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# ON THE SPECTRUM OF A SECOND-ORDER PERIODIC DIFFERENTIAL EQUATION

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ABSTRACT. In this paper we derive asymptotic approximations for the periodic and semi-periodic eigenvalues for a second-order periodic differential equation known as Hill's equation. Our results are sharper than the existing results in the literature in that they give sharper error bounds whilst relaxing the smoothness assumptions. For some particular potentials, including that of Mathieu equation, we provide estimates for the corresponding eigenvalues using the symbolic manipulator package, Maple.

**1. Introduction.** We consider the differential equation

$$y''(t) + (\lambda - q(t))y(t) = 0, \tag{1.1}$$

where $\lambda$ is a real-parameter, $q$ is a real-valued periodic function with period $\pi$. For some $N \geq 2$ we assume that $q^{(N-1)}(t)$ exists and is integrable on $[0, \pi]$.

We associate two types of boundary conditions with (1.1) on the interval $[0, \pi]$. The periodic boundary conditions $y(0) = y(\pi)$, $y'(0) = y'(\pi)$; the semi-periodic boundary conditions $y(0) = -y(\pi)$, $y'(0) = -y'(\pi)$. We denote the periodic eigenvalues by $\{\lambda_n\}$ and the semi-periodic eigenvalues by $\{\mu_n\}$. It is known [**3**] that the two sets of eigenvalues satisfy the relation

$$-\infty < \lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \cdots .$$

The instability intervals of (1.1) are defined to be $I_0 = (-\infty, \lambda_0)$, $I_{2m+1} = (\mu_{2m}, \mu_{2m+1})$ and $I_{2m+2} = (\lambda_{2m+1}, \lambda_{2m+2})$.

We make the point that a more general second-order periodic differential equation

$$\{p(t)y'(t)\}' + \{\lambda s(t) - q(t)\}y(t) = 0 \tag{1.2}$$

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can be reduced to an equation of type (1.1) by using the Liouville transformation if $p''$ and $s''$ exist and are piecewise continuous. In this case, the periodic and the semi-periodic eigenvalues of the reduced equation are the same as with those of (1.1), [**3**].

Many authors have computed estimates to the periodic and the semi-periodic eigenvalues and used those to solve the forward and inverse scattering problems. We refer in particular to [**3**] and [**6**]. A feature of the estimates is that they become increasingly accurate the more times that $p(t), q(t)$ and $s(t)$ are differentiable. In this paper, we derive asymptotic estimates for the periodic and the semi-periodic eigenvalues of (1.1) with an error term of order $O(m^{-(N+1)})$ under the condition that $q^{(N-1)}(t)$ exist and be integrable on $[0,\pi]$. The error term in the corresponding estimates derived in [**3**] is $o(m^{-N})$ under the condition that $q^{(N-1)}(t)$ exists and is piecewise continuous on the same interval. Besides the improvement in the error term, the computations are carried out to all orders in the spectral parameter and coefficients are given recursively while a few of them are explicitly given in [**3**]. Also examples including the Mathieu equation are carried out by Maple.

We suppose without loss of generality that $q$ has a mean value zero, i.e.,

$$\int_0^\pi q(t)\,dt = 0.$$

As an illustration of our results we show that if $q^{(N-1)}(t)$ exists and is integrable on $[0,\pi]$ then the periodic and the semi-periodic eigenvalues of (1.1) satisfy, as $m \to \infty$

i) $N$ is even

(1.3)

$$\mu_{2m}^{1/2}, \mu_{2m+1}^{1/2} = (2m+1) + \frac{i}{\pi}\sum_{k=1}^{\frac{N}{2}} \frac{1}{(2m+1)^{2k-1}} \int_0^\pi \alpha_{2k-1}(t)\,dt \mp \frac{1}{2^N \pi (2m+1)^N}\Bigg[\bigg(\int_0^\pi q^{(N-1)}(t)\sin(2(2m+1)t)\,dt\bigg)^2 + \bigg(\int_0^\pi q^{(N-1)}(t)\cos(2(2m+1)t)\,dt\bigg)^2\Bigg]^{1/2} + O(m^{-(N+1)})$$

and

(1.4)

$$\lambda_{2m+1}^{1/2}, \lambda_{2m+2}^{1/2} = 2(m+1) + \frac{i}{\pi}\sum_{k=1}^{N/2} \frac{1}{2^{2k-1}(m+1)^{2k-1}} \int_0^\pi \alpha_{2k-1}(t)\,dt$$
$$\mp \frac{1}{2^{2N}\pi(m+1)^N}\left[\left(\int_0^\pi q^{(N-1)}(t)\sin(4(m+1)t)\,dt\right)^2\right.$$
$$\left.+\left(\int_0^\pi q^{(N-1)}(t)\cos(4(m+1)t)\,dt\right)^2\right]^{1/2} + O(m^{-(N+1)}).$$

