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ON A THEOREM OF BANACH AND KURATOWSKI AND K-LUSIN SETS

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ABSTRACT. In a paper of 1929, Banach and Kuratowski proved—assuming the continuum hypothesis—a combinatorial theorem which implies that there is no non-vanishing σ-additive finite measure μ on $\mathbf{R}$ which is defined for every set of reals. It will be shown that the combinatorial theorem is equivalent to the existence of a K-Lusin set of size $2^{\aleph_0}$ and that the existence of such sets is independent of ZFC + $\neg$CH.

0. Introduction. In [**1**], Stefan Banach and Kazimierz Kuratowski investigated the following problem in measure theory:

Problem. *Does there exist a non-vanishing finite measure μ on $[0,1]$ defined for every $X \subseteq [0,1]$, which is σ-additive and such that for each $x \in [0,1]$, $\mu(\{x\}) = 0$?*

They showed that such a measure does not exist if one assumes the continuum hypothesis, denoted by CH. More precisely, assuming CH, they proved a combinatorial theorem [**1**, Théorème II] and showed that this theorem implies the nonexistence of such a measure. The combinatorial result is as follows:

Banach-Kuratowski theorem. *Under the assumption of* CH*, there is an infinite matrix $A^i_k \subseteq [0,1]$ (where $i, k \in \omega$) such that*:

(i) *For each $i \in \omega$, $[0,1] = \cup_{k\in\omega} A^i_k$.*

(ii) *For each $i \in \omega$, if $k \neq k'$, then $A^i_k \cap A^i_{k'} = \varnothing$.*

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(iii) *For every sequence* $k_0, k_1, \dots, k_i, \dots$ *of* ω, *the set* $\bigcap_{i\in\omega}(A^i_0 \cup A^i_1 \cup \dots \cup A^i_{k_i})$ *is at most countable.*

In the following we call an infinite matrix $A^i_k \subseteq [0,1]$ (where $i, k \in \omega$) for which (i), (ii) and (iii) hold, a *BK-matrix*.

Wacław-Sierpiński proved—assuming CH—in [**11**] and [**12**] two theorems involving sequences of functions on $[0,1]$ and showed in [**11**] and [**13**] that these two theorems are equivalent to the Banach-Kuratowski theorem, or equivalently, to the existence of a BK-matrix.

Remark. Concerning the problem in measure theory mentioned above, we like to recall the well-known theorem of Stanisław Ulam (cf. [**15**] or [**7**, Theorem 5.6]), who showed that each σ-additive finite measure μ on ω_1, defined for every set $X \subseteq \omega_1$ with $\mu(\{x\}) = 0$ for each $x \in \omega_1$, vanishes identically. This result implies that if CH holds, then there is no non-vanishing σ-additive finite measure on $[0,1]$.

In the sequel we show that even if CH fails, a BK-matrix—which will be shown to be equivalent to the existence of a K-Lusin set of size $2^{\aleph_0}$—may still exist.

Our set-theoretical terminology (including forcing) is standard and may be found in textbooks like [**2**], [**4**] and [**6**].

1. The Banach-Kuratowski theorem revisited. Before we give a slightly modified version of the Banach-Kuratowski proof of their theorem, we introduce some notation.

For two functions $f, g \in {}^\omega\omega$, let $f \preceq g$ if and only if for each $n \in \omega$, $f(n) \leq g(n)$.

For $\mathcal{F} \subseteq {}^\omega\omega$, let $\lambda(\mathcal{F})$ denote the least cardinality such that, for each $g \in {}^\omega\omega$, the cardinality of $\{f \in \mathcal{F} : f \preceq g\}$ is strictly less than $\lambda(\mathcal{F})$. If $\mathcal{F} \subseteq {}^\omega\omega$ is a family of size $\mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of the continuum, then we obviously have $\aleph_1 \leq \lambda(\mathcal{F}) \leq \mathfrak{c}^+$. This leads to the following definition:

$$\mathfrak{l} := \min\{\lambda(\mathcal{F}) : \mathcal{F} \subseteq^\omega \omega \wedge |\mathcal{F}| = \mathfrak{c}\}.$$

If one assumes CH, then one can easily construct a family $\mathcal{F} \subseteq^\omega \omega$ of cardinality $\mathfrak{c}$ such that $\lambda(\mathcal{F}) = \aleph_1$ and therefore CH implies that $\mathfrak{l} = \aleph_1$.

