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BOUNDED AND ALMOST PERIODIC ULTRADISTRIBUTIONS AS BOUNDARY VALUES OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. We represent every bounded ultradistribution of Beurling or of Roumieu type on **R** as the boundary value of a holomorphic function. In particular, each almost periodic ultradistribution admits such a representation and we characterize the almost periodic ultradistributions that are boundary values of holomorphic functions in the upper (or lower) half-plane in terms of their spectra.

1. Introduction. The representation of several classes of distributions and ultradistributions as boundary values of holomorphic functions has been studied by several authors. We mention Bengel [**1**], Carmichael [**3, 4**], Luszczki and Zielezny [**10**], Meise [**11**], Petzsche and Vogt [**13**], Tillmann [**15, 16**] and Vogt [**17**].

In a recent paper Pilipović [**14**] showed that appropriate L^∞ estimates on holomorphic functions imply that they have a boundary value which is a bounded ultradistribution. However the methods of [**14**] did not permit to represent every bounded ultradistribution as the boundary value of such a holomorphic function (see [**14**, Section 3]). He worked in the context of ultradistributions as they were defined by Komatsu [**8**] and, for technical reasons, he required the strong nonquasianalytic condition on the defining sequence.

In this paper we show that every bounded ultradistribution on **R** is the boundary value of a holomorphic function on $\mathbf{C}\setminus\mathbf{R}$ satisfying some growth estimates. More precisely, we consider the smallest algebra of holomorphic functions in $\mathbf{C}\setminus\mathbf{R}$ which contains all bounded holomorphic functions in $\mathbf{C}\setminus\mathbf{R}$, all the exponentials $\exp(i\lambda z)$, $\lambda\in\mathbf{R}$, and is stable under differential operators of infinite order. Using results of Petzsche and Vogt [**13**] and an extension of the characterization of bounded

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ultradistribution given by Cioranescu [**6**] (see [**7**]), we show that each function in this class has a bounded ultradistribution as boundary value. Our methods are different from the ones of Pilipović [**14**] and allow us to avoid the strong nonquasianalytic condition. Moreover, we show that the boundary value map is surjective.

Finally we characterize the almost periodic ultradistributions that are boundary values of holomorphic functions in the upper (or lower) half-plane in terms of their spectra. Our approach to ultradistributions is the one of Braun, Meise and Taylor [**2**].

1. Preliminaries. First we introduce the spaces of functions and ultradistributions and most of the notation that will be used in the sequel.

Definition 1. A weight function is an increasing continuous function $\omega : [0,\infty[\to [0,\infty[$ with the following properties:

(α) $L \geq 0$ exists with $\omega(2t) \leq L(\omega(t)+1)$ for all $t \geq 0$,

(β) $\int_1^\infty (\omega(t)/t^2)\, dt < \infty$,

(γ) $\log(t) = o(\omega(t))$ as t tends to ∞,

(δ) $\varphi : t \to \omega(e^t)$ is convex.

For most of the results of the paper we have to replace the condition (α) by the stronger condition

(α_1) $\sup_{\lambda\geq 1} \limsup_{t\to\infty}(\omega(\lambda t)/\lambda\omega(t)) < \infty$.

If the weight ω satisfies the additional condition

(ϵ) $C \geq 1$ exists such that for all $y > 0$

$$\int_1^\infty \frac{\omega(ty)}{t^2}\, dt \leq C\omega(y) + C$$

we will say that ω is a *strong weight*. Examples of weight functions with and without property (ϵ) can be found in [**12**].

The radial extension of ω is $\omega(z) := \omega(|z|)$, $z \in \mathbf{C}$.

The *Young conjugate* φ^* : $[0,\infty[\to \mathbf{R}$ of φ is given by $\varphi^*(s) := \sup\{st - \varphi(t), t \geq 0\}$.

There is no loss of generality to assume that ω vanishes on $[0,1]$. Then φ^* has only nonnegative values and $\varphi^{**} = \varphi$.

Remark. If the weight $\omega(t)$ is concave for t large enough then every equivalent weight satisfies (α_1). See [**13**, 1.1] for details.

Definition 2 [**2**]. Let ω be a weight function. For a compact set $K \subset \mathbf{R}$ we let

$$\mathcal{D}_{(\omega)}(K) := \{f \in \mathcal{D}(K) : \| f \|_{K,\lambda} < \infty \quad \text{for every} \quad \lambda > 0\},$$

and

$$\mathcal{D}_{\{\omega\}}(K) := \{f \in \mathcal{D}(K) : \| f \|_{K,\lambda} < \infty \quad \text{for some} \quad \lambda > 0\},$$

where $\| f \|_{K,\lambda} := \sup_{x\in K} \sup_{n\in\mathbf{N}_0} |f^{(n)}(x)| \exp(-\lambda\varphi^*(n/\lambda))$.

Then $\mathcal{D}_{(\omega)}(K)$, endowed with its natural topology, is a Fréchet space, while $\mathcal{D}_{\{\omega\}}(K)$ is an (LB)-space.

We write $\mathcal{D}_*(K)$, where $*$ can be either (ω) or $\{\omega\}$.