ii) $N$ is odd

(1.5)

$$\mu_{2m}^{1/2}, \mu_{2m+1}^{1/2} = (2m+1) + \frac{i}{\pi}\sum_{k=1}^{(N-1)/2} \frac{1}{(2m+1)^{2k-1}} \int_0^\pi \alpha_{2k-1}(t)\,dt$$
$$- \frac{1}{2\pi(2m+1)^N}\sum_{k=1}^{N-2}\int_0^\pi \alpha_k(t)\alpha_{N-1-k}(t)\,dt$$
$$\mp \frac{1}{2^N\pi(2m+1)^N}\left[\left(\int_0^\pi q^{(N-1)}(t)\sin(2(2m+1)t)\,dt\right)^2\right.$$
$$\left.+\left(\int_0^\pi q^{(N-1)}(t)\cos(2(2m+1)t)\,dt\right)^2\right]^{1/2} + O(m^{-(N+1)})$$

and

(1.6)

$$\lambda_{2m+1}^{1/2}, \lambda_{2m+2}^{1/2} = 2(m+1) + \frac{i}{\pi}\sum_{k=1}^{(N-1)/2} \frac{1}{2^{2k-1}(m+1)^{2k-1}} \int_0^\pi \alpha_{2k-1}(t)\,dt$$
$$- \frac{1}{2^{N+1}\pi(m+1)^N}\sum_{k=1}^{N-2}\int_0^\pi \alpha_k(t)\alpha_{N-1-k}(t)\,dt$$
$$\mp \frac{1}{2^{2N}\pi(m+1)^N}\left[\left(\int_0^\pi q^{(N-1)}(t)\sin(4(m+1)t)\,dt\right)^2\right.$$
$$\left.+\left(\int_0^\pi q^{(N-1)}(t)\cos(4(m+1)t)\,dt\right)^2\right]^{1/2} + O(m^{-(N+1)}),$$

where

$$\alpha_1(t) = \frac{-i}{2}\, q(t)$$

and for $n = 1, 2, \dots, N-1$

$$\alpha_{n+1}(t) = i\Big[\alpha_n'(t) + \sum_{k=1}^{n-1} \alpha_k(t)\alpha_{n-k}(t)\Big], \tag{1.7}$$

with $i^2 = -1$. As a result of these estimates, instability intervals are given explicitly as follows:

$$I_{2m+1} = \frac{1}{2^{N-1}\pi(2m+1)^{N-1}}\bigg[\bigg(\int_0^\pi q^{(N-1)}(t)\sin(2(2m+1)t)\,dt\bigg)^2 + \bigg(\int_0^\pi q^{(N-1)}(t)\cos(2(2m+1)t)\,dt\bigg)^2\bigg]^{1/2} + O(m^{-N})$$

$$I_{2m+2} = \frac{1}{2^{2N-1}\pi(m+1)^{N-1}}\bigg[\bigg(\int_0^\pi q^{(N-1)}(t)\sin(4(m+1)t)\,dt\bigg)^2 + \bigg(\int_0^\pi q^{(N-1)}(t)\cos(4(m+1)t)\,dt\bigg)^2\bigg]^{1/2} + O(m^{-N}).$$

Central to our analysis is the following theorem of Hochstadt [**5**], which involves $\Lambda_n(\tau)$, the eigenvalues of (1.1) considered on the interval $[\tau, \tau+\pi]$, where $0 \le \tau < \pi$, with the Dirichlet boundary conditions

$$y(\tau) = y(\tau+\pi) = 0. \tag{1.8}$$

**Theorem A.** *The ranges of* $\Lambda_{2m}(\tau)$ *and* $\Lambda_{2m+1}(\tau)$*, as function of* $\tau$ *for* $\tau \in [0, \pi]$ *are* $[\mu_{2m}, \mu_{2m+1}]$ *and* $[\lambda_{2m+1}, \lambda_{2m+2}]$*, respectively.*

*We also note* (*see* [**5**]) *that* (1.1) *with* (1.8) *is equivalent to*

$$y''(t) + (\lambda - q(t+\tau))y(t) = 0 \tag{1.9}$$

*with the boundary condition*

$$y(0) = y(\pi) = 0. \tag{1.10}$$

The method is an exploitation of the well-known Ricatti equation associated to Hill's equation, to compute the expansion of $\Lambda_n(\tau)$ and then use Theorem A.