The crucial point in the Banach-Kuratowski proof of their theorem is [**1**, Théorème II′]. In our notation it reads as follows:

Proposition 1.1. *The existence of a* BK*-matrix is equivalent to* $\mathfrak{l} = \aleph_1$.

For the sake of completeness and for the reader's convenience, we give the Banach-Kuratowski proof of Proposition 1.1.

Proof. ($\Leftarrow$). Let $\mathcal{F} \subseteq^\omega \omega$ be a family of cardinality $\mathfrak{c}$ with $\lambda(\mathcal{F}) = \aleph_1$. In particular, for each $g \in {}^\omega\omega$, the set $\{f \in \mathcal{F} : f \preceq g\}$ is at most countable. Let f_α, $\alpha < \mathfrak{c}$, be an enumeration of $\mathcal{F}$. Since the interval $[0,1]$ has cardinality $\mathfrak{c}$, there is a one-to-one function Ξ from $[0,1]$ onto $\mathcal{F}$. For $x \in [0,1]$, let $n_i^x := \Xi(x)(i)$. Now for $i,k \in \omega$, define the sets $A_k^i \subseteq [0,1]$ as follows:

$$x \in A_k^i \quad \text{if and only if} \quad k = n_i^x.$$

It is easy to see that these sets satisfy the conditions (i) and (ii) of a BK-matrix. For (iii), take any sequence $k_0, k_1, \ldots, k_i, \ldots$ of ω and pick an arbitrary $x \in \cap_{i\in\omega} (A_0^i \cup A_1^i \cup \cdots \cup A_{k_i}^i)$. By definition, for each $i \in \omega$, x is in $A_0^i \cup A_0^i \cup A_1^i \cup \cdots \cup A_{k_i}^i$. Hence for each $i \in \omega$ we get $n_i^x \le k_i$, which implies that for $g \in {}^\omega\omega$ with $g(i) := k_i$ we have $\Xi(x) \preceq g$. Now, since $\lambda(\mathcal{F}) = \aleph_1$, $\Xi(x) \in \mathcal{F}$ and x was arbitrary, the set $\{x \in [0,1] : \Xi(x) \preceq g\} = \cap_{i\in\omega} (A_0^i \cup A_1^i \cup \cdots \cup A_{k_i}^i)$ is at most countable.

($\Rightarrow$). Let $A_k^i \subseteq [0,1]$ (where $i,k \in \omega$), be a BK-matrix, and let $\mathcal{F} \subseteq {}^\omega\omega$ be the family of all functions $f \in {}^\omega\omega$ such that $\cap_{i\in\omega} A_{f(i)}^i$ is nonempty. It is easy to see that $\mathcal{F}$ has cardinality $\mathfrak{c}$. Now, for any sequence $k_0, k_1, \ldots, k_i, \ldots$ of ω, the set $\cap_{i\in\omega} (A_0^i \cup A_1^i \cup \cdots \cup A_{k_i}^i)$ is at most countable, which implies that for $g \in {}^\omega\omega$ with $g(i) := k_i$, the set $\{f \in \mathcal{F} : f \preceq g\}$ is at most countable. Hence, $\lambda(\mathcal{F}) = \aleph_1$.

2. ***K*-Lusin sets.** In this section we show that $\mathfrak{l} = \aleph_1$ is equivalent to the existence of a K-Lusin set of size $\mathfrak{c}$.

We work in the Polish space ${}^\omega\omega$.

Fact 2.1. A closed set $K \subseteq {}^\omega\omega$ is compact if and only if there is a function $f \in {}^\omega\omega$ such that $K \subseteq \{g \in {}^\omega\omega : g \preceq f\}$.

