact that $\begin{pmatrix} 0 & B \\ 0 & 0\end{pmatrix} \in \operatorname{Alg}\operatorname{Lat}\mathcal{A}$ if and only if $B \in \operatorname{ref}\mathcal{S}$. To show that $B \in \operatorname{ref}[\mathcal{S},\mathcal{D}]$ it suffices to prove that $\begin{pmatrix} 0 & B \\ 0 & 0\end{pmatrix} \in \operatorname{Alg}\operatorname{Lat}\mathcal{W}(\mathcal{A},\tilde{\mathcal{D}})$, where $\tilde{\mathcal{D}}$ is the appropriate diagonal algebra. By [**1**, Proposition 5.6] we have $\operatorname{Alg}\operatorname{Lat}\mathcal{W}(\mathcal{A},\tilde{\mathcal{D}}) = \int_\Lambda^\oplus \operatorname{Alg}\operatorname{Lat}\mathcal{A}(\lambda)\, d\mu(\lambda)$, where all $\operatorname{Alg}\operatorname{Lat}\mathcal{A}(\lambda)$ are equal to

$$\left\{\begin{pmatrix} a(\lambda)I & T(\lambda) \\ 0 & b(\lambda)I\end{pmatrix} : a(\lambda),\, b(\lambda) \in \mathbf{C}, T(\lambda) \in \operatorname{ref}\mathcal{S}(\lambda)\right\}.$$

Note that $\begin{pmatrix} 0 & B(\lambda) \\ 0 & 0\end{pmatrix} \in \operatorname{Alg}\operatorname{Lat}\mathcal{A}(\lambda)$ for almost all $\lambda \in \Lambda$. Hence by [**1**, Proposition 3.3] $\begin{pmatrix} 0 & B \\ 0 & 0\end{pmatrix} \in \operatorname{Alg}\operatorname{Lat}\mathcal{W}(\mathcal{A},\tilde{\mathcal{D}})$. Thus $B \in \operatorname{ref}[\mathcal{S},\mathcal{D}]$.

Assume now that $B \in \operatorname{ref}[\mathcal{S},\mathcal{D}]$. Let $\mathcal{U}$ denote the weak*-closed algebra spanned by $\left\{\begin{pmatrix} 0 & S \\ 0 & 0\end{pmatrix} : S \in \mathcal{S}\right\}$ and $\tilde{\mathcal{D}}$. Then $\tilde{B} = \begin{pmatrix} 0 & B \\ 0 & 0\end{pmatrix} \in \operatorname{Alg}\operatorname{Lat}\mathcal{U}$. Hence the operator $\tilde{B}$ is decomposable by [**1**, Proposition 5.6] and $\tilde{B} = \int_\Lambda^\oplus \tilde{B}(\lambda)\, d\mu(\lambda)$, where $\tilde{B}(\lambda) \in \operatorname{Alg}\operatorname{Lat}\mathcal{U}(\lambda)$ almost everywhere. Moreover, $\tilde{B}(\lambda) = \begin{pmatrix} a(\lambda)I & B(\lambda) \\ 0 & b(\lambda)I \end{pmatrix}$ for $a(\lambda), b(\lambda) \in \mathbf{C}$ and $B(\lambda) \in \operatorname{ref}\mathcal{S}(\lambda)$ for almost all $\lambda \in \Lambda$. Therefore $B = \int_\Lambda^\oplus B(\lambda)\, d\mu(\lambda)$ and $B \in \int_\Lambda^\oplus \operatorname{ref}\mathcal{S}(\lambda)\, d\mu(\lambda)$. The last sentence of the Theorem is a consequence of Proposition 2.

3. Applications to quasinormal operators. A subspace of operators $\mathcal{S} \subset B(\mathcal{H})$ has *property* $\mathbf{A}_1(1)$ if for all weak*-continuous linear functionals ϕ on $B(\mathcal{H})$ and $\varepsilon > 0$, there are $a, b \in \mathcal{H}$ such that $\|a\| \cdot \|b\| \leq (1+\varepsilon)\|\phi\|$ and $\phi(A) = (Aa, b)$ for all $A \in \mathcal{S}$. If we do not require the control of the norms $\|a\| \cdot \|b\|$, the subspace $\mathcal{S}$ is called *elementary*.

Theorem 3.4 of [**2**] shows that every intransitive subspace $\mathcal{B}$ of all Toeplitz operators in the Hardy space H^2 (i.e., $\operatorname{ref}\mathcal{B} \neq B(H^2)$) is elementary. A careful reader can notice that $\mathcal{B}$ has in fact property $\mathbf{A}_1(1)$. Namely, using the notation from [**2**], take $f \in L^1$ which represents the weak* continuous functional $\hat{f}$ on $\mathcal{B}$ with $\|f\|_1 = 1$. By intransitivity we can fix a non-trivial rank-one operator $g_1 \otimes h_1$ ($g_1, h_1 \in H^2$) in the preannihilator of $\mathcal{B}$ with norm smaller then ε. Corollary 3.3 of [**2**] shows that $\log|f + \alpha g_1 \bar{h}_1| \in L^1$ for almost all α ($|\alpha| = 1$). By [**4**] as it was shown in [**2**, Corollary 3.3] there is $g_2 \otimes h_2$ ($g_2, h_2 \in H^2$) which agree with functional given by f on $\mathcal{B}$. Moreover, by [**4**, p. 48, first sentence], $g_2 \otimes h_2$ can be chosen such that $\|g_2\|_2 \|h_2\|_2 = \|f + \alpha g_1 \bar{h}_1\|_1 \leq 1 + \varepsilon$. Hence we obtained the following

Proposition 5. *Every intransitive weak* or weakly closed subspace of all Toeplitz operators on the Hardy space has property* $\mathbf{A}_1(1)$.