**2. The technique.** We only consider the case that $N$ is even and define

$$v_\tau(t,\Lambda) = \frac{y'(t,\Lambda)}{y(t,\Lambda)} - r_\tau(t,\Lambda), \tag{2.1}$$

where $r_\tau(t,\Lambda)$ is a complex-valued differentiable function which will be determined, and $y(t,\Lambda)$ is a complex-valued solution of (1.9) with $y'(0,\Lambda)/y(0,\Lambda) = r_\tau(0,\Lambda)$. Substitution of (2.1) into (1.9) yields

$$v'_\tau = -v_\tau^2 - 2v_\tau r_\tau - Q_\tau, \tag{2.2}$$

where

$$Q_\tau = r_\tau^2 + r'_\tau + (\lambda - q(t+\tau)). \tag{2.3}$$

Let

$$r_\tau(t,\Lambda) = i\Lambda^{1/2} + \sum_{n=1}^{N-1} \Lambda^{-n/2}\alpha_n(t+\tau) + \rho_\tau(t,\Lambda) \tag{2.4}$$

with $\rho_\tau(0,\Lambda) = 0$. We choose

$$\alpha_1(t+\tau) = -\frac{i}{2}\, q(t+2),$$
$$\alpha_2(t+\tau) = \frac{i}{2}\,\alpha'_1(t+\tau) = \frac{1}{4}\, q'(t+\tau)$$

and for $n = 2, \dots, N-2$,

$$\alpha_{n+1}(t+\tau) = \frac{i}{2}\Big[\alpha'_n(t+\tau) + \sum_{k=1}^{n-1} \alpha_k(t+\tau)\alpha_{n-k}(t+\tau)\Big]. \tag{2.5}$$

Then (2.3) becomes

(2.6)
$$\begin{aligned} Q_\tau(t,\Lambda) = \sum_{n=N-1}^{2N-2} &\Lambda^{-n/2}\Big(\sum_{r+s=n} \alpha_r(t+\tau)\alpha_s(t+\tau)\Big) \\ &+ \Lambda^{-(N-1)/2}\alpha'_{N-1}(t+\tau) + 2\rho_\tau(t,\Lambda) \\ &\times \Big(\sum_{n=1}^{N-1} \Lambda^{-n/2}\alpha_n(t+\tau)\Big) + \rho_\tau^2(t,\Lambda) + 2i\Lambda^{1/2}\rho_\tau(t,\Lambda) + \rho'_\tau(t,\Lambda). \end{aligned}$$

To determine $\rho_\tau(t,\Lambda)$ we solve the following first-order linear differential equation

(2.7)
$$\rho_\tau'(t,\Lambda)+2\Big(i\Lambda^{1/2}+\sum_{n=1}^{N-1}\Lambda^{-n/2}\alpha_n(t+\tau,\Lambda)\Big)\rho_\tau(t,\Lambda) = -\Big[\Lambda^{-(N-1)/2}\alpha_{N-1}'(t+\tau)+\sum_{n=N-1}^{2N-2}\Lambda^{-n/2}\Big(\sum_{r+s=n}\alpha_r(t+\tau)\alpha_s(t+\tau)\Big)\Big]$$

and find that

$$\rho_\tau(t,\Lambda) = -\int_0^t \Big[\Lambda^{-(N-1)/2}\alpha_{N-1}'(x+\tau) + \sum_{n=N-1}^{2N-2}\Lambda^{-n/2}\Big(\sum_{r+s=n}\alpha_r(x+\tau)\alpha_s(x+\tau)\Big)\Big] \times e^{-2\int_x^t (i\Lambda^{1/2}+\sum_{n=1}^{N-1}\Lambda^{-n/2}\alpha_n(s+\tau))\,ds}\,dx. \tag{2.8}$$

After $\rho_\tau(t,\Lambda)$ has been determined, (2.6) reduces to

$$Q_\tau(t,\Lambda) = \rho_\tau^2(t,\Lambda). \tag{2.9}$$

Since $\alpha_i(t+\tau)$ involves no derivative of $q(t+\tau)$ and hence of $q(t)$ with a degree greater than $i-1$, $\rho_\tau(t,\Lambda)$ given by (2.8) is bounded. Hence there exists a $K_N < \infty$ such that

$$|\rho_\tau(t,\Lambda)| < K_N\Lambda^{-(N-1)/2} \tag{2.10}$$

and

$$Q_\tau(t,\Lambda) = \rho_\tau^2(t,\Lambda) < K_N^2\Lambda^{-(N-1)} = C_N\Lambda^{-(N-1)}. \tag{2.11}$$

Using this bound on $Q_\tau(t,\Lambda)$ we prove that $v_\tau(t,\Lambda)$ is bounded. First, we define the following functions:

$$A_\tau(x,\Lambda) := \sup_{0\le\xi\le x}\left|\int_0^\xi Q_\tau(t,\Lambda)e^{-2\int_t^x r_\tau(s,\Lambda)\,ds}\,dt\right|,$$
$$B_\tau(x,\Lambda) := \int_0^x |e^{-2\int_t^x r_\tau(s,\Lambda)\,ds}|\,dt,$$
$$V_\tau(x,\Lambda) := \sup_{0\le t\le x}|v_\tau(t,\Lambda)|.$$

**Lemma 1.** $|v_\tau(t,\Lambda)| < 2A_\tau(\pi,\Lambda) = O(\Lambda^{-(N-1)})$, $0 \le t \le \pi$.

We prove the lemma in two steps.