For a fundamental sequence $(\mathrm{K}_j)_{j\in\mathbf{N}}$ of compact subsets of $\mathbf{R}$ we let

$$\mathcal{D}_*(\mathbf{R}) := \mathrm{ind}_{j\to}\mathcal{D}_*(\mathrm{K_j}).$$

The elements of $\mathcal{D}'_{(\omega)}(\mathbf{R})$ (respectively $\mathcal{D}'_{\{\omega\}}(\mathbf{R})$) are called *ultradistributions of Beurling* (*respectively Roumieu*) *type.*

Definition 3 [**7**]. We denote

$$\mathcal{D}_{L^1,\omega,\lambda} := \left\{f \in \mathcal{D}_{L^1}(\mathbf{R}) : |f|_\lambda := \sup_{n\in\mathbf{N}_0} \| f^{(n)} \|_{L^1} \exp\left(-\lambda\varphi^*\left(\frac{n}{\lambda}\right)\right) < \infty\right\}.$$

We define $\mathcal{D}_{L^1,(\omega)}(\mathbf{R}) := \mathrm{proj}_{\leftarrow j}\mathcal{D}_{L^1,\omega,j}$ and $\mathcal{D}_{L^1,\{\omega\}}(\mathbf{R}) := \mathrm{ind}_{j\to}\mathcal{D}_{L^1,\omega,(1/j)}$.

Then $\mathcal{D}_{L^1,(\omega)}(\mathbf{R})$ is a Fréchet space and $\mathcal{D}_{L^1,\{\omega\}}(\mathbf{R})$ is an (LB)-space. As usual, we write $\mathcal{D}_{L^1,*}(\mathbf{R})$ to denote $\mathcal{D}_{L^1,(\omega)}(\mathbf{R})$ or $\mathcal{D}_{L^1,\{\omega\}}(\mathbf{R})$.

The space $\mathcal{D}_*(\mathbf{R})$ is dense in $\mathcal{D}_{L^1,*}(\mathbf{R})$ and the inclusion $\mathcal{D}_*(\mathbf{R}) \subset \mathcal{D}_{L^1,*}(\mathbf{R})$ is continuous.

The elements of $\mathcal{D}'_{L^1,*}(\mathbf{R})$ are known as *bounded $*$-ultradistributions*. In what follows we will always consider $\mathcal{D}'_{L^1,*}(\mathbf{R})$ equipped with the strong topology $\beta(\mathcal{D}'_{L^1,*}(\mathbf{R}), \mathcal{D}_{L^1,*}(\mathbf{R}))$. An element T of $\mathcal{D}'_{L^1,*}(\mathbf{R})$ is said to be *almost periodic* if T is limit in $\mathcal{D}'_{L^1,*}(\mathbf{R})$ of a sequence of trigonometric polynomials.

The classical case $\mathcal{D}_{L^1}(\mathbf{R})$ is formally not a particular case of what we present here since $\omega(t) = \log(1+t)$ does not satisfy property (γ). However, all our results also hold in this case after some modifications.

Let $G \in \mathcal{H}(\mathbf{C})$ be an entire function such that $\log|G(z)| = \mathcal{O}(\omega(|z|))$ (respectively $\log|G(z)| = o(\omega(|z|))$) as $|z|$ tends to infinity. Then

$$T_G(\varphi) := \sum_{n\in\mathbf{N}_0} (-i)^n \, \frac{G^{(n)}(0)}{n!} \, \varphi^{(n)}(0)$$

defines an ultradistribution $T_G \in \mathcal{D}'_{(\omega)}(\mathbf{R})$ (respectively $T_G \in \mathcal{D}'_{\{\omega\}}(\mathbf{R})$) whose support reduces to $\{0\}$. The operator

$$G(D) : \mathcal{D}'_*(\mathbf{R}) \to \mathcal{D}'_*(\mathbf{R}), \qquad G(D)\nu := T_G * \nu$$

is called an *ultradifferential operator* of class $*$. We note that, for every $f \in \mathcal{D}_{L^1,*}(\mathbf{R})$,

$$(G(D)f)(x) = \sum_{n\in\mathbf{N}_0} i^n \, \frac{G^{(n)}(0)}{n!} \, f^{(n)}(x).$$

The Cauchy inequalities and the convexity of φ^* imply that, if $G \in \mathcal{H}(\mathbf{C})$ and $\log|G(z)| = \mathcal{O}(\omega(|z|))$, then $|G^{(n)}(0)/n!| \le C \inf_{r>0} \exp(k\varphi(r) - nr) = C\exp(-k\varphi^*(n/k))$, for some $k \in \mathbf{N}, C > 0$ and all $n \in \mathbf{N}$, whereas, if $\log|G(z)| = o(\omega(|z|))$, then for every $m \in \mathbf{N}$ there is $C_m > 0$ such that $|G^{(n)}(0)/n!| \le C_m \exp(-(1/m)\varphi^*(nm))$. Accordingly, each ultradifferential operator defines a continuous linear operator (see [**7**], 2.4)

$$G(D) : \mathcal{D}'_{L^1,*}(\mathbf{R}) \longrightarrow \mathcal{D}'_{L^1,*}(\mathbf{R}).$$

Definition 4. Let $\omega : [0,\infty[\to [0,\infty[$ be a weight function. Then ω^* is defined by
$$\omega^*(s) := \sup_{t\geq 0}\{\omega(t) - st\}.$$

The function ω^* is continuous, convex, decreasing and $\omega^*(s) < \infty$ for all $s > 0$ since each weight function ω satisfies that $\lim_{t\to\infty}(\omega(t)/t) = 0$ by [**12**, 1.2(b)].