Let ω be a proper subset of the set of all integers and $T \in B(\mathcal{H})$. Denote by $\mathcal{S}(T, \omega)$ the set $\overline{span}\{T^{(n)} : n \in \omega\}$.

Lemma 6. *Assume that V is an isometry and ω is a proper subset of the set of all integers. Then the subspace $\mathcal{S}(V, \omega)$ is reflexive and has property* $\mathbf{A}_1(1)$.

Proof. Let $S \oplus U$ be the Wold decomposition of an isometry V, i.e., S is a shift of certain multiplicity and U is a unitary operator on $\mathcal{H}$.

If the multiplicity is one then [**2**, Theorem 1.1] shows that $\mathcal{S}(S,\omega)$ is reflexive. As to property $\mathbf{A}_1(1)$, the required factorization was shown in Proposition 5. The case of any multiplicity follows from the case of the multiplicity one by Theorem 4 and [**5**, Corollary 3.7]. The subspace $\overline{span}(\mathcal{P}_U)$ has the same properties as a subspace of commutative von Neumann algebra generated by U. Also, by Theorem 4 and [**5**, Corollary 3.7] it is easy to see that

$$\mathcal{S}(S,\omega) \oplus \overline{span}(\mathcal{P}_U)$$

is reflexive and has property $\mathbf{A}_1(1)$ and its subspace $\mathcal{S}(V,\omega)$ has the same properties (see [**5**]).

Theorem 7. *Let $T \in B(\mathcal{H})$ be a quasinormal operator. A weak*-closed subspace generated by any proper subset of $\{(T^*)^n, T^n : n \in \mathbf{N}\}$ is reflexive and has property $\mathbf{A}_1(1)$.*

Proof. Using the notations introduced before Lemma 6, the considered subspace can be denoted by $\mathcal{S}(T,\omega)$.

Let $\mathcal{Z}$ be the commutative von Neumann algebra generated by T^*T. By direct integral theory (see [**5**, Remark 3.1(b)] and [6, Theorem 6, p.19]) $\mathcal{Z}$ can be seen as an algebra of an orthogonal sum of multiplication operators by the scalar function from $L^\infty(\mu_i)$ on the spaces $L^2(\mu_i, \mathcal{H}_i)$, where $(\Lambda_i, \mathcal{M}_i, \mu_i)$ is a complete regular σ-finite Borel measure space and i runs through a countable set. Moreover, every operator in $\mathcal{S}(T,\omega)$ can be seen as an orthogonal sum of the multiplication operators A_i such that $(A_i f)(\lambda_i) = A_i(\lambda_i) f(\lambda_i)$, where $A_i \in L^\infty(\mu_i, B(\mathcal{H}_i))$.

Let us identify T with $\oplus_i T_i(\cdot)$. Since T is quasinormal thus $T_i^*(\lambda_i)T_i(\lambda_i)$ is a scalar multiple of the identity operator for almost all $\lambda_i \in \Lambda_i$. Therefore $T_i(\lambda_i)$ is a scalar multiple of an isometry.

By Lemma 6 we can see that for each $\lambda_i \in \Lambda_i$ the subspace $\mathcal{S}(T_i(\lambda_i),\omega)$ is reflexive and has property $\mathbf{A}_1(1)$. Hence by Theorem 4 and [**5**, Corollary 3.7]

$$\int_{\Lambda_i}^{\oplus} \mathcal{S}(T_i(\lambda_i),\omega)\, d\mu_i(\lambda_i)$$

is reflexive and has property $\mathbf{A}_1(1)$. Therefore, as above,

$$\bigoplus_i \int_{\Lambda_i}^{\oplus} \mathcal{S}(T_i(\lambda_i), \omega)\, d\mu_i(\lambda_i)$$

is reflexive and has property $\mathbf{A}_1(1)$. Thus $\mathcal{S}(T, \omega)$ as a subspace of above has the same properties (by [**5**, Proposition 2.5]).

Corollary 8. *Let T be a quasinormal operator and let k be an integer, then subspaces $\overline{span}\{T^{(n)} : n \geq k\}$ and $\overline{span}\{T^{(n)} : n \neq k\}$ are reflexive and have property $\mathbf{A}_1(1)$.*

Remark 9. Note that Theorem 6 remains true if we consider weak operator topology instead of weak* operator topology.

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