*Step* 1.

$$4A_\tau(\pi,\Lambda)B_\tau(\pi,\Lambda) < 1 \tag{2.12}$$

for sufficiently large $\Lambda$.

*Proof of Step* 1. We note that $r_\tau(t,\Lambda)$, given by (2.4), is a complex-valued function and hence can be written as

$$r_\tau(t,\Lambda) = r_{1,\tau}(t,\lambda) + ir_{2,\tau}(t,\Lambda), \tag{2.13}$$

where both $r_{1,\tau}(t,\Lambda)$ and $r_{2,\tau}(t,\Lambda)$ are real-valued. Now, by construction $r_\tau(t,\Lambda)$, and hence $r_{1,\tau}(t,\Lambda)$, is the sum of powers of derivatives of $q(t+\tau)$. Because of the assumptions on $q(t+\tau)$, $r_\tau(t,\Lambda)$, hence of $r_{1,\tau}(t,\Lambda)$, is bounded. Therefore, for some $M > 0$

$$-2r_{1,\tau}(t,\Lambda) < M. \tag{2.14}$$

Using (2.14) we bound $B_\tau(x,\Lambda)$ for any $x$ in $[0,\pi]$ as follows:

$$\begin{aligned} B_\tau(x,\Lambda) &= \int_0^x |e^{-2\int_t^x r_\tau(s,\Lambda)\,ds}|\,dt \\ &= \int_0^x |e^{-2\int_t^x (r_{1,\tau}(s,\Lambda)+ir_{2,\tau}(s,\Lambda))\,ds}|\,dt \\ &= \int_0^x |e^{-2\int_t^x r_{1,\tau}(s,\Lambda)\,ds}|\,dt \\ &< \int_0^x |e^{\int_t^x M\,ds}|\,dt \\ &= \int_0^x e^{M(x-t)}\,dt < e^{M\pi}\int_0^x dt \\ &< \pi e^{M\pi}. \end{aligned} \tag{2.15}$$

Next, we bound $A_\tau(x,\Lambda)$ for any $x$ in $[0,\pi]$. From (2.11) and (2.14) we see that

$$\begin{aligned} A_\tau(x,\Lambda) &< C_N\Lambda^{-(N-1)}\int_0^\xi e^{M\pi}\,dt \\ &< C_N\Lambda^{-(N-1)}\pi e^{M\pi}. \end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we conclude that

$$A_\tau(x,\Lambda)B_\tau(x,\Lambda) < \pi^2 C_N \Lambda^{-(N-1)} e^{2M\pi}. \tag{2.17}$$

Now, by choosing $\Lambda$ large enough we find that

$$A_\tau(x,\Lambda)B_\tau(x,\Lambda) < \frac{1}{4}.$$

Letting $x=\pi$ we get (2.12).

*Step* 2. If $4A_\tau(\pi,\Lambda)B_\tau(\pi,\Lambda) < 1$, then

$$|v_\tau(x,\Lambda)| < 2A_\tau(\pi,\Lambda). \tag{2.18}$$

*Proof of Step* 2. First, solving the equation (2.2) with $v_\tau(0,\Lambda)=0$ we find that

$$\begin{aligned} v_\tau(x,\Lambda) = -\Big[\int_0^x Q_\tau(t,\Lambda)e^{-2\int_t^x r_\tau(s,\Lambda)\,ds}\,dt \\ + \int_0^x v_\tau^2(t,\Lambda)e^{-2\int_t^x r_\tau(s,\Lambda)\,ds}\,dt\Big]. \end{aligned} \tag{2.19}$$

Therefore

$$\begin{aligned} |v_\tau(x,\Lambda)| &\le \Big|\int_0^x Q_\tau(t,\Lambda)e^{-2\int_t^x r_\tau(s,\Lambda)\,ds}\,dt\Big| \\ &\quad + \Big|\int_0^x v_\tau^2(t,\Lambda)e^{-2\int_t^x r_\tau(s,\Lambda)\,ds}\,dt\Big| \\ &\le A_\tau(\pi,\Lambda) + \Big[\sup_{0\le t\le x}|v_\tau(t,\Lambda)|\Big]^2 \\ &\quad \times \int_0^x |e^{-2\int_t^x r_\tau(s,\Lambda)\,ds}|\,dt \\ &\le A_\tau(\pi,\Lambda) + V_\tau^2(x,\Lambda)B_\tau(\pi,\Lambda) \end{aligned} \tag{2.20}$$

for any $x$ in $[0,\pi]$.

