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# COHOMOLOGICAL PROPERTIES OF MULTIPLE COVERINGS OF SMOOTH PROJECTIVE CURVES

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ABSTRACT. Let $X$, respectively $C$, be a smooth projective curve of genus $g$, respectively $q$, and $f : X \to C$ a degree $k$ finite morphism. Set $E := f_*(\mathbf{O}_X)/\mathbf{O}_C$. Hence $E$ is a rank $k-1$ vector bundle on $C$ with $\deg(E) = kq - k + 1 - g$. Here we study the cohomological properties of $E$ and in particular the integers $h^0(C, E(tP))$, $P \in C$ and $t \in \mathbf{N}$. We use these integers to define the notion of $f$-Weierstrass points.

**1. Introduction.** Let $X$, respectively $C$, be a smooth connected projective curve of genus $g$, respectively $q$, defined over an algebraically closed base field $\mathbf{K}$ and $f : X \to C$ a degree $k$ covering. Set $E := f_*(\mathbf{O}_X)/\mathbf{O}_C$. The sheaf $E$ is a rank $k-1$ vector bundle on $C$. We will say that $E$ is the bundle associated to $f$. Many geometrical properties of $X$ are detected by the cohomological properties of $E$. If $q = 0$, then $E$ is a direct sum of line bundles and the degrees of the rank 1 summands of $E$ uniquely determine the so-called scrollar invariants of the pencil $f$ (see [**12**, Section 2]). In this paper we will consider the case $q > 0$. If either $\mathrm{char}\,(\mathbf{K}) = 0$ or $\mathrm{char}\,(\mathbf{K}) > k$, the trace map induces a splitting $f_*(\mathbf{O}_X) \cong \mathbf{O}_C \oplus E$; since $X$ is connected, in this case we have $h^0(C, E) = 0$. For any $P \in C$ and any integer $t$, set $n(f, P, t) := h^0(C, E(tP))$ and $N(f, P, t) := h^0(C, f_*(\mathbf{O}_X)(tP)) = h^0(X, (f^{-1}(P))^{\otimes t})$ (projection formula). The sequence $n(f, P) := \{n(f, P, t)\}_{t \geq 0}$, respectively $N(F, P) := \{N(f, P, t)\}_{t \geq 0}$, will be called the numerical sequence, respectively the total numerical sequence, of $f$ at $P$. Set $n(f, t) := n(f, P, t)$ and $N(f, t) := N(f, P, t)$ for general $P \in C$. The sequence $n(f) := \{n(f, t)\}_{t \geq 0}$, respectively $N(f) := \{N(f, t)\}_{t \geq 0}$, will be called the numerical sequence, respectively the total numerical sequence, of $f$. If $P \in C$ and $n(f, P, t) \neq f(f, t)$ for

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some $t$, i.e., if the numerical sequence of $f$ at $P$ is not the numerical sequence of $f$ at $P$ is not the numerical sequence of $f$, then $P$ is called a Weierstrass point of $f$ or an $f$-Weierstrass point. In the same way we define the total Weierstrass points of $f$. If $f_*(\mathbf{O}_X) \cong \mathbf{O}_C \oplus E$, every Weierstrass point of $C$ is a total Weierstrass point of $f$. If $q = 0$, then there is no $f$-Weierstrass point (see Remark 4.1), and the knowledge of the numerical sequence of $f$ is equivalent to the knowledge of the scrollar invariants of the pencil $f$.

*Remark* 1.1. Since $f$ is finite, we have $h^0(C, f_*(\mathbf{O}_X)) = h^0(X, \mathbf{O}_X) = 1$ and $h^1(C, f_*(\mathbf{O}_X)) = h^1(X, \mathbf{O}_X) = g$. Hence $\chi(E) = \chi(f_*(\mathbf{O}_X)) - \chi(\mathbf{O}_C) = 1 - g + q - 1 = q - g$. Thus, by Riemann-Roch we have $\deg(E) = \chi(E) + (k-1)(q-1) = kq - k - g + 1$. For any $P \in C$ and any integer $t$ we have $\deg(E(tP)) = kq - k - g + 1 + t(k-1)$. Hence by Riemann-Roch for any $P \in C$ and any integer $t$ we have $h^0(C, E(tP)) \geq \max\{0, q-g+t(k-1)\}$ and $h^1(C, E(tP)) \geq \max\{0, (k-1)(q-1) - kq + k + g - 1 - t(k-1)\} = \max\{0, g - q - t(k-1)\}$.

Motivated by Remark 1.1, we introduce the following definition

**Definition 1.2.** The covering $f$ is called *cohomologically balanced* if, for a general $P \in C$, we have $h^0(C, E(tP)) = \max\{0, q-k+t(k-1)-g\}$, or equivalently, $h^1(C, E(tP)) = \max\{0, g-q-t(k-1)\}$, for every integer $t > 0$.

The covering $f$ is cohomologically balanced if and only if, for a general $P \in C$ and every integer $t$, or just every positive integer, we have $h^0(C, E(tP)) \cdot h^1(C, E(tP)) = 0$.

**Example 1.3.** If $q = 0$, then $E$ is the direct sum of $k-1$ line bundles, say of degree $a_1, \dots, a_{k-1}$ with $a_1 \geq \cdots \geq a_{k-1}$. The pencil $f$ is cohomologically balanced if and only if $a_1 \leq a_{k-1} + 1$, i.e., if and only if the vector bundle $E$ is rigid. By [**2**] for a general $k$-gonal curve of genus $g \geq 2k-1$, the unique associated degree $k$ pencil $f : X \to \mathbf{P}^1$ is cohomologically balanced.

In Section 2 we will prove the following result.