(See [**2**, Lemma 1.2.3].)

An uncountable set $X \subseteq^\omega \omega$ is a *Lusin set* if, for each meager set $M \subseteq^\omega \omega$, $X \cap M$ is countable.

An uncountable set $X \subseteq^\omega \omega$ is a K-*Lusin set* if, for each compact set $K \subseteq^\omega \omega$, $X \cap K$ is countable.

Lemma 2.2. *Every Lusin set is a K-Lusin set.*

Proof. By Fact 2.1 every compact set $K \subseteq^\omega \omega$ is meager (even nowhere dense), and therefore, every Lusin set is a K-Lusin set. □

Lemma 2.3. *The following are equivalent*:

(a) $\mathfrak{l} = \aleph_1$.

(b) *There is a K-Lusin set of cardinality* $\mathfrak{c}$.

Proof. This follows immediately from the definitions and Fact 2.1. □

Remark. Concerning Lusin sets we would like to mention that Sierpiński gave in [**14**] a combinatorial result which is equivalent to the existence of a Lusin set of cardinality $\mathfrak{c}$.

For $f, g \in {}^\omega\omega$, define $f \preceq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. The cardinal numbers $\mathfrak{b}$ and $\mathfrak{d}$ are defined as follows:

$$\mathfrak{b} := \min\{|\mathcal{F}| : \mathcal{F} \subseteq^\omega \omega \text{ and } \forall\, g \in {}^\omega\omega \exists f \in \mathcal{F}(f \not\preceq^* g)\}$$

$$\mathfrak{d} := \min\{|\mathcal{F}| : \mathcal{F} \subseteq^\omega \omega \text{ and } \forall\, g \in {}^\omega\omega \exists f \in \mathcal{F}(g \preceq^* f)\}.$$

Lemma 2.4. $\mathfrak{l} = \aleph_1$ *implies* $\mathfrak{b} = \aleph_1$ *and* $\mathfrak{d} = \mathfrak{c}$. *Moreover, K-Lusin sets are exactly those* (*uncountable*) *subsets of* ${}^\omega\omega$ *whose all uncountable subsets are unbounded.* (*Families like that are also called* strongly unbounded *and they play an important role in preserving unbounded families in iterations, see, e.g.,* [**2**] *for details.*)

Proof. Assume $\mathfrak{l} = \aleph_1$, then, by Lemma 2.3, there exists a K-Lusin set $X \subseteq^\omega \omega$ of cardinality $\mathfrak{c}$. It is easy to see that every uncountable subset of X is unbounded, so $\mathfrak{b} = \aleph_1$.On the other hand, every function $g \in {}^\omega\omega$ dominates only countably many elements of X. Hence no family $\mathcal{F} \subseteq {}^\omega\omega$ of cardinality strictly less than $\mathfrak{c}$ can dominate all elements of X, and thus $\mathfrak{d} = \mathfrak{c}$. □

Proposition 2.5. *Adding κ many Cohen reals produces a Lusin set of size κ.*

(See [**2**, Lemma 8.2.6].)

Theorem 2.6. *The existence of a K-Lusin set of cardinality $\mathfrak{c}$ is independent of* ZFC $+\neg$ CH.

Proof. By Proposition 2.5 and Lemma 2.2 it is consistent with ZFC that there is a K-Lusin set of cardinality $\mathfrak{c}$.

On the other hand, it is consistent with ZFC that $\mathfrak{b} > \aleph_1$ or that $\mathfrak{d} < \mathfrak{c}$ (cf. [**2**]). Therefore, by Lemma 2.4, it is consistent with ZFC that there are no K-Lusin sets of cardinality $\mathfrak{c}$. □

By Lemma 2.3 and Proposition 1.1, as an immediate consequence of Theorem 2.6, we get the following.

Corollary 2.7. *The existence of a* BK-*matrix is independent of* ZFC $+\neg$ CH.