We now claim that if $4A_\tau(\pi,\Lambda)B_\tau(\pi,\Lambda) < 1$, then $|v_\tau(x,\Lambda)| < 2A_\tau(\pi,\Lambda)$. We need to establish that

$$V_\tau(x,\Lambda) < 2A_\tau(\pi,\Lambda).$$

Clearly, $V_\tau(0,\Lambda) = 0$. Assume that the claim is false, and let $x_0 \in (0,\pi]$ be the smallest such that

$$V_\tau(x_0,\Lambda) = 2A_\tau(\pi,\Lambda). \tag{2.21}$$

Using (2.20) and the definition of $V_\tau(x,\Lambda)$ we see that

$$V_\tau(x_0,\Lambda) \le A_\tau(\pi,\Lambda) + V_\tau^2(x_0,\Lambda)B_\tau(\pi,\Lambda). \tag{2.22}$$

Substituting (2.21) into (2.22) we observe that

$$2A_\tau(\pi,\Lambda) \le A_\tau(\pi,\Lambda)(1 + 4A_\tau(\pi,\Lambda)B_\tau(\pi,\Lambda)). \tag{2.23}$$

From (2.12) and (2.23) we get that

$$2A_\tau(\pi,\Lambda) < 2A_\tau(\pi,\Lambda),$$

which is a contradiction, and therefore the proof of Step 2 follows.

We also know from (2.16) that

$$A_\tau(x,\Lambda) = O(\Lambda^{-(N-1)}).$$

This together with Step 1 and Step 2 proves the lemma.

Combining the lemma with (2.1) we find that

$$\begin{aligned}\frac{y'(t,\Lambda)}{y(t,\Lambda)} &= r_\tau(t,\Lambda) + O(\Lambda^{-(N-1)}) \\ &= r_{1,\tau}(t,\Lambda) + ir_{2,\tau}(t,\Lambda) + O(\Lambda^{-(N-1)}).\end{aligned} \tag{2.24}$$

On the other hand $y(t,\Lambda)$ is a complex-valued solution of (1.9) which can be written as

$$y(t,\Lambda) = R(t,\Lambda)\exp(i\theta(t,\Lambda)), \tag{2.25}$$

where $R(t,\Lambda)$ and $\theta(t,\Lambda)$ are both real-valued. It follows from (2.25) that

$$\frac{y'}{y} = \frac{R'}{R} + i\theta. \tag{2.26}$$

Substitution of (2.26) into (2.24) yields

$$\frac{R'(t,\Lambda)}{R(t,\Lambda)} = r_{1,\tau}(t,\Lambda) + O(\Lambda^{-(N-1)}), \tag{2.27}$$

$$\Theta'(t,\Lambda) = r_{2,\tau}(t,\Lambda) + O(\Lambda^{-(N-1)}). \tag{2.28}$$

From (2.28)

$$(n+1)\pi = \int_0^\pi r_{2,\tau}(t,\Lambda) + O(\Lambda^{-(N-1)}), \tag{2.29}$$

where $r_{2,\tau}(t,\Lambda)$ is the imaginary part of $r_\tau(t,\Lambda)$. (For details, see [**4**], [**2**]). To separate $r_\tau(t,\Lambda)$ into the real and imaginary parts, we need to know which elements of the sum in (2.4) are real and which are imaginary. We also need to separate $\rho_\tau(t,\Lambda)$ into the real and imaginary parts. To this end we give the following lemma:

**Lemma 2.** *$\alpha_k(t+\tau)$ is real if $k = 2m$ and pure imaginary if $k = 2m+1$.*

We prove the lemma by induction. Clearly the lemma is true for $k = 1$ and $k = 2$ since

$$\alpha_1(t+\tau) = -\frac{i}{2}\, q(t+\tau)$$

is pure imaginary, and

$$\alpha_2(t+\tau) = -\frac{1}{2i}\, \alpha_1'(t+\tau) = \frac{1}{4}\, q'(t+\tau)$$

is real. We assume that the lemma is true for $k \le n$, and we prove that it is also true for $k = n+1$. For the case of $n = 2m$, from (1.7)

$$\alpha_{2m+1}(t+\tau) = \frac{i}{2}\Big[\alpha_{2m}'(t+\tau) + \sum_{s=1}^{2m-1} \alpha_s(t+\tau)\alpha_{2m-s}(t+\tau)\Big].$$

Now, $\alpha_{2m}'(t+\tau)$ is real since $\alpha_{2m}(t+\tau)$ is so by the induction hypothesis. It suffices to show that each term of the sum is also real. To see this:

• if $s$ is even, so is $2m-s$. Hence both $\alpha_s(t+\tau)$ and $\alpha_{2m-s}(t+\tau)$ are real and so is their product.