Definition 5. Let ω be a weight function. We define

$$\mathcal{H}_{(\omega^*)} := \{F \in \mathcal{H}(\mathbf{C}\setminus\mathbf{R}) : \sup_{z\in\mathbf{C}\setminus\mathbf{R}} |F(z)|\mathrm{e}^{-n|\mathrm{Im}\, z|-n\omega^*(|\mathrm{Im}\, z|/n)} < \infty \text{ for some } n \in \mathbf{N}\}$$

and

$$\mathcal{H}_{\{\omega^*\}} := \{F \in \mathcal{H}(\mathbf{C}\setminus\mathbf{R}) : \text{ there is } n \in \mathbf{N} \text{ such that } \sup_{z\in\mathbf{C}\setminus\mathbf{R}} |F(z)|\mathrm{e}^{-n|\mathrm{Im}\, z|-(1/k)\omega^*(k|\mathrm{Im}\, z|)} < \infty \text{ for every } k \in \mathbf{N}\}.$$

Then $\mathcal{H}_{(\omega^*)}$ is an (LB)-space and $\mathcal{H}_{\{\omega^*\}}$ is an (LF)-space. We write $\mathcal{H}_{\omega^*}$ to denote either $\mathcal{H}_{(\omega^*)}$ or $\mathcal{H}_{\{\omega^*\}}$.

Proposition 1. *For each $F \in \mathcal{H}_{\omega^*}$ and each ultradifferential operator $G(D)$ of class $*$, we have that $G(D)_x F \in \mathcal{H}_{\omega^*}$.*

Proof. We only consider the Beurling case since the Roumieu case can be treated similarly. Let F be a holomorphic function in $\mathbf{C}\setminus\mathbf{R}$ with
$$|F(z)| \leq C\mathrm{e}^{k\omega^*(|\mathrm{Im}\, z|/k)+k|\mathrm{Im}\, z|}$$
for some $C > 0$ and $k \in \mathbf{N}$. Let $G(D)$ be an ultradifferential operator of class (ω). We put
$$H(x+iy) = G(D)_x F(x+iy), \quad y \neq 0.$$

Then H is holomorphic in $\mathbf{C}\setminus\mathbf{R}$ and $H(z)=\sum_{n=0}^{\infty}a_nF^{(n)}(z)$ where $|a_n|\leq C\mathrm{e}^{-m\varphi^*(n/m)}$ for some constants $C>0$ and $m\in\mathbf{N}$. The Cauchy's inequalities give

$$|F^{(n)}(z)|\leq n!\;C\,\frac{\exp\Big(2k\;|\mathrm{Im}\,z|+k\omega^*(|\mathrm{Im}\,z|/2k)\Big)}{\big(|\mathrm{Im}\,z|/2\big)^n}.$$

For $0<|\mathrm{Im}\,z|<2$, we have, by Stirling's formula and on account that $\varphi^{**}=\varphi$,

$$\begin{aligned}\sum_{n=0}^{\infty}|a_n|\Big(\frac{2}{|\mathrm{Im}\,z|}\Big)^n n! &\leq C\sum_{n=0}^{\infty}n!\,\mathrm{e}^{-m\varphi^*(n/m)}\Big(\frac{2}{|\mathrm{Im}\,z|}\Big)^n\\ &\leq C\sum_{n=0}^{\infty}\frac{\sqrt{2\pi n}}{\mathrm{e}^{n/2}}\frac{\exp\Big(n\log(2n/|\mathrm{Im}\,z|)-m\varphi^*(n/m)\Big)}{\mathrm{e}^{n/2}}\\ &\leq C\sum_{n=0}^{\infty}\frac{\sqrt{2\pi n}}{\mathrm{e}^{n/2}}\frac{\exp\Big(m\varphi^{**}(\log(2n/|\mathrm{Im\,z}|))\Big)}{\mathrm{e}^{n/2}}\\ &\leq C\sum_{n=0}^{\infty}\frac{\sqrt{2\pi n}}{\mathrm{e}^{n/2}}\frac{\exp\Big(m\omega(2n/|\mathrm{Im\,z}|)\Big)}{\mathrm{e}^{n/2}}\\ &\leq C\sup_n\exp\Big(m\omega\Big(\frac{2n}{|\mathrm{Im\,z}|}\Big)-\frac{n}{2}\Big)\sum_{n=0}^{\infty}\frac{\sqrt{2\pi n}}{\mathrm{e}^{n/2}}.\end{aligned}$$

for some constant $C>0$ which is not the same at each occurrence. Consequently, for some constant $C>0$, we obtain

$$\begin{aligned}&|H(z)|\exp\Big(-2k|\mathrm{Im}\,z|-2k\omega^*\Big(\frac{|\mathrm{Im}\,z|}{2k}\Big)\Big)\\ &\qquad\leq C\sup_n\exp\Big(m\omega\Big(\frac{2n}{|\mathrm{Im\,z}|}\Big)-\frac{n}{2}\Big)\\ &\qquad\leq C\mathrm{e}^{m\omega^*\big(|\mathrm{Im\,z}|/4m\big)},\end{aligned}$$

whenever $0<|\mathrm{Im}\,z|<2$. In case $2\leq|\mathrm{Im}\,z|$ the same argument gives the convergence of $\sum_{n=0}^{\infty}|a_n|n!$, from which it easily follows that $H\in\mathcal{H}_{(\omega^*)}$. $\square$

2. Boundary values. From now on, we will only consider weight functions ω satisfying the property (α_1).

Our aim is to study which $*$-ultradistributions can be obtained as boundary values of functions in the space $\mathcal{H}_{\omega *}$. First, we observe that for the boundary value, only what happens near the real axis is relevant and therefore we may apply [**13**, 4.5, 4.6] to conclude the following result

Lemma 2. *If ω is a weight function with the property (α_1), then the boundary value operator $T : \mathcal{H}_{\omega^*} \to \mathcal{D}'_*(\mathbf{R})$ given by*

$$\langle T(F), \varphi \rangle = \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}} (F(x + i\varepsilon) - F(x - i\varepsilon))\varphi(x)\, dx,$$

is a well-defined, continuous and linear mapping.

What we will see in our following result is that the range of T consists of bounded $*$-ultradistributions and that, in fact, each bounded $*$-ultradistribution is the boundary value of a function in $\mathcal{H}_{\omega^*}$.