**Theorem 1.4.** *Fix integers $k, \gamma, b$ with $k \geq 2$, $b \geq 2$, $\gamma \geq 2k$, $\gamma - k + 1 \equiv 0 \pmod{k-1}$, a smooth projective curve $C$ with $p_a(C) = q$ and a degree $b$ morphism $\beta : C \to \mathbf{P}^1$. Then there exists a degree $k$ covering $f : X \to C$ with $X$ smooth curve of genus $k(q-1)+1+b(\gamma-k+1)$ such that the bundle associated to $f$ is the direct sum of $k-1$ isomorphic line bundles of degree $b(\gamma-k+1)/(k-1)$. If either* $\mathrm{char}\,(\mathbf{K}) = 0$ *or* $\mathrm{char}\,(\mathbf{K}) > b(\gamma+k-1)$*, then $f$ is cohomologically balanced.*

*Remark* 1.5. Let $C$ be a smooth projective curve of genus $q$. The condition $g - k(q-1) - 1 \equiv 0 \pmod{k-1}$ is a necessary condition for the existence of a degree $k$ covering $f : X \to C$ with $X$ of genus $g, C$ of genus $q$ and such that the associated bundle is the direct sum of $k-1$ line bundles of the same degree. For every integer $b \geq q+1$ there are many nonspecial degree $b$ pencils $\beta : C \to \mathbf{P}^1$; if $b > q+1$, the associated linear system is not complete, but this does not matter from the point of view of Theorem 1.4. Hence we see that $g \geq 4q+1$; then the numerical assumptions of Theorem 1.4 are necessary for the existence of a degree $k$ covering $X \to C$ with $X$ of genus $g$ and such that the associated bundle is a direct sum of line bundles with the same degree.

Now we will extend the definition of associated bundle to the case of coverings $f : X \to C$ in which $X$ is a singular curve (see Remark 1.6). In this more general set-up, we are able to give a very strong generalization of Theorem 1.4 (see Theorem 1.7).

*Remark* 1.6. Let $X$ be an integral projective curve, $C$ a smooth projective curve and $f : X \to C$ a finite morphism. Set $k := \deg(f)$. Since $f$ is finite, $C$ smooth and $X$ locally Cohen-Macaulay, the morphism $f$ is flat and the coherent sheaf is locally free. Furthermore, $E := f_*(\mathbf{O}_X)/\mathbf{O}_X$ is a rank $k-1$ vector bundle on $C$. We will call $E$ the associated bundle of $f$. Set $g := p_a(X)$ and $q := p_a(C)$. As in Remark 1.1 we obtain $\deg(E) = kq - k - g + 1 + t(k-1)$. We extend the definition of cohomologically balanced to this set-up. Since $C$ is smooth, every $Q \in C$ is a Cartier divisor of $C$. Hence, for every $Q \in C$, the scheme theoretic fiber $f^{-1}(Q)$ is a Cartier divisor of $X$. We will say that $Q \in C$ is $f$-Weierstrass if there is an integer $t \geq 1$ such that $h^0(C, E(tQ)) > h^0(C, E(tP))$ for a general $P \in C$.

**Theorem 1.7.** *Fix integers $k, \gamma, b, g$ with $k \geq 2$, $b \geq 2$, $\gamma \geq 2k$, $\gamma - k + 1 \equiv 0 \pmod{k-1}$ and $k(q-1) + 1 + b(\gamma + k - 1) \leq g \leq k(q-1) + k - 1 + b(\gamma + k - 1)$. Assume either* $\mathrm{char}\,(\mathbf{K}) = 0$ *or* $\mathrm{char}\,(\mathbf{K}) > b(\gamma + k - 1)$. *Fix a smooth projective curve $C$ with $p_a(C) = q$ and a degree $b$ morphism $\beta : C \to \mathbf{P}^1$. Then there exist an integral projective nodal curve $Z$ with $p_a(Z) = g$ and exactly $g - k(q-1) - 1 - b(\gamma+k-1)$ ordinary nodes as only singularities and a degree $k$ covering $f : X \to C$ such that, calling $F$ the associated bundle of $f$ and $\pi : Z \to X$ the normalization map, the degree $k$ covering $f \circ \pi : Z \to C$ has as associated vector bundle the direct sum, $E$, of $k - 1$ isomorphic line bundles; and, for a general $P \in C$ and every $t \leq b(\gamma-k+1)/(k-1)+q-1$ we have $h^0(C, F(tP)) = h^0(C, E(tP)) = 0$.*

In Section 3 we obtain several degree $k$ coverings $f : X \to C$ with good cohomological properties taking as $X$ the normalization of a nodal curve $Y \subset C \times \mathbf{P}^1$ and as $f$ the morphism induced by the projection of $C \times \mathbf{P}^1$ onto its first factor (see 3.1). If $q := p_a(C) = 1$, then we obtain a cohomologically balanced covering, see 3.3. In Section 4 we prove that if $g \equiv 1 \pmod{k-1}$ and $q > 0$, then every degree $k$ covering has a Weierstrass point. In Section 5 we show that if $q \geq 4$ a sufficiently general stable bundle does not occur as associated bundle (see 5.12 and 5.13). In the same section we prove other results concerning the existence or nonexistence of coverings with certain bundles as associated bundle if $k = 3$ or $q = 1$.

## 2. Examples and proofs of 1.4 and 1.7.

*Remark* 2.1. Assume $q = 0$ and use the notation of Example 1.3. In arbitrary characteristic we have $f_*(\mathbf{O}_X) \cong \mathbf{O}_C \oplus E$ for the following reason. Since $X$ is connected, we have $h^0(C, f_*(\mathbf{O}_X)) = h^0(X, \mathbf{O}_X) = 1$. Since $h^1(C, \mathbf{O}_C) = q = 0$, we obtain $h^0(\mathbf{P}^1, E) = 0$. Hence $a_i < 0$ for every $i$. Since any extension of a degree $a_i < 0$ line bundle on $\mathbf{P}^1$ by the trivial line bundle splits, we obtain $f_*(\mathbf{O}_X) \cong \mathbf{O}_C \oplus E$.