• if $s$ is odd, so is $2m-s$. Hence both $\alpha_s(t+\tau)$ and $\alpha_{2m-s}(t+\tau)$ are pure imaginary and hence their product is real.

The other case is similar.

We also find $\mathrm{Im}\,(\rho_\tau(t,\Lambda))$ with an error term $O(\Lambda^{-(N+1)/2})$ which is (see [**1**])

(2.30)

$$\begin{aligned}\mathrm{Im}\,(\rho_\tau(t,\Lambda)) = \Lambda^{-\frac{(N-1)}{2}} \int_0^t \Big[ & i\alpha'_{N-1}(x+\tau) \\ & + i\sum_{k=1}^{N-2}\alpha_k(x+\tau)\alpha_{N-1-k}(x+\tau)\Big] \cos(2\Lambda^{\frac{1}{2}}(t-x))\,dx \\ & -2\Lambda^{-N/2}\int_0^t \Big[\alpha'_{N-1}(x+\tau)+\sum_{k=1}^{N-2}\alpha_k(x+\tau)\alpha_{N-1-k}(x+\tau)\Big] \\ & \times \Big(\int_x^t \alpha_1(s+\tau)\,ds\Big)\sin(2\Lambda^{1/2}(t-x))\,dx \\ & +\Lambda^{-N/2}\int_0^t \Big(\sum_{k=1}^{N-1}\alpha_k(x+\tau)\alpha_{N-k}(x+\tau)\Big)\sin(2\Lambda^{1/2}(t-x))\,dx \\ & +O(\Lambda^{-(N+1)/2})\end{aligned}$$

and hence

$$\begin{aligned} r_{2,\tau}(t,\Lambda) = \Lambda^{1/2} - i\sum_{k=1}^{N/2} & \alpha_{2k-1}(t+\tau)\Lambda^{-(2k-1)/2} \\ & +\Lambda^{-(N-1)/2}\int_0^t \Big[i\alpha'_{N-1}(x+\tau)+i\sum_{k=1}^{N-2}\alpha_k(x+\tau)\alpha_{N-1-k}(x+\tau)\Big] \\ & \cos(2\Lambda^{1/2}(t-x))\,dx \\ & -2\Lambda^{-N/2}\int_0^t \Big[\alpha'_{N-1}(x+\tau)+\sum_{k=1}^{N-2}\alpha_k(x+\tau)\alpha_{N-1-k}(x+\tau)\Big] \end{aligned} \tag{2.31}$$

$$
\begin{aligned}
&\times \Big(\int_x^t \alpha_1(s+\tau)\,ds\Big)\sin(2\Lambda^{1/2}(t-x))\,dx \\
&+\Lambda^{-N/2}\int_0^t \Big[\sum_{k=1}^{N-1}\alpha_k(x+\tau)\alpha_{N-k}(x+\tau)\Big]\sin(2\Lambda^{1/2}(t-x))\,dx \\
&+O(\Lambda^{-(N+1)/2}).
\end{aligned}
$$

Substituting (2.4) into (2.29) and rearranging yields

$$
\begin{aligned}
(n+1)\pi = {}& \Lambda^{\frac{1}{2}}\pi - i\sum_{k=1}^{\frac{N}{2}}\Lambda^{-\frac{2k-1}{2}}\int_0^\pi \alpha_{2k-1}(t+\tau)\,dt \\
&+\Lambda^{-\frac{N-1}{2}}\int_0^\pi \Big(\int_0^t i\alpha'_{N-1}(x+\tau)\cos(2\Lambda^{\frac{1}{2}}(t-x))\,dx\Big)\,dt \\
&+\Lambda^{-\frac{N-1}{2}}\int_0^\pi \Big(\int_0^t (i\sum_{k=1}^{N-2}\alpha_k(x+\tau)\alpha_{N-1-k}(x+\tau)\Big) \\
&\cos(2\Lambda^{\frac{1}{2}}(t-x))\,dx)\,dt \\
&-2\Lambda^{-\frac{N}{2}}\int_0^\pi \Big(\int_0^t \alpha'_{N-1}(x+\tau)\Big(\int_x^t \alpha_1(s+\tau)\,ds\Big) \\
&\sin(2\Lambda^{\frac{1}{2}}(t-x))\,dx\Big)\,dt \\
&-2\Lambda^{-\frac{N}{2}}\int_0^\pi \Big(\int_0^t \Big[\sum_{k=1}^{N-2}\alpha_k(x+\tau)\alpha_{N-1-k}(x+\tau)\Big] \\
&\times \Big(\int_x^t \alpha_1(s+\tau)\,ds\Big)\sin(2\Lambda^{\frac{1}{2}}(t-x))\,dx\Big)\,dt \\
&+\Lambda^{-\frac{N}{2}}\int_0^\pi \Big(\int_0^t \Big[\sum_{k=1}^{N-1}\alpha_k(x+\tau)\alpha_{N-k}(x+\tau)\Big] \\
&\sin(2\Lambda^{1/2}(t-x))\,dx\Big)\,dt \\
&+O(\Lambda^{-(N+1)/2}).
\end{aligned}
\tag{2.32}
$$