Theorem 3. *If $F \in \mathcal{H}_{\omega^*}$, then $T(F)$ is a bounded $*$-ultradistribution and the boundary value map $T : \mathcal{H}_{\omega^*} \to \mathcal{D}'_{L^1,*}(\mathbf{R})$ is linear, continuous and surjective.*

Proof. Given $F \in \mathcal{H}_{\omega^*}$, we define for $x \in \mathbf{R}$, $F_x(z) := F(x + z)$. The set of translations $\{F_x : x \in \mathbf{R}\}$ is bounded in $\mathcal{H}_{\omega^*}$; therefore, $\{T(F_x) : x \in \mathbf{R}\}$ is bounded in $\mathcal{D}'_*(\mathbf{R})$; that is, for every $\varphi \in \mathcal{D}_*(\mathbf{R})$ the set $\{\langle T(F_x), \varphi \rangle : x \in \mathbf{R}\}$ is bounded. Consequently $T(F) * \varphi \in \mathcal{C}_b(\mathbf{R})$. We apply [**7**, 3.2] to conclude that $T(F) \in \mathcal{D}'_{L^1,*}(\mathbf{R})$. Moreover, $T : \mathcal{H}_{\omega^*} \to \mathcal{D}'_{L^1,*}(\mathbf{R})$ is continuous by the closed graph theorem.

To show that T is surjective, the first step is to prove that the derivative S' of $S \in \mathcal{D}'_{L^1,*}(\mathbf{R})$, is the boundary value of some function in $\mathcal{H}_{\omega^*}$.

By [**7**, 3.2], it suffices to consider the case $S = G(D)h$ where $G(D)$ is an ultradifferential operator of class $*$ and $h \in C_b(\mathbf{R})$. We put

$$H(z) = \frac{1}{2\pi i} \int_{\mathbf{R}} h(t)\, \frac{1}{(t - z)^2}\, dt.$$

Then H is holomorphic in $\mathbf{C}\setminus\mathbf{R}$ and $|H(z)| \leq \| h \|_\infty/|y|$, where $z = x+iy$. Moreover, given $m \in \mathbf{N}$ there is a $K_m > 0$ such that

$$\begin{aligned}|H(z)| &\leq \| h \|_\infty \mathrm{e}^{\log(1/|y|)} \leq K_m \mathrm{e}^{\log(1/|y|)-(1/m)\varphi^*(m)}\\ &\leq K_m \mathrm{e}^{(1/m)\omega(1/|y|)} \leq K_m \mathrm{e}^{(1/m)\omega^*(m|y|)+1},\end{aligned}$$

if $|y| < 1$ while $|H(z)| \leq \| h \|_\infty$ whenever $|y| \geq 1$. Hence $H \in \mathcal{H}_{\{\omega^*\}} \subset \mathcal{H}_{(\omega^*)}$.

Given $\varphi \in \mathcal{D}_*(\mathbf{R})$ we have, by Tonelli-Hobson's and Fubini's theorems,

$$\begin{aligned}\int_{\mathbf{R}} H(x+iy)\varphi(x)\,dx &= \frac{1}{2\pi i}\int_{\mathbf{R}} h(t)\bigg(\int_{\mathbf{R}} \frac{\varphi(x)}{(t-x-iy)^2}\,dx\bigg)\,dt\\ &= -\frac{1}{2\pi i}\int_{\mathbf{R}} h(t)\bigg(\int_{\mathbf{R}} \frac{\varphi'(x)}{t-x-iy}\,dx\bigg)\,dt.\end{aligned}$$

For $y > 0$,

$$\int_{\mathbf{R}} \Big(H(x+iy)-H(x-iy)\Big)\varphi(x)\,dx = -\int_{\mathbf{R}} h(t)(P_y * \varphi')(t)\,dt,$$

where $P_y(t) := \dfrac{y}{\pi}\,\dfrac{1}{t^2+y^2}$.

Since $\varphi' \in L^1$, $P_y * \varphi'$ tends to φ' as $y \to 0$ in the L^1-norm then

$$\lim_{\varepsilon\to 0^+}\int_{\mathbf{R}} \Big(H(x+i\varepsilon)-H(x-i\varepsilon)\Big)\varphi(x)\,dx = -\int_{\mathbf{R}} h(t)\varphi'(t)\,dt = \langle h', \varphi\rangle.$$

To finish we define $F(z) = G(D)_x H(z)$ which is an element of $\mathcal{H}_{\omega^*}$. Then, for every $\varphi \in \mathcal{D}_*(\mathbf{R})$ we obtain

$$\begin{aligned}S'(\varphi) &= \lim_{\varepsilon\to 0^+}\int_{\mathbf{R}} (H(x+i\varepsilon)-H(x-i\varepsilon))G(-D)\varphi(x)\,dx\\ &= \lim_{\varepsilon\to 0^+}\int_{\mathbf{R}} (F(x+i\varepsilon)-F(x-i\varepsilon))\varphi(x)\,dx,\end{aligned}$$

and we are done.