**Example 2.2.** Here we assume $k = 2$ and $\mathrm{char}\,(\mathbf{K}) \neq 2$. Hence $E \in \mathrm{Pic}\,(C)$ and $f_*(\mathbf{O}_X) \cong \mathbf{O}_C \oplus E$. Set $N := E^*$. The double covering $f$ is uniquely determined by $C, N$ and the choice of an effective reduced divisor $B \in |N^{\otimes 2}|$. Vice versa, any such triple $(C, N, B)$ gives a unique double covering $f : X \to C$ with $X$ smooth; the curve $X$ is connected if and only if $N \neq \mathbf{O}_C$. By the Riemann-Hurwitz formula we have $\deg(N) = g - 2q + 1$. A point $p \in C$ is a Weierstrass point of $f$ if and only if it is a Weierstrass point in the sense of [**6**] for the complete linear system on $C$ associated to $\omega_C \otimes N$. Hence in this way we may associate weights to any Weierstrass point. The double covering $f$ is cohomologically balanced if and only if the complete linear system on $C$ associated to $\omega_C \otimes N$ has classical generic Hermite invariants, i.e., for every integer $t$ with $1 \leq t \leq h^0(C, \omega_C \otimes N) - 2$ the generic $t$-dimensional osculating subspace of the image curve has order of contact $t + 1$ at the osculating point. Since $N$ is nontrivial, by [**6**, Theorem 15], this is always the case if either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) > \deg(\omega_C \otimes N)$, but it may fail for low $\mathrm{char}(\mathbf{K})$ if $q > 0$. By the Brill-Segre formula [**6**, Theorem 9 and Theorem 15], if either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) > \deg(\omega_C \otimes N)$ the total number of $f$-Weierstrass points (counting multiplicities) is $h^0(C, \omega_C \otimes N)(h^0(C, \omega_C \otimes N) - 1)(q - 1) + h^0(C, \omega_C \otimes N)\deg(\omega_C \otimes N)$.

*Remark* 2.3. Assume $\mathrm{char}(\mathbf{K}) = 0$. Let $f : X \to C$ be a degree $k$ covering whose associated bundle is a direct sum of $k - 1$ line bundles $L_1, \ldots, L_{k-1}$ with $0 \geq \deg(L_1) \geq \ldots \geq \deg(L_{k-1})$. As in 2.1 or 1.3 we see that $f$ is cohomologically balanced if and only if $\deg(L_1) \leq \deg(L_{k-1}) + 1$. Set $b := \deg(L_1) - \deg(L_{k-1})$. As in 2.2, by [**6**, Theorem 15], for a general $P \in C$ we have $h^0(C, E(tP)) = 0$ if and only if $t + \deg(L_1) \leq q - 1$ and $h^1(C, E(tP)) = 0$ if and only if $t + \deg(L_{k-1}) \geq q - 1$. Hence if $b \geq 2$ there are exactly $b - 1$ integers $t$, all consecutive, such that $h^0(C, E(tP)) \cdot h^1(C, E(tP)) \neq 0$ for a general $P \in C$.

**Example 2.4.** Here we assume that $f : X \to C$ is a simple degree $k$ cyclic covering in the sense of [**4**, Example 1.1]. We assume either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) > k$. hence there is an $N \in \mathrm{Pic}(C)$ and a reduced divisor $B \in |N^{\otimes k}|$ such that $E^* \cong \oplus_{1 \leq i < k} N^{\otimes i}$ and $B$ is the branch locus of $f$. Hence $\deg(E) = -k(k-1)\deg(N)/2$. Vice versa, any such pair $(N, B)$ gives a degree $k$ simple cyclic covering $f : X \to C$ with $X$ smooth. $X$ is connected, i.e., $X$ is not the disjoint union of $k$ copies of $C$, each of them mapped isomorphically onto $C$ by $f$, if and only if $N \neq \mathbf{O}_C$. By the Riemann-Hurwitz formula or Remark a1 we have $g = kq - k + 1 + k(k-1)\deg(N)/2$. The set of all $f$-Weierstrass points is the union of the sets of all Weierstrass points in the sense of [**6**] of the complete linear systems associated to $\omega_C \otimes N^{\otimes i}$, $1 \leq i \leq k-1$.

*Remark* 2.5. Assume $\mathrm{char}(\mathbf{K}) = 0$. We saw in Example 2.1 that if $k \geq 3$ a simple degree $k$ cyclic covering is cohomologically balanced if and only if it is étale. A similar proof works for a tower of cyclic extensions, none of them étale or of degree 2.

**Example 2.6.** Fix integers $k, \gamma, b$ with $k \geq 2$, $b \geq 2$, $\gamma \geq 2k$, $\gamma - k + 1 \equiv 0 \pmod{k-1}$, a smooth projective curve $C$ with $p_a(C) = q$, a degree $b$ morphism $\beta : C \to \mathbf{P}^1$ and a degree $k$ covering $\tau : Y \to \mathbf{P}^1$ such that $Y$ is a smooth curve of genus $\gamma$ and the pencil $\tau$ is cohomologically balanced. By Remark 2.1 we have $\gamma_*(\mathbf{O}_Y) \cong \mathbf{O}_{\mathbf{P}^1} \oplus F$ with $F$ isomorphic to a direct sum of line bundles of degree $a_1, \ldots, a_{k-1}$ with $0 > a_1 \geq \cdots \geq a_{k-1}$; for instance, by [**2**] as pair $(Y, \tau)$ we may take a general $k$-gonal curve of genus $\gamma$ with its associated pencil. Since $\tau$ is cohomologically balanced, we have $a_1 \leq a_{k-1} + 1$, Example 1.3. Since $a_1 + \cdots + a_{k-1} = -\gamma + k - 1$, Remark 1.1, and $\gamma - k + 1 \equiv 0 \pmod{k-1}$, we have $a_j = a_1$ for every $j$, i.e., $F$ is a direct sum of $k-1$ isomorphic line bundles. Assume that the branch loci of $\tau$ and of $\beta$ are disjoint and that the ramification of $\tau$ is ordinary and over $2\gamma + 2k - 2$ distinct points of $\mathbf{P}^1$. Take the fiber product of the morphisms $\tau$ and $\beta$ and call $f : X \to C$ the associated degree covering. $X$ is a smooth curve by the universal property of the normalization. The curve $X$ has genus $k(q-1) + 1 + b(\gamma + k - 1)$ by the Riemann-Hurwitz formula. We have $f_*(\mathbf{O}_X) = \mathbf{O}_C \oplus E$ with $E \cong \tau^*(F)$. Hence $E$ is the direct sum of $k-1$ isomorphic line bundles. Hence, as in Example b1, if either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K})$ is large, then $f$ is cohomologically balanced. Furthermore, we see that if in this case each $f$-Weierstrass point must be counted with weight at least $k-1$.