3. Odds and ends. An uncountable set $X \subseteq [0,1]$ is a *Sierpiński set* if, for each measure zero set $N \subseteq [0,1]$, $X \cap N$ is countable.

Proposition 3.1. *The following are equivalent:*

(a) CH.

(b) *There exists a Lusin set of cardinality $\mathfrak{c}$ and an uncountable Sierpiński set.*

(c) *There exists a Sierpiński set of cardinality* $\mathfrak{c}$ *and an uncountable Lusin set.*

(See [**9**, p. 217].)

Proposition 3.2. *It is consistent with* ZFC *that there exists a* K*-Lusin set of cardinality* $\mathfrak{c}$*, but there are neither Lusin nor Sierpiński sets.*

Proof. Let $\mathbf{M}_{\omega_2}$ denote the ω_2-iteration of Miller forcing—also called "rational perfect set forcing"—with countable support. Let us start with a model V in which CH holds, and let $G_{\omega_2} = \langle m_\iota : \iota < \omega_2 \rangle$ be the corresponding generic sequence of Miller reals. Then, in $V[G_{\omega_2}]$, G_{ω_2} is a K-Lusin set of cardinality $\mathfrak{c} = \aleph_2$. For this we have to show the following property:

$$\text{For all } f \in {}^\omega\omega \cap V[G_{\omega_2}], \text{ the set } \{\iota : m_\iota \preceq f\} \text{ is countable.}$$

Suppose not, and let $f \in {}^\omega\omega \cap V[G_{\omega_2}]$ be a witness. Further, let p be an $\mathbf{M}_{\omega_2}$-condition such that

$$p \Vdash_{\mathbf{M}_{\omega_2}} \text{"for some } n_0 \in \omega, \text{ the set } \{\iota : \forall k \geq n_0 (m_\iota(k) < \dot{f}(k))\} \text{ is uncountable."}$$

We can assume that these dominated reals m_ι are among $\{m_\alpha : \alpha < \beta < \omega_2\}$ and that β is minimal. This way, f is added after step β of the iteration. Let $a^* := \mathrm{cl}(\dot{f})$ be the (countable) set of ordinals such that, if we know $\{m_\iota : \iota \in a\}$, then we can compute $\dot{f}$. (Notice that a^* is much more than just the support of $\dot{f}$, since it contains also supports of all conditions that are involved in conditions involved in $\dot{f}$, and so on.) Let N be a countable model such that p, $\dot{f} \in N$, $a^* \subseteq N$, and let $\mathbf{M}_{\alpha^*}$ be the iteration of Miller forcing, where we put the empty forcing at stages $\alpha \notin a^*$ (essentially, $\mathbf{M}_{a^*}$ is the same as $\mathbf{M}_{\mathrm{o.t.}(a^*)}$).

The crucial lemma—which is done in [**10**, Lemma 3.1] for Mathias forcing, but also works for Miller forcing—is the following: If $N \models p \in \mathbf{M}_{\alpha^*}$, then there exists a $q \in \mathbf{M}_{\omega_2}$ which is stronger than p such that $\mathrm{cl}(q) = a^*$ and q is $(N, \mathbf{M}_{\alpha^*})$-generic over N. In particular, if $\{m_\iota : \iota < \omega_2\}$ is a generic sequence of Miller reals consistent with q, then $\{m_\iota : \iota \in a^*\}$ is $\mathbf{M}_{\alpha^*}$-generic over N (consistent with p).

So, fix such a q. Now we claim that for $\gamma \in \beta \setminus N$, q forces that $\dot{f}(k) > m_\gamma(k)$ for some $k \geq n_0$: Take any $\gamma \in \beta \setminus N$ and let q^* be a condition stronger than q. Let $q_1^* = q_1 \mid \beta$, and let $q_2^* = q^* \mid a$. Without loss of generality, we may assume that $q_2^* = q$. Now, first we strengthen q_1^* to determine the length of stem of $q_1^*(\gamma)$ and make it equal to some $k > n_0$. Next we shrink q_2^* to determine the first k digits of $\dot{f}$. Finally we shrink $q_1^*(\gamma)$ such that $q_1^*(\gamma)(k) > \dot{f}(k)$. Why can we do this? Although f is added after m_γ, from the point of view of model N, it was added before. So, working below condition q_2^* (in $\mathbf{M}_{\alpha^*}$) we can compute as many digits of $\dot{f}$ as we want without making any commitments on m_γ and vice versa. Even though the computation is in N, it is absolute. This completes the first part of the proof.