Now, let $S \in \mathcal{D}'_{L^1,*}(\mathbf{R})$ be given and observe that $S = -ie^{-ix}(e^{ix}S)' + iS'$. As we have shown, there are $F, G \in \mathcal{H}_{\omega^*}$ such that $T(G) = (e^{ix}S)'$ and $T(F) = iS'$. Therefore $H(z) := -ie^{-iz}G(z) + F(z) \in \mathcal{H}_{\omega^*}$ and has S as its boundary value. $\square$

Remark. If a sequence $(M_p)_{p\in\mathbf{N}_0}$ satisfies the conditions $(M1)$, $(M2)$ and $(M3)$ of Komatsu [**8**] then there is a concave strong weight κ such that $\mathcal{D}_{(M_j)}(\mathbf{R}) = \mathcal{D}_{(\kappa)}(\mathbf{R})$ [**12**]. Here $\mathcal{D}_{(M_j)}(\mathbf{R})$ is the set of all functions $f \in \mathcal{C}^\infty(\mathbf{R})$ with compact support such that $\sup_{j\in\mathbf{N}_0}\sup_{x\in\mathbf{R}} |f^{(j)}(x)|/h^j M_j < \infty$ for each $h > 0$. It also holds that $\mathcal{D}_{L^1}^{(M_p)}$ and $\mathcal{D}_{L^\infty}^{\prime(M_p)}$ in [**14**] coincide precisely with $\mathcal{D}_{L^1,(\kappa)}$ and $\mathcal{D}'_{L^1,(\kappa)}$, respectively. Moreover $H_{L^\infty}^{(M_p)}$ strictly contains $H_{(\kappa^*)}$ and hence our boundary value map $H_{(\kappa^*)} \to \mathcal{D}'_{L^1,(\kappa)}$ is a proper restriction of the one considered by Pilipovic [**14**]. Nevertheless our approach gives the surjectivity of the boundary value operator, a fact that was not obtained in [**14**]. A similar result holds in the Roumieu case. Therefore our results properly extend the case of bounded ultradistributions in [**14**].

Our next aim is to justify that $\mathcal{H}_{\omega*}$ is, in some sense, the smallest reasonable class that can be considered. This is done in Proposition 6. First we need some auxiliary results.

Lemma 4. *Let $F \in \mathcal{H}_{\omega^*}$ be given. Then*

$$T(F) = \lim_{\varepsilon\to 0^+} (F(\cdot + i\varepsilon) - F(\cdot - i\varepsilon))$$

in the weak topology $\sigma(\mathcal{D}'_{L^1,*}(\mathbf{R}), \mathcal{D}_{L^1,*}(\mathbf{R}))$.

Proof. We have to check that for each $F \in \mathcal{H}_{\omega^*}$ and each $\psi \in \mathcal{D}_{L^1,*}(\mathbf{R})$, the limit

$$\lim_{\varepsilon\to 0^+} \int_{\mathbf{R}} (F(x + i\varepsilon) - F(x - i\varepsilon))\psi(x)\,dx$$

exists. Since every $F \in \mathcal{H}_{\omega*}$ can be decomposed as $F = F_1 + F_2$ with $F_1, F_2 \in \mathcal{H}_{\omega*}$, F_1 identically zero in $\operatorname{Im} z < 0$ and F_2 vanishing on $\operatorname{Im} z > 0$, we can assume that F vanishes on the lower half-plane. Since the limit exists for each ψ in $\mathcal{D}_*(\mathbf{R})$ which is dense in $\mathcal{D}_{L^1,*}(\mathbf{R})$, it is enough to show that $\{F(x + i\varepsilon) : 0 < \varepsilon < 1\}$ is bounded in

$\mathcal{D}'_{L^1,*}(\mathbf{R})$. As a consequence of [**7**, 3.3], this happens if and only if $\{F_{i\varepsilon} * \varphi : 0 < \varepsilon < 1\}$ is a bounded set in $\mathcal{C}_b(\mathbf{R})$ for each $\varphi \in \mathcal{D}_*(\mathbf{R})$. Here $F_{i\varepsilon}(x) := F(x + i\varepsilon)$.

We first consider the Beurling case. Since $F \in \mathcal{H}_{(\omega^*)}$, there are $C > 0$, $k \in \mathbf{N}$ such that

$$|F(x+iy)| \leq C\mathrm{e}^{k\omega^*(y/k)} \quad \text{for} \quad 0 < y < 2.$$

Let $\varphi \in \mathcal{D}_{(\omega)}(\mathbf{R})$ be given and let $b > 0$ be such that $\mathrm{supp}\varphi \subset [-\mathrm{b}, \mathrm{b}]$. By [**13**, 3.4] we find $\phi \in \mathcal{D}((-b, b) \times (-1/2, 1/2)$ such that

(i) $\phi_{|\mathbf{R}} = \varphi$

(ii) $\sup_{z \in \mathbf{C} \setminus \mathbf{R}} |(\partial/\partial\bar{z})\phi(x+iy)\mathrm{e}^{k\omega^*(|y|/k)}| < \infty$.

We apply Stokes' theorem to the function $\theta_x(\xi) := F(\xi + i\varepsilon)\phi(x - \xi)$ in the rectangle $D := [x - 2b, x + 2b] \times [0, 1]$ to get

$$\begin{aligned}\int_{\mathbf{R}} F(x + i\varepsilon - t)\varphi(t)\,dt &= \int_{\mathbf{R}} F(t + i\varepsilon)\varphi(x - t\,)dt \\ &= 2i \int_D F(\xi + i\varepsilon)\frac{\partial}{\partial\bar{\xi}}\phi(x - \xi)\,dm_2(\xi)\end{aligned}$$

from which it follows that

$$|(F_{i\varepsilon} * \varphi)(x)| \leq Cm_2(D)$$

for some constant $C > 0$ since ω^* is decreasing.

We now consider the Roumieu case. Since $F \in \mathcal{H}_{\{\omega^*\}}$, for every $m \in \mathbf{N}$ there is a constant $C > 0$ such that

$$|F(x+iy)| \leq C\mathrm{e}^{(1/m)\omega^*(ym)} \quad \text{for} \quad 0 < y < 2.$$

Let $\varphi \in \mathcal{D}_{\{\omega\}}(\mathbf{R})$ be given. We apply [**13**, 3.3] to find $\phi \in \mathcal{D}(\mathbf{R} \times (-1/2, 1/2))$ and $\rho > 0$ such that

(i) $\phi_{|\mathbf{R}} = \varphi$

(ii) $\sup_{z \in \mathbf{C} \setminus \mathbf{R}} |(\partial/\partial\bar{z})\phi(x+iy)\mathrm{e}^{\rho\omega^*(|y|/\rho)}| < \infty$.