*Proof of Theorem* 1.4. As in Example 2.6 take as $f$ a fiber product of $\beta$ with a degree $k$ covering $u$: $Y \to \mathbf{P}^1$ with $Y$ general smooth $k$-gonal curve of genus $\gamma$. Since $\gamma - k + 1 \equiv 0 \pmod{k-1}$, the bundle associated to $u$ is the direct sum of isomorphic line bundles [**2**]. For a general choice of the pair $(Y, u)$ we obtain the connectedness of $X$.

The last assertion follows from the discussion in Example 2.2.

*Proof of Theorem* 1.7. By Theorem 1.4 there exist a smooth curve $X$ of genus $k(q-1)+1+b(\gamma+k-1)$ and a degree $k$ covering $h : Z \to C$ such that the associated vector bundle $E$ is the direct sum of $k-1$ copies of a line bundle of degree $b(\gamma-k+1)/(k-1)$. Hence for general $P \in C$ we have $h^0(C, E(tP)) = 0$ if $t \le b(\gamma-k+1)/(k-1)+q-1$. Take as $X$ any nodal curve obtained from $X$ by pinching together $g-k(q-1)-1-b(\gamma+k-1)$ pairs of points of $X$, say $\{P_i, Q_i\}$, $1 \le i \le g-k(q-1)-1-b(\gamma+k-1)$ such that $h(P_i) = h(Q_i)$ for every $i$ and $h(P_i) \ne h(P_j)$ for $i \ne j$. By construction $Z$ is the normalization of $C$ and the covering $h : Z \to C$ descends to a covering $f : X \to C$. Call $F$ the bundle associated to $f$. Since $\mathbf{O}_X$ is a subsheaf of $\pi_*(\mathbf{O}_Z)$, $F$ is a subsheaf of $E$. Hence for a general $P \in C$ we have $h^0(C, F(tP)) \le h^0(C, E(tP)) = 0$ for every $t \le b(\gamma-k+1)/(k-1)+q-1$.

**3. Nodal curves in a ruled surface.** In this section we will use the following notation. Let $C$ be a smooth projective curve of genus $q$. Set $M := C \times \mathbf{P}^1$. We have $\mathrm{Pic}\,(M) \cong \mathrm{Pic}\,(C) \times \mathbf{Z}$ and we will write $(L, b)$ for the element of $\mathrm{Pic}\,(M)$ induced by $L \in \mathrm{Pic}\,(C)$ and the degree $b$ line bundle on $\mathbf{P}^1$. If $\deg(L) = a$ we will say that $(L, b)$ has numerical type $(a, b)$. Similarly we will say that an effective Cartier divisor on $M$ is of type $(L, b)$ if $Y \in |(L, b)|$ and $(a, b)$, $a := \deg(L)$, will be called the numerical type of $Y$. The canonical sheaf $\omega_M$ of $M$ is $(\mathbf{O}_C, -2)$. Hence by Künneth formula we have $h^0(M, \omega_M) = 0$ and $h^1(M, \omega_M) = q$. Fix an integer $k \ge 2$. Let $Y \subset M$ be an integral projective curve with numerical type $(a, k)$ for some $k$ and $\pi : X \to Y$ its normalization. The composition of $\pi$ with the projection onto $C$ induces a degree $k$ covering $f : X \to C$. By the adjunction formula we have $\omega_Y \cong \omega_M(Y) \mid Y$. Thus $2p_a(Y) - 2 = (a+2q-2)k + (k-2)a$. Since $h^0(M, \omega_M) = 0$ and $h^1(M, \omega_M) = q$, the restriction map $\rho : H^0(M, \omega_M(Y)) \to H^0(Y, \omega_Y)$ is injective and $\mathrm{diam}\,(\mathrm{Coker}\,(\rho)) = q$. Now assume that $Y$ has only ordinary nodes as singularities and set $S := \mathrm{Sing}\,(Y)$ and $z := \mathrm{Card}\,(S)$. Since the conductor of $\mathbf{O}_Y$ in $\mathbf{O}_X$ is just $\mathbf{I}_{S,Y}$, we see that $p_a(X) = p_a(Y) - z$ and that $H^0(M, \mathbf{I}_S \otimes \omega_M(Y))$ is naturally isomorphic to a subspace, $V$, of $H^0(X, \omega_X)$. Furthermore, if $S$ imposes $z$ independent conditions to

$H^0(M, \omega_M(Y))$, i.e., if $h^0(M, \mathbf{I}_S \otimes \omega_M(Y)) = h^0(M, \omega_M(Y)) - z$, then $V$ has codimension $q$ in $H^0(X, \omega_X)$. For any $N \in \mathrm{Pic}\,(C)$, $t \in \mathbf{Z}$ and any finite subset $S$ of $M$, set $W(S, N, t) := \mathbf{P}(H^0(M, \mathbf{I}_S \otimes (N, t)))$ and $V(S, N, t) := \mathbf{P}(H^0(M, (\mathbf{I}_S)^2 \otimes (N, t)))$.

**Theorem 3.1.** *Assume* $\mathrm{char}\,(\mathbf{K}) = 0$. *Fix integers* $a, k, z$ *with* $k \geq 2$, $a \geq 4q + 2$ *and* $0 \leq z \leq k - 1$, $N \in \mathrm{Pic}^{\,a}(C)$ *with* $\deg(N) = a$ *and a general subset* $S$ *of* $M$ *with card*$(S) = z$. *There exists an irreducible nodal curve* $Y \subset M$ *of type* $(N, K)$ *with Sing*$(Y) = S$. *Fix any such curve* $Y$ *and call* $X$ *the normalization of* $Y$ *and* $f : X \to C$ *the degree* $k$ *morphism induced by the projection* $M \to C$. *Let* $E$ *be the bundle associated to* $f$. *Hence* $p_a(X) = (a + k - 2)a/2 + (q - 1)k + 1$ *and* $\deg(E) = q - p_a(X) = -(a+k-2)a/2-(q-2)k-1$. *For a general* $P \in c$ *we have* $h^0(C, E(tP)) = 0$ *and* $h^1(C, E(tP)) = h^1(C, E) - t(k-1) = (k-1)p_a(X) - \deg(E) - t(k-1) = k(a+k-2)a/2 + (q-1)k^2 + k - q - t(k-1)$ *for every integer* $t$ *with* $0 \leq t(k-1) \leq k(a+k-2)a/2+(q-1)k^2+k-q$.

*Proof.* We first prove the existence of the nodal curve $Y$.