After some calculations (changing the order of integration and using integration by parts repeatedly) (2.32) reduces to the following:

$$
\begin{aligned}
(n+1)\pi = \Lambda^{1/2}\pi - i\sum_{k=1}^{N/2}\Lambda^{-(2k-1)/2}\int_0^\pi \alpha_{2k-1}(t+\tau)\,dt \\
+\frac{i^{N-2}}{2^N}\Lambda^{-N/2}\int_0^\pi q^{(N-1)}(t+\tau)\sin(2\Lambda^{1/2}(\pi-t))\,dt \\
+O(\Lambda^{-(N+1)/2}).
\end{aligned}
\tag{2.33}
$$

Using reversion on (2.33) we get

(2.34)

$$
\begin{aligned}
\Lambda_n^{1/2}(\tau) = (n+1) + \frac{i}{\pi}\sum_{k=1}^{N/2}\frac{1}{(n+1)^{2k-1}}\int_0^\pi \alpha_{2k-1}(t)\,dt \\
+\frac{i^{N-2}}{2^N\pi(n+1)^N}\int_0^\pi q^{(N-1)}(t+\tau)\sin(2(n+1)t)\,dt + O(n^{-(N+1)}).
\end{aligned}
$$

The following lemma will be needed to find the periodic and the semi-periodic eigenvalues.

**Lemma 3.** *Let*

$$
I_1(n,\tau) := \int_0^\pi q^{(N-1)}(t+\tau)\sin(2(n+1)t)\,dt.
$$

*Then*

$$
\begin{aligned}
\min_{0\le\tau<\pi} I_1(n,\tau) = -\Bigg[\Big(\int_0^\pi q^{(N-1)}(t)\sin(2(n+1)t)\,dt\Big)^2 \\
+\Big(\int_0^\pi q^{(N-1)}(t)\cos(2(n+1)t)\,dt\Big)^2\Bigg]^{1/2}
\end{aligned}
\tag{2.35}
$$

*and*

$$
\begin{aligned}
\max_{0\le\tau<\pi} I_1(n,\tau) = \Bigg[\Big(\int_0^\pi q^{(N-1)}(t)\sin(2(n+1)t)\,dt\Big)^2 \\
+\Big(\int_0^\pi q^{(N-1)}(t)\cos(2(n+1)t)\,dt\Big)^2\Bigg]^{1/2}.
\end{aligned}
\tag{2.36}
$$

*Proof.* By a change of variable, $t+\tau=u$, we see that

$$\begin{aligned} I_1(n,\tau) &= \cos(2(n+1)\tau)\int_{\tau}^{\tau+\pi} q^{(N-1)}(t)\sin(2(n+1)t))\,dt \\ &\quad - \sin(2(n+1)\tau)\int_{\tau}^{\tau+\pi} q^{(N-1)}(t)\cos(2(n+1)t))\,dt \\ &= \cos(2(n+1)\tau)\int_{0}^{\pi} q^{(N-1)}(t)\sin(2(n+1)t))\,dt \\ &\quad - \sin(2(n+1)\tau)\int_{0}^{\pi} q^{(N-1)}(t)\cos(2(n+1)t))\,dt. \end{aligned} \tag{2.37}$$

The last equality holds since

$$q^{(N-1)}(t)\overset{\cos}{\sin}(2(n+1)t))$$

is periodic with period $\pi$.

Let

$$\begin{aligned} B_1(n) &:= \int_0^{\pi} q^{(N-1)}(t)\sin(2(n+1)t))\,dt, \\ B_2(n) &:= \int_0^{\pi} q^{(N-1)}(t)\cos(2(n+1)t))\,dt. \end{aligned}$$

Then (2.37) becomes

$$\begin{aligned} I_1(n,\tau) &= B_1(n)\cos(2(n+1)\tau)) - B_2(n)\sin(2(n+1)\tau)) \\ &= \sqrt{B_1^2(n)+B_2^2(n)}\,\sin(2(n+1)\tau)+\psi), \end{aligned} \tag{2.38}$$

where $\psi$ is chosen so that

$$\sin\psi = \frac{B_1(n)}{\sqrt{B_1^2(n)+B_2^2(n)}}, \qquad \cos\psi = \frac{-B_2(n)}{\sqrt{B_1^2(n)+B_2^2(n)}}.$$

Hence a value of $\tau$ which makes (2.38) a minimum is given by

$$\tau_{\min}(n) = \frac{1}{2(n+1)}\Big(\frac{3\pi}{2}-\psi\Big),$$

and a value of $\tau$ that makes (2.38) a maximum is given by

$$\tau_{\max}(n) = \frac{1}{2(n+1)}\Big(\frac{\pi}{2} - \psi\Big).$$

Replacing $\tau$ in (2.38) by $\tau_{\min}(n)$ and $\tau_{\max}(n)$ the lemma is proved.