Then

$$F(\xi + i\varepsilon)\frac{\partial}{\partial\bar{\xi}}\,\phi(x - \xi)$$

is bounded in D and we can proceed as in the Beurling case to get the conclusion. $\square$

Proposition 5. *The kernel of T consists exactly of those functions $F \in \mathcal{H}_{\omega^*}$ which can be extended to an entire function.*

Proof. We first observe that the boundary value map $\mathcal{H}_{(\omega^*)} \to \mathcal{D}'_{L^1,(\omega)}(\mathbf{R})$ is an extension of $\mathcal{H}_{\{\omega^*\}} \to \mathcal{D}'_{L^1,\{\omega\}}(\mathbf{R})$. Consequently we only have to consider the Beurling case.

Let $F \in \mathcal{H}_{(\omega^*)}$ be such that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbf{R}} (F(x+i\varepsilon) - F(x-i\varepsilon))\varphi(x)\,dx = 0$$

for every $\varphi \in \mathcal{D}_{L^1,(\omega)}(\mathbf{R})$. We fix $z \in \mathbf{C}$ with $0 < \operatorname{Im} z < 1$ and we take $0 < \delta < (\operatorname{Im} z/2)$ and $R > 0$ with $|\operatorname{Re} z| < R$. We consider $\phi(\xi) := [1/(\xi - z)^2]$ for $\xi \neq z$, $\xi \in \mathbf{C}$. We fix $\varepsilon > 0$ small enough and apply Cauchy's theorem to the function $g(\xi) := F(\xi + i\varepsilon)\phi(\xi)$ on the rectangle $[-R, R] \times [0, \delta]$, keeping in mind that $F_{i\varepsilon}$ is bounded on the vertical segments. After taking limits as R goes to infinity we get

$$\int_{\mathbf{R}} F(x+i\varepsilon)\phi(x)\,dx = \int_{\mathbf{R}} F(x+i\varepsilon+i\delta)\phi(x+i\delta)\,dx$$

and similarly

$$\int_{\mathbf{R}} F(x-i\varepsilon)\phi(x)\,dx = \int_{\mathbf{R}} F(x-i\varepsilon-i\delta)\phi(x-i\delta)\,dx.$$

Since $\phi_{|\mathbf{R}} \in \mathcal{D}_{L^1,(\omega)}(\mathbf{R})$ we conclude

$$0 = \langle T(F), \phi_{|\mathbf{R}}\rangle = \int_{\mathbf{R}} F(x+i\delta)\phi(x+i\delta)\,dx - \int_{\mathbf{R}} F(x-i\delta)\phi(x-i\delta)\,dx.$$

On the other hand, denoting by γ_R the boundary of $[-R, R] \times [\delta, 1]$ positively oriented, we have $F'(z) = (1/2\pi i)\int_{\gamma_R} (F(\xi)/(\xi - z)^2)\,d\xi$.

Since F is bounded on $\{z \in \mathbf{C} \ : \delta \le \operatorname{Im} z \le 1\}$ the integrals on the vertical segments go to 0 as R goes to infinity, therefore

$$F'(z) = \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{F(t+i\delta)}{(t+i\delta-z)^2}\,dt - \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{F(t+i)}{(t+i-z)^2}\,dt.$$

Since $[F(\xi)/(\xi - z)^2]$ is a holomorphic function on the lower half-plane, proceeding as before we obtain

$$0 = \frac{-1}{2\pi i}\int_{\mathbf{R}} \frac{F(t-i\delta)}{(t-i\delta-z)^2}\,dt + \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{F(t-i)}{(t-i-z)^2}\,dt.$$

Consequently, for $0 < \operatorname{Im} z < 1$,

$$F'(z) = -\frac{1}{2\pi i}\left(\int_{\mathbf{R}+i} \frac{F(\xi)}{(\xi-z)^2}\,d\xi - \int_{\mathbf{R}-i} \frac{F(\xi)}{(\xi-z)^2}\,d\xi\right).$$

The same holds in case $-1 < \operatorname{Im} z < 0$. The righthand side defines a holomorphic function on $\{z \in \mathbf{C} : |\operatorname{Im} z| < 1\}$. Then F' can be extended to an entire function and consequently we find two constants A_1, A_2 and an entire function H such that

$$\begin{aligned} F(z) &= H(z) + A_1 \quad \text{for } \operatorname{Im} z > 0,\\ F(z) &= H(z) + A_2 \quad \text{for } \operatorname{Im} z < 0. \end{aligned}$$

Since the boundary value of F is zero, we conclude $A_1 = A_2$ and F can be extended to an entire function. □

Proposition 6. *$\mathcal{H}_{\omega^*}$ is the smallest algebra of holomorphic functions on $\mathbf{C} \setminus \mathbf{R}$, X, with the following properties*

(i) *X contains all bounded holomorphic functions in $\mathbf{C} \setminus \mathbf{R}$ and all the exponentials* $\exp(i\lambda z)$, $\lambda \in \mathbf{R}$,

(ii) *for each $F \in X$ and each $G(D)$ ultradifferential operator of class $*$, we have that $G(D)_x F \in X$.*

Proof. We already know that $\mathcal{H}_{\omega^*}$ verifies (i) and (ii).