**First claim.** *Fix integers* $a, k, z$ *with* $k \geq 2$, $a \geq 4q + 2$ *and* $0 \leq z \leq k - 1$, $N \in \mathrm{Pic}^{\,a}(C)$ *with* $\deg(N) = a$ *and a general subset* $S$ *of* $M$ *with card*$(S) = z$. *We have* $h^0(M, (\mathbf{I}_S)^2 \otimes (N, k)) = (k + 1)(a + 1 - q) - 3z$ *and* $h^1(M, (\mathbf{I}_S)^2 \otimes (N, k)) = 0$. *We have* $\mathrm{diam}\,(V(S, N, k)) = (k+1)(a+1-q)-3z > 0$ *and a general* $Y \in V(S)$ *is an irreducible nodal curve with* $S = $ *Sing*$(Y)$.

*Proof of the first claim.* Since $a \geq 4q + 2$, there are very ample line bundles $A, B$ on $C$ with $N \cong A \otimes B$, $\deg(A) \geq 2q+1$, $\deg(B) \geq 2q+1$ and hence with $h^1(C, A) = h^1(C, B) = 0$. By the generality of $S$ we have $\mathrm{diam}\,(W(S, A, [k/2])) = (\deg(A) + 1 - q)([(k/2] + 1) - z$ and $\mathrm{diam}\,(W(S, B, [(k + 1)/2])) = (\deg(B) + 1 - q)([(k + 1)/2])\mathbf{Z}$. It is easy to see that the linear systems $W(S, A, [k/2])$ and $W(S, B, [(k + 1)/2])$ have $S$ as scheme-theoretic base locus and that they induce a local embedding of $M \setminus S$. Hence by a characteristic free form of Bertini's theorem the general $E \in W(S, A, [k/2])$ and the general $F \in W(S, B, [(k+1)/2])$ are smooth. Since the linear systems $W(S, A, [k/2])$ and $W(S, B, [(k+1)/2])$ have $S$ as scheme-theoretic base locus and they induce a local embedding of $M \setminus S$, the general $E \in W(S, A, [k/2])$ and the general $F \in W(S, B, [(k+1)/2])$ are transverse, i.e., the curve $E \cup F \in V(S, N, k)$ is nodal and $\mathrm{Sing}\,(E \cup F) = S$. In this way we easily obtain $h^1(M, (\mathbf{I}_S)^2 \otimes (N, k)) = 0$ and hence $h^0(M, (\mathbf{I}_S)^2 \otimes (N, k)) = (a + 1 - q)(k + 1) - 3z$. To prove the first claim it is sufficient to show that a general $Y \in W(S, N, k)$ is not of the form $E \cup F$ with $E \in W(S, A, [k/2])$ and $F \in W(S, B, [(k+1)/2])$. This is true because $(a+1-q)(k+1)-3z-1 > (\deg(A)+1-q)([k/2]+1)-z+(\deg(B)+1-q)([(k+1)/2]+1)-z$ since $z \leq k-1$ and $\deg(A) + \deg(B) = a$.

Fix any nodal and irreducible $Y \in V(S, a, k)$ and call $X$ and $E$ the associated objects. The numerical invariants of $X$ and $E$ were computed before the statement of 3.1. Since $\mathrm{char}\,(\mathbf{K}) = 0$, the complete linear system of the line bundle $\omega_C \otimes N$ has classical Hermite invariants at a general $P \in C$, i.e., for a general $P \in C$ and every integer $t \geq 0$ we have $h^0(C, \omega_C \otimes N(-tP)) = \max\{0, h^0(C, \omega_C \otimes N) - t) = \max\{a + q - 1, t\}$ [**6**, Theorem 15]. Hence the result follows from the injectivity of the map $\rho : H^0(M, \omega_M(Y)) \to H^0(Y, \omega_Y)$, the equality $\dim(\mathrm{Coker}\,(\rho)) = q$ and the corresponding result for $X$ proved before the statement of 3.1.

*Remark* 3.2. Assume $\mathrm{char}\,(\mathbf{K}) > 0$. We may obtain verbatim Theorem 3.1 if we assume that the complete linear systems $\omega_C$ and $\omega_C \otimes N$ have classical invariants at the generic point of $C$. For instance it is sufficient to assume $\mathrm{char}\,(\mathbf{K}) > 2q - 2 + a$.

**Theorem 3.3.** *Fix integers $g, k$ with $k \geq 2$, $g \geq 3k + 13$ and $g \equiv 2$ (mod $k - 1$). Assume either $\mathrm{char}\,(\mathbf{K}) = 0$ or $\mathrm{char}\,(\mathbf{K}) > k$. Let $C$ be an elliptic curve. Then there exists a cohomologically balanced degree $k$ covering $f : X \to C$ with $X$ a smooth curve of genus $g$.*

*Proof.* Apply Theorem 3.1 to the curve $C$ and let $f : X \to C$ be the corresponding covering. Call $E$ the bundle associated to $f$. We have $\deg(E) = 1 - g$. By Theorem 3.1 for a general $P \in C$ and every integer $t \leq (g - 2)/(k - 1)$, we have $h^0(C, E(tP)) = 0$, i.e., $h^1(C, E(tP)) = h^1(C, E) - t(k-1)$. Hence $h^1(C, E((g-2)/(k-1))) = 1$. Since $P$ is general we obtain $h^1(C, E(1 + (g - 2)/(k - 1))) = 0$. Thus $h^1(C, E(y)) = 0$ for every $y \geq 1 + (g - 2)/(k - 1)$. Hence $f$ is cohomologically balanced.

## 4. Existence of $f$-Weierstrass points.

*Remark* 4.1. Take $f, X, C, E, k, g$ and $q$ as usual. If $q = 0$, then there is no $f$-Weierstrass point because $\mathbf{P}^1$ is homogeneous, every vector bundle on $\mathbf{P}^1$ is a direct sum of line bundles and every line bundle on $\mathbf{P}^1$ is homogeneous.