Finally, asymptotic estimates for the periodic and the semi-periodic eigenvalues stated at the beginning follow from (2.34)–(2.36).

We also note that the error term at the end of (2.34) can be improved to $o(n^{-(N+1)})$ by further manipulations in (2.8). In this case

$$\begin{aligned}\Lambda_n^{1/2}(\tau) = (n+1) &+ \frac{i}{\pi}\sum_{k=1}^{N/2}\frac{1}{(n+1)^{2k-1}}\int_0^\pi \alpha_{2k-1}(t)\,dt\\ &+ \frac{i^{N-2}}{2^N\pi(n+1)^N}\int_0^\pi q^{(N-1)}(t+\tau)\sin(2(n+1)t)\,dt\\ &+ \frac{1}{\pi(n+1)^{N+1}}\int_0^\pi \alpha_1(t)\alpha_{N-1}(t)\,dt\\ &- \frac{1}{2\pi(n+1)^{N+1}}\sum_{k=1}^{N-1}\int_0^\pi \alpha_k(t)\alpha_{N-k}(t)\,dt\\ &+ o(n^{-(N+1)})\end{aligned}$$

as $n \to \infty$.

In the following examples we only find the semi-periodic eigenvalues. The first equation is known as Mathieu equation.

**Example 1.** $q(t) = 2a\cos 2t$, $N = 2$, $a$ is constant.

$$\sqrt{\mu_{2m}} = \frac{1}{8}\,\frac{64\pi m^5+160\pi m^4+144\pi m^3+56\pi m^2+8\pi m-\sqrt{2}a\Delta}{m(1+m)\pi(4m^2+4m+1)} + O(m^{-3}),$$
$$\sqrt{\mu_{2m+1}} = \frac{1}{8}\,\frac{64\pi m^5+160\pi m^4+144\pi m^3+56\pi m^2+8\pi m+\sqrt{2}a\Delta}{m(1+m)\pi(4m^2+4m+1)} + O(m^{-3}),$$

where

$$\Delta = \sqrt{(1+m+m^2-4mcos\pi-4cos\pi m^2)}.$$

**Example 2.** $q(t) = t - (\pi/2)$, $N = 2$.

$$\sqrt{\mu_{2m}} = \frac{64\pi m^3 + 96\pi m^2 + 48\pi m + 8\pi}{32\pi m^2 + 32\pi m + 8\pi} - \frac{\sqrt{(-2\cos(4\pi m + 2\pi) + 2)/(4m^2 + 4m + 1)}}{32\pi m^2 + 32\pi m + 8\pi} + O(m^{-3})$$

$$\sqrt{\mu_{2m+1}} = \frac{64\pi m^3 + 96\pi m^2 + 48\pi m + 8\pi}{32\pi m^2 + 32\pi m + 8\pi} + \frac{\sqrt{(-2cos(4\pi m + 2\pi) + 2)/(4m^2 + 4m + 1)}}{32\pi m^2 + 32\pi m + 8\pi} + O(m^{-3}).$$

**Example 3.** $q(t) = t^{1/2} - 2/3\pi^{1/2}$, $N = 2$,

$$\sqrt{\mu_{2m}} = \frac{192\pi m^3 + 288\pi m^2 + 144\pi m + 24\pi}{96\pi m^2 + 96\pi m + 24\pi} - \frac{3\sqrt{(\pi(\text{Fresnel}\, S(8m+4) + \text{Fresnel}\, C(8m+4))/(2m+1))}}{96\pi m^2 + 96\pi m + 24\pi} + O(m^{-3}),$$

$$\sqrt{\mu_{2m+1}} = \frac{192\pi m^3 + 288\pi m^2 + 144\pi m + 24\pi}{96\pi m^2 + 96\pi m + 24\pi} + \frac{3\sqrt{(\pi(\text{Fresnel}\, S(8m+4) + \text{Fresnel}\, C(8m+4))/(2m+1))}}{96\pi m^2 + 96\pi m + 24\pi} + O(m^{-3}),$$

where

$$\text{Fresnel}\, S(8m+4) = \int_0^{8m+4} \sin\left(\frac{\pi}{2} t^2\right) dt,$$
$$\text{Fresnel}\, C(8m+4) = \int_0^{8m+4} \cos\left(\frac{\pi}{2} t^2\right) dt.$$

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