On the other hand, it is known (cf. [**5**]) that in $V[G_{\omega_2}]$, there are neither Lusin nor Sierpiński sets of any uncountable size, which completes the proof. $\square$

Proposition 3.3. *It is consistent with* ZFC *that* $\mathfrak{b} = \aleph_1$ *and* $\mathfrak{d} = \mathfrak{c}$*, but there is no* K*-Lusin set of cardinality* $\mathfrak{c}$.

Proof. Take a model M in which we have $\mathfrak{c} = \aleph_2$ and in which Martin's Axiom MA holds. Let $G = \langle c_\beta : \beta < \omega_1 \rangle$ be a generic sequence of Cohen reals of length ω_1. In the resulting model $M[G]$ we have $\mathfrak{b} = \aleph_1$ (since the set of Cohen reals forms an unbounded family) and $\mathfrak{d} = \aleph_2$. On the other hand, there is no K-Lusin set of cardinality $\mathfrak{c}$ in $M[G]$. Why? Suppose $X \subseteq^\omega \omega$ has cardinality $\aleph_2$. Take a countable ordinal α and a subset $X' \subseteq X$ of cardinality $\aleph_2$ such that $X' \subseteq M[G_\alpha]$, where $G_\alpha := \langle c_\beta : \beta \leq \alpha \rangle$. Now $M[G_\alpha] = M[c]$ (for some Cohen real c) and $M[c] \models$ MA (σ-centered) (cf. [**8**] or [**2**, Theorem 3.3.8]). In particular, since MA (σ-centered) implies $\mathfrak{p} = \mathfrak{c}$ and $\mathfrak{p} \leq \mathfrak{b}$, we have $M[c] \models \mathfrak{b} = \aleph_2$. Thus there is a function which bounds uncountably many elements of X'. Hence, by Lemma 2.4, X cannot be a K-Lusin set. $\square$

Let Q be a countable dense subset of the interval $[0, 1]$. Then $X \subseteq [0, 1]$ is *concentrated on* Q if every open set of $[0, 1]$ containing Q contains all but countably many elements of X.

Proposition 3.4. *The following are equivalent*:

(a) *There exists a K-Lusin set of cardinality* $\mathfrak{c}$.

(b) *There exists a concentrated set of cardinality* $\mathfrak{c}$.

Proof. (b) $\rightarrow$ (a). Let Q be a countable dense set in $[0,1]$, and let $\varphi : [0,1] \setminus Q \rightarrow {}^\omega\omega$ be a homeomorphism. If $U \subseteq^\omega \omega$ is compact, then $\varphi^{-1}[U]$ is compact, so closed in $[0,1]$ and $[0,1] \setminus \varphi^{-1}[U]$ is an open set containing Q. Hence, the image under φ of an uncountable set $X \subseteq [0,1]$ concentrated on Q is a K-Lusin set of the same cardinality as X.

(a) $\rightarrow$ (b). The preimage under φ of a K-Lusin set of cardinality $\mathfrak{c}$ is a set concentrated on Q of the same cardinality. $\square$

Remark. A Lusin set is concentrated on every countable dense set, and concentrated sets always have strong measure zero. However, the existence of a strong measure zero set of size $\mathfrak{c}$ does not imply the existence of a concentrated sets of size $\mathfrak{c}$. In fact, the existence of a strong measure zero set of size $\mathfrak{c}$ is consistent with $\mathfrak{d} = \aleph_1$ (see [**3**]).

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