Now let X satisfy (i) and (ii). We consider the Beurling case, and we assume that H is a holomorphic function in $\mathbf{C} \setminus \mathbf{R}$ with

$$|H(z)| \le C e^{k\omega^*(|\operatorname{Im} z|/k)}$$

for some constants $C > 0$ and $k \in \mathbf{N}$, and let $S \in \mathcal{D}'_{L^1,(\omega)}(\mathbf{R})$ be the boundary value of H. Without loss of generality we assume that H vanishes on $\operatorname{Im} z < 0$. We denote by $\mathcal{D}^m_{(\omega)}(-1,1)$ the set of $\mathcal{C}^\infty$ functions Γ with compact support in $(-1,1)$ such that $\| \Gamma \|_{K,m} < \infty$. There is $m \in \mathbf{N}$ such that $S * \psi \in \mathcal{C}_b(\mathbf{R})$ for each $\psi \in \mathcal{D}^m_{(\omega)}(-1,1)$ (see the proof of (3) implies (2) in [**7**, 3.2]), and there are an ultradifferential operator $G(D)$ of class (ω), $\Gamma \in \mathcal{D}^m_{(\omega)}(-1,1)$ and $\chi \in \mathcal{D}_{(\omega)}(\mathbf{R})$ with $\operatorname{supp} \chi \subset [-1,1]$ such that

$$S = G(D)(S * \Gamma) + S * \chi.$$

Let $P(t,z) = \dfrac{1}{2\pi i}\Big[\dfrac{1}{t-z} - \dfrac{1}{t-\bar{z}}\Big]$, $t \in \mathbf{R}$, $z \in \mathbf{C} \setminus \mathbf{R}$, be the Poisson kernel. Then $P(t,z) \in \mathcal{D}_{L^1,(\omega)}(\mathbf{R})$ for each $z \in \mathbf{C} \setminus \mathbf{R}$ and

$$\langle S_t, P(t,z)\rangle = \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}} H(t+i\varepsilon) P(t,z)\, dt.$$

But $\int_{\mathbf{R}} H(t+i\varepsilon)P(t,z)\,dt$ is the harmonic extension of the bounded holomorphic function on $\operatorname{Im} z > 0$ with continuous extension to the boundary, $H_{i\varepsilon}(t) := H(t+i\varepsilon)$, hence, it is just $H(z+i\varepsilon)$ and therefore

$$\langle S_t, P(t,z)\rangle = H(z).$$

Now, the harmonic extension of $f := S * \Gamma$ is

$$\begin{aligned} F(z) = \big(f * P_y\big)(x) &= \int_{\mathbf{R}} (S * P_y)(x-t)\Gamma(t)\, dt \\ &= \int_{-1}^{1} H(z-t)\Gamma(t)\, dt \end{aligned}$$

for $z = x + iy$. Therefore $F(z)$ is a bounded holomorphic function in the upper half-plane.

If G denotes the harmonic extension of $g := S * \chi$ to the upper half-plane, then the same argument shows that G is a bounded holomorphic function in $\operatorname{Im} z > 0$. Then, by (i) and (ii),

$$\tilde{H} := G(D)_x F + G \in X \cap \mathcal{H}_{(\omega^*)}$$

and its boundary value is S. Therefore the function defined as $H(z) - \tilde{H}(z)$ if $\operatorname{Im} z > 0$ and vanishing in the lower half-plane has boundary value 0 and it can be extended to an entire function from where it follows that $H = \tilde{H} \in X$.

Finally we observe that for each $H \in \mathcal{H}_{(\omega^*)}$ such that H vanishes on $\operatorname{Im} z < 0$, there is $\lambda \in \mathbf{R}$ such that

$$|\mathrm{e}^{i\lambda z} H(z)| \leq C \mathrm{e}^{k\omega^*(\operatorname{Im} z/k)}, \quad \operatorname{Im} z > 0$$

for some $C > 0$, $k \in \mathbf{N}$. Hence $\mathrm{e}^{i\lambda z} H(z)$, and consequently H, belongs to X.

To finish the proof we discuss the Roumieu case. We only have to prove that a holomorphic function H in $\mathbf{C} \setminus \mathbf{R}$ with

$$|H(z)| \leq C_n \mathrm{e}^{(1/n)\omega^*(n|\operatorname{Im} z|)}$$

for every $n \in \mathbf{N}$ and vanishing on the lower half-plane is in X. To do this we put $S := T(H) \in \mathcal{D}'_{L^1,\{\omega\}}(\mathbf{R})$, the boundary value of H, and we find a weight σ, not necessarily with the property (α_1), such that $\sigma = o(\omega)$ and $S \in \mathcal{D}'_{L^1,(\sigma)}(\mathbf{R})$ [**7**, 3.1]. It follows from the proof of the proposition in the Beurling case that there are an ultradifferential operator $G(D)$ of class (σ) and two bounded holomorphic functions F and G in the upper half-plane such that S is the boundary value in $\mathcal{D}'_{L^1,\{\omega\}}(\mathbf{R})$ of $G(D)_x F + G$. Consequently H and $G(D)_x F + G$ have the same boundary value in $\mathcal{D}'_{L^1,\{\omega\}}(\mathbf{R})$ and therefore $H = G(D)_x F + G$. Since $G(D)$ is an ultradifferential operator of class $\{\omega\}$ we get that $H \in X$. □

Remark. Let F be a holomorphic function in $\mathbf{C} \setminus \mathbf{R}$. Then $F \in \mathcal{H}_{\{\omega^*\}}$ if and only if there is a weight σ, not necessarily satisfying the property (α_1), such that $F \in \mathcal{H}_{(\sigma^*)}$.

To show that the boundary value map is surjective, we have used that the exponential $\exp(-iz)$ belongs to $\mathcal{H}_{\omega^*}$. As the following shows, this is essential for surjectivity.