**Theorem 4.2.** *Assume* $\mathrm{char}\,(\mathbf{K}) = 0$. *Let* $C$ *be a smooth projective curve of genus* $q \geq 1$, $X$ *a smooth projective curve of genus* $g$ *and* $f : X \to C$ *a cohomologically balanced degree* $k$ *covering. Assume* $g - 1 \equiv 0 \pmod{k-1}$. *Then there exist* $f$*-Weierstrass points.*

*Proof*. Let $E$ be the bundle associated to $f$. By Remark 1.1 we have $\deg(E) = kq - q + 1 - g$. Hence the condition $g - 1 \equiv 0 \pmod{k-1}$ is equivalent to the conditions that the slope $\mu(E) := \deg(E)/\mathrm{rank}\,(E)$ of $E$ is an integer. Set $x := \mu(E) + 1 - q$. Set $W(E) := \{L \in \mathrm{Pic}^x(C) : h^0(C, E \otimes L^*) > 0\}$. Since $\mu(E)$ is an integer, $x$ is an integer and hence $W(E)$ is well defined. By [**11**, Remark 1.6 and Lemma 2.2], the condition $\mu(E) \in \mathbf{Z}$ implies that $W(E)$ is either $\mathrm{Pic}^x(C)$ or a divisor of $\mathrm{Pic}^x(C)$ whose cohomology class is a nonzero multiple of the $\Theta$-divisor of $\mathrm{Pic}^x(C)$. For this computation we do not need the stability of $E$. Since $f$ is cohomologically balanced, $W(E) \neq \mathrm{Pic}^x(C)$. Since the $\Theta$-divisor of $\mathrm{Pic}^x(C)$ is ample and $C$ is embedded in $\mathrm{Pic}^x(C)$ as the curve $j(C) := \{\mathbf{O}_C(xQ)\}Q \in C$, we have $j(C) \cap W(E) \neq \varnothing$. The points $Q \in C$ such that $\mathbf{O}_C(xQ) \in j(C) \cap W(E)$ are exactly the $f$-Weierstrass points.

*Remark* 4.3. Under the assumptions of Theorem 4.2, we even obtained an enumerative formula for the set of all $f$-Weierstrass points: set $p = 0$, $g = q$, $r = k - 1$ and $\delta = q - 1$ in the formula in the statement of [**11**, Lemma 2.2] and obtain $W(E) = (k-1)\Theta$. The discussion in [**11**, Remark 1.6] shows that to be sure of the existence of $f$-Weierstrass points we need to assume $g - 1 \equiv 0 \pmod{k-1}$: if $\mu(E) \notin \mathbf{Z}$ the expected codimension of $W(E)$ is at least two and hence even when $f$-Weierstrass points do exist, no enumerative formula for their weighted number exists (at least from this point of view).

**5. Stable associated bundle.** For any vector bundle $F$ on a smooth projective curve, set $\mu(F) := \deg(F)/\mathrm{rank}(F)$. Let $f : X \to C$ be a covering with $X, C$ smooth projective curves. We will say that $f$ is composite if there is a smooth projective curve $Y$ and coverings $a : X \to Y$, $b : Y \to C$, with $\deg(a) \geq 2$, $\deg(b) \geq 2$ and $f = b \circ a$. Hence, if $\deg(f)$ is prime, then $f$ is not composite. Take a factorization $f = b \circ a$ of the covering $f$. The sheaf $b_*(\mathbf{O}_Y)$ is a locally free subsheaf of $f * (\mathbf{O}_X)$ with $\mathrm{rank}(b_*(\mathbf{O}_Y)) = \deg(b)$. The sheaf $b_*(\mathbf{O}_Y)$ is an $\mathbf{O}_C$-subalgebra of $f_*(\mathbf{O}_X)$ with unity, i.e., containing $\mathbf{O}_C$. If either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) > \deg(f)$, then $b_*(\mathbf{O}_Y)$ is a direct summand of $f_*(\mathbf{O}_X)$. Vice versa, let $B$ be a locally free subsheaf of $f_*(\mathbf{O}_X)$ containing $\mathbf{O}_C$ and closed under the algebra product of $f_*(\mathbf{O}_X)$. Since any finite $\mathbf{O}_C$-algebra induces a finite covering of $C$, the algebra $B$ induces a factorization of $f$ as $f = \alpha \circ \beta$ with $\beta : D \to C$ and $\deg(\beta) = \mathrm{rank}(B)$. In general $D$ may be singular, but since $C$ and $X$ are smooth, there is a factorization $f = a \circ b$ of $f$ with $b : Y \to C$, $Y$ normalization of $D$ and $b$ composition of $\beta$ with the normalization map $Y \to D$. The $\mathbf{O}_C$-algebra $b_*(\mathbf{O}_Y)$ is the saturation of $B$ in $f_*(\mathbf{O}_X)$, i.e., the only subsheaf of $f_*(\mathbf{O}_X)$ with $B \subseteq b_*(\mathbf{O}_Y)$, $\mathrm{rank}(b_*(\mathbf{O}_Y)) = \mathrm{rank}(B)$ and $f_*(\mathbf{O}_X)/b_*(\mathbf{O}_Y)$ torsion-free (i.e., locally free because $C$ is a smooth curve).

**Lemma 5.1.** *Let $C$ be a smooth projective curve of genus $q$, $P \in C$ and $A, B$ vector bundles on $C$ with $0 > \mu(A) > 1 + 1/\mathrm{rank}(A) + \mu(B)$. Then there exists an integer $t > 0$ such that $h^0(C, A(tP)) \neq 0$ and $h^1(C, B(tP)) \neq 0$.*

*Proof.* For any vector bundle $F$ on $C$ and any integer $x$ we have $\mu(F(xP)) = \mu(F) + x$. Hence there is an integer $t > 0$ such that $q + 1/\mathrm{rank}(A) \geq \mu(A(tP)) > q-1$ and $\mu(B(tP)) < q-1$. By Riemann-Roch we have $h^0(C, A(tP)) \neq 0$ and $h^1(C, B(tP)) \neq 0$. $\square$

**Proposition 5.2.** *Let $C, X, Z$ be smooth projective curves and $f : X \to C$, $h : Z \to X$ finite coverings with* $\deg(f) \geq 2$ *and* $\deg(h) \geq 2$. *Set $G := p_a(Z)$, $g := p_a(X)$, $q := p_a(C)$, $k :=$* $\deg(f)$ *and*

$x := \deg(h)$. *Assume that* $G > g(x+1)+kx-k$ *and either* $\mathrm{char}(\mathbf{K}) = 0$ *or* $\mathrm{char}(\mathbf{K}) > \max\{k, x\}$. *Then the covering* $h \circ f : Z \to C$ *is not cohomologically balanced.*

*Proof.* Let $F$, respectively $H$, respectively $E$, be the bundle associated to $f$, respectively $h$, respectively $f \circ h$. We have $E \cong F \oplus f_*(H)$. By Remark 1.1 we have $\deg(F) = kq - q - g + 1$ and $\deg(f_*(H)) = \deg(H) - \deg(F) = kxq - kq + g - G$. Hence $\mu(F) - \mu(f_*(H)) = (1-g)/(k-1) - (g-G)/(kx-k) > 1 + 1/\mathrm{rank}(F)$. Hence we conclude by Lemma 5.1. $\square$

Several examples of towers of the étale double coverings discussed in Example 2.2 show that in the statement of Proposition 5.2 we need to assume some conditions on $G$, $g$ and $q$, i.e., we need enough ramification.