Example. The characteristic function of the interval $]-\infty, 0]$ is not the boundary value in $\mathcal{D}'_{L^1,*}(\mathbf{R})$ of any function in $\{F \in \mathcal{H}(\mathbf{C} \setminus \mathbf{R}) : \sup_{z \in \mathbf{C} \setminus \mathbf{R}} |F(z)| \mathrm{e}^{-n\omega^*(|\operatorname{Im} z|/n)} < \infty \text{ for some } n \in \mathbf{N}\}$.

Assume that the characteristic function of the interval $]-\infty, 0]$ is the boundary value of a function F in the space above. Then, proceeding as in the proof of Proposition 5, it is easy to see that for $0 < |\mathrm{Im}\, z| < 1$,

$$2\pi i F'(z) = G(z) + \frac{1}{z}$$

where

$$G(z) = \int_{\mathbf{R}-i} \frac{F(\xi)}{(\xi - z)^2}\, d\xi - \int_{\mathbf{R}+i} \frac{F(\xi)}{(\xi - z)^2}\, d\xi.$$

Observe that $|F'(z)| \le C\mathrm{e}^{n\omega^*(|\mathrm{Im}\, z|/n)}$, for some constants C, n, therefore it is bounded in $(1/2) < |\mathrm{Im}\, z|$, and that G is holomorphic in $|\mathrm{Im}\, z| < 1$ and bounded in $|\mathrm{Im}\, z| < 2/3$. Hence $\tilde{G}(z) := G(z)$ for $|\mathrm{Im}\, z| < 1$ and $\tilde{G}(z) := 2\pi i F'(z) - 1/z$ for $z \in \mathbf{C} \setminus \mathbf{R}$ is a bounded entire function, therefore it is constant. This implies that there is a $D \in \mathbf{C}$ such that $2\pi i F(z) - Dz - L(z)$ is constant in the upper, or lower, half-plane, where L denotes a continuous branch of the logarithm in the upper, or lower, half-plane, which contradicts the estimates for the growth of F. $\square$

Finally we characterize the almost periodic $*$-ultradistributions that are boundary values of holomorphic functions in $\mathcal{H}_{\omega^*}$ vanishing in a half-plane. If $S \in \mathcal{D}'_{L^1,*}(\mathbf{R})$ is an almost periodic ultradistribution, then the Fourier coefficients of S are defined as

$$C_S(\lambda) := \frac{M(\mathrm{e}^{-i\lambda x} S * \varphi)}{\hat{\varphi}(0)}$$

where φ is a fixed test function $\varphi \in \mathcal{D}_*(\mathbf{R})$ such that $\hat{\varphi}(0) \neq 0$ and $M(\mathrm{e}^{-i\lambda x} S * \varphi)$ denotes the mean value of the almost periodic function $\mathrm{e}^{-i\lambda x} S * \varphi$ (see [**7**] for details). The former definition does not depend on the fixed function φ, and the spectrum of S,

$$\Lambda_S := \{\lambda \in \mathbf{R} : C_S(\lambda) \neq 0\}$$

is countable.

Proposition 7. *Let S be an almost periodic $*$-ultradistribution. Then S is the boundary value of $F \in \mathcal{H}_{\omega^*}$ vanishing on* $\mathrm{Im}\, z < 0$

(*respectively* $\operatorname{Im} z > 0$), *if and only if* Λ_S *is bounded below* (*respectively* Λ_S *is bounded above*).

Proof. Let us assume that the spectrum Λ_S is bounded below. It easily follows from [**7**, 4.2] that we can represent $S = G(D)f + h$ where f and h are almost periodic functions, $G(D)$ is an $*$-ultradifferential operator and $\Lambda_S = \Lambda_f \cup \Lambda_h$. Therefore f and h are almost periodic functions whose spectra are bounded below. Applying [**9**, IV, 1.1] we find F and H holomorphic functions in the upper half-plane and continuous in its closure such that $F_{|\mathbf{R}} = f$ and $H_{|\mathbf{R}} = h$ and

$$|F(z)| \le C\mathrm{e}^{\lambda \operatorname{Im} z}, \quad |H(z)| \le C\mathrm{e}^{\lambda \operatorname{Im} z} \quad \text{for } \operatorname{Im} z \ge 0$$

and for some constants $C > 0$, $\lambda > 0$. For $\operatorname{Im} z < 0$, we put $F(z) = H(z) = 0$. Then $F, H \in \mathcal{H}_{\omega^*}$ and consequently

$$G(D)_x F + H \in \mathcal{H}_{\omega^*}$$

and it has S as its boundary value.

Conversely, let $F \in \mathcal{H}_{\omega *}$ be such that $F(x + i\varepsilon)$ converges to S as $\varepsilon \to 0^+$. There is $\mu > 0$ such that if $H(z) := \mathrm{e}^{i\mu z}F(z)$, then $|H(z)| \le C\mathrm{e}^{k\omega^*\left(\operatorname{Im} z/k\right)}$ for $\operatorname{Im} z > 0$ and some $C, k > 0$. Let $\varphi \in \mathcal{D}_*(\mathbf{R})$ be given. Proceeding as in Lemma 4, we show that $(H * \varphi)$ is bounded in the upper half-plane and has boundary value $\mathrm{e}^{i\mu x} S * \varphi$. Applying [**9**, IV, 1.1] we have that the spectrum of $\mathrm{e}^{i\mu x} S * \varphi$ is bounded below by 0. Since this holds for arbitrary $\varphi \in \mathcal{D}_*(\mathbf{R})$ we conclude that the spectrum of $\mathrm{e}^{i\mu x} S$ consists of nonnegative real numbers, hence $\Lambda_S \subset [-\mu, +\infty[$ and the proposition is proved. □

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