Now we will study the numerical sequence of a covering.

*Remark* 5.3. Let $A_0, A_1, \ldots, A_x$, $1 \le x \le k$, be the Harder-Narasimhan filtration of the vector bundle $f_*(\mathbf{O}_X)$; for the reader's sake we recall the meaning of this filtration (see, e.g., [**8**, Section 1]); $A_0 = \{0\}$; $A_x = f_*(\mathbf{O}_X)$; $x = 1$ if and only if $f_*(\mathbf{O}_X)$ is semistable; $A_1$ is a semistable subsheaf of $f_*(\mathbf{O}_X)$ with maximal slope and among the subsheaves of $f_*(\mathbf{O}_X)$ with this property $A_1$ is only one with maximal rank and contains all other ones; for every integer $i$ with $1 \le i < x$ the vector bundle $A_i$ is a proper saturated subbundle of $A_{i+1}$ and the vector bundles $A_{i+1}/A_i$, $1 \le i < x$, are semistable; if $1 \le i < x$, then $\mu(A_{i+1}/A_i) < \mu(A_i/A_{i-1})$. Let $y$ be the maximal integer $i$ with $1 \le i \le x$ and such that $\mu(A_i/A_{i-1}) \ge 0$. By the very definition of $y$ we have $\mathbf{O}_C \subseteq A_y$. Now assume either $\mathrm{char}(\mathbf{K}) = 0$ or $q = 1$. Under one of these assumptions, if $A, B$ are semistable, then $A \otimes B$ is semistable (if $\mathrm{char}(\mathbf{K}) > 0$ and $q = 1$ one has to use Atiyah's classification of all vector bundles on an elliptic curve [**1**]; if $\mathrm{char}(\mathbf{K}) = 0$, this is due to Maruyama [**8**, Theorem 2.5]). For any two semistable vector bundles $A, B$ on $C$ with $\mu(A) > \mu(B)$, we have $h^0(C, \mathrm{Hom}(A, B)) = 0$. Hence we see that the multiplication map $A_y \otimes f_*(\mathbf{O}_X) \to f_*(\mathbf{O}_X)$ has an image contained in $A_y$. Thus $A_y$ is a proper saturated subalgebra of

$f_*(\mathbf{O}_X)$. Thus $A_y$ corresponds to a finite covering $b : Y \to C$ such that $f$ factors through $b$. We have $A_y = \mathbf{O}_C$ if and only if $b$ is the identity. By Remark 1.1, we have $\deg(A_y) \le 0$. Hence $\deg(A_y) = 0$ and $b$ is étale.

**5.4.** Here we consider the case $q = 1$ because only for $q = 1$ are we allowed to use Remark 5.3 in arbitrary characteristic. We use the notation of Remark 5.3 and set $B_i := A_i/A_{i-1}$, $1 \le i \le y$. Since $C$ is an elliptic curve, we have $h^1(C, \mathbf{O}_C(tQ)) = 0$ for every $Q \in C$ and every $t > 0$. Hence $N(f,Q,t) = n(f,Q,t) + t$ for every $Q \in C$ and every $t > 0$. Let $A$ be a semistable vector bundle on $C$. For a general $P \in C$ we have $h^0(C, A(tP)) = \deg(A) + t(\mathrm{rank}(A))$ and $h^1(C, A(tP)) = 0$ if $\deg(A) + t(\mathrm{rank}(A)) \ge 0$ and $h^1(C, A(tP)) = -\deg(A) - t(\mathrm{rank}(A))$ and $h^0(C, A(tP)) = 0$ if $\deg(A) + t(\mathrm{rank}(A)) \le 0$. We have $\mathrm{rank}(A_1) = N(f, 1)$. More generally, the knowledge of all pairs of integers $(\mathrm{rank}(A_i), \deg(A_i))$, $1 \le i \le y$, is equivalent to the knowledge of the numerical sequence $n(f)$ of $f$. Fix any semistable vector bundle $A$ on $C$, any $t \in \mathbf{Z}$ and any $P, Q \in C$; we have $h^0(C, A(tP)) = \deg(A) + t(\mathrm{rank}(A)) = h^0(C, A(tQ))$ if $\deg(A) + t(\mathrm{rank}(A))\deg(A) + t(\mathrm{rank}(A)) > 0$, while $h^0(C, A(tP)) = 0 = h^0(C, A(tQ))$ if $\deg(A) + t(\mathrm{rank}(A)) < 0$ [**1**, Lemma 15]; now fix $P_0 \in C$; if $\deg(A) + t(\mathrm{rank}(A)) = 0$ for a general $P' \in C$ we have $h^0(C, A(tP')) = 0$ but there exists a unique $Q' \in C$ with $h^0(C, A(tQ')) \neq 0$ (and indeed $h^0(C, A(tQ')) = 1$) [**1**, Theorem 5]. Hence there are $f$-Weierstrass points if and only if some bundle $B_i$, $1 \le i \le x$, has integer slope $\mu(B_i) = \deg(B_i)/\mathrm{rank}(B_i)$. Using Atiyah's classification of vector bundles on elliptic curves, we may restate the result we just proved in the following way.

**Proposition 5.5.** *Let $f : X \to C$ be a covering with $C$ elliptic curve and $E$ the bundle associated to $f$. There is no $f$-Weierstrass point if and only if every semistable graded subquotient of the Harder-Narasimhan of $E$ has rank at least two and it is stable.*

In particular if $k = 2$ or $k = 3$ and $g$ is even, every degree $k$ covering has an $f$-Weierstrass point, while if $k = 3$ and $g$ is odd a covering $f$ has an $f$-Weierstrass point if and only if its associated bundle is the direct sum of two line bundles. Notice that for any rank 2 indecomposable vector bundle, $E$, on $C$ with $\deg(E)$ odd and $\deg(E) \leq -3$, we have $h^0(C, \mathrm{Hom}(\mathbf{S}^3(E), \det(E))) \neq 0$ because $\mathbf{S}^3(E)^*$ is semistable and with slope $-3\mu(E)$, while $\det(E)$ has slope $-2\mu(E)$. Hence by [**9**, Theorem 3], if $g$ is odd we may find a degree 3 covering $f : X \to C$ with $X$ of genus $g$ and stable associated bundle; we need to check the smoothness of $X$; here we need to use [**9**, 5.1 and 5.2], and that the semistable sheaf $\mathrm{Hom}(S^3(E), \det(E))$ is spanned by its global sections. For every $g$ it is easy using [**9**, Section 6] to find triple coverings of the elliptic curve $C$ with a decomposable rank 2 vector bundle as associated bundle and with total space smooth and of genus $g$.

*Remark* 5.6. Using the quoted result [**1**, Theorem 5], we see that the knowledge of the numerical sequences $n(f, Q)$ for all $Q \in C$ also gives for all integers $u$ the number of the indecomposable factors of $E$ with slope $u$.

**5.7** Here we consider the case $k = 3$, $q \geq 1$. We assume either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) \geq 5$. We will heavily use the description of triple coverings made in [**9**]. Let $E$ be the bundle associated to the triple coverings $f : X \to C$. By Remark 1.1 or [**9**, (9.1)], we have $\deg(E) = 3q - 2 - g$. First assume $E$ decomposable, say $E \cong L^* \oplus R^*$ with $L \in \mathrm{Pic}(C)$, $R \in \mathrm{Pic}(C)$ and $\deg(L) \leq \deg(M)$. By [**9**, Section 6], both $L^{\otimes 2} \otimes M^*$ and $M^{\otimes 2} \otimes L^*$ are effective. In particular we have $\deg(M) \leq 2(\deg(L))$. It is well known that in the case $q = 0$ (i.e., in the case of trigonal pencils) these inequalities are sharp and that for all pairs of integers $a, b$ with $-a \geq -b \geq -2b > 0$ there is a degree 3 pencil $f : X \to \mathbf{P}^1$ with $X$ smooth and irreducible and such that the associated bundle is the direct sum of a line bundle of degree $a$ and a line bundle of degree $b$ [**9**, Section 9]. Here we study the general case for $q > 0$ allowing the case in which the associated bundle is indecomposable.

**5.8** Let $C$ be a smooth curve of genus $q$ and $E$ is a rank 2 vector bundle on $C$. The case of étale triple coverings of $C$ (i.e., the case $g = 3q - 2$) is classical. Hence we will study only coverings which are not étale. Since 3 is prime, by Remark 5.3 the Harder-Narasimhan filtration of the associated bundle has all slopes $< 0$. Hence, since every rank two vector bundle on $C$ is an extension of two line bundles, without losing any generality we may assume that $E$ fits in an exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0 \tag{1}$$

with $\deg(A) < 0$. Set $a := \deg(A)$ and $b := \deg(B)$. Hence $\deg(E) = a + b$. We will always assume that $a$ is maximal among all the degrees of the rank 1 subbundles of $E$. Hence if $a > b$, then (1) is the Harder-Narasimhan filtration of $E$ and $A$ is uniquely determined by $E$, while if $a \le b$, then $A$ may not be uniquely determined by $E$, but $E$ fits at most in a one-dimensional family of extensions (1) with invariants $a$ and $B$ except in the case $E \cong A \oplus A$ ([**7**, Corollary 4.6] and a similar analysis in the case $a = b$ in which $E$ is semistable but not stable). The multiplication of map $f_*(\mathbf{O}_X) \otimes f_*(\mathbf{O}_X) \to f_*(\mathbf{O}_X)$ induces two maps $\alpha : A \otimes A \to E$ and $\beta : A \otimes A \to \mathbf{O}_C$. Composing $\alpha$ with the surjection $E \to B$ given by (1) we obtain a map $\gamma : A \otimes A \to B$. If $\gamma = 0$, then $\alpha$ induces a map $\delta : A \otimes A \to A$ and $\alpha = 0$ if and only if $\gamma = \delta = 0$.

(a) Here we assume $\gamma = \delta = 0$, i.e., $\alpha = 0$. Hence $\beta \ne 0$. Thus $h^0(C, (A^*)^{\otimes 2}) \ne 0$ and the map $\beta$ is uniquely determined, up to a multiplicative constant, by the choice of an effective divisor $D \in |(A^*)^{\otimes 2}|$. The multiplication map $f_*(\mathbf{O}_X) \otimes f_*(\mathbf{O}_X) \to f_*(\mathbf{O}_X)$ sends the saturated subsheaf $\mathbf{O}_C \oplus A$ of $f_*(\mathbf{O}_X)$ into $\mathbf{O}_C$ and in particular into $\mathbf{O}_C \oplus A$. Thus $(\mathbf{O}_C \oplus A, D)$ gives a double covering of $C$ and the triple covering $f$ factors through this double covering (see the beginning of this section). Hence this case cannot occur.

(b) Here we assume $\gamma = 0$ and $\delta \ne 0$. Hence $h^0(C, A^*) \ne 0$. The multiplication map $f_*(\mathbf{O}_X) \otimes f_*(\mathbf{O}_X) \to f_*(\mathbf{O}_X)$ sends the saturated subsheaf $\mathbf{O}_C \oplus A$ of $f_*(\mathbf{O}_X)$ into $\mathbf{O}_C$ and in particular into $\mathbf{O}_C \oplus A$. Hence, as in case (b), this case cannot occur.

(c) By the analysis of cases (a) and (b) we may assume $\gamma \ne 0$. Hence $h^0(C, B \otimes (A^*)^{\otimes 2}) > 0$. In particular we have $-2a \ge -b$ and $-2a = -b$ if and only if $B \cong A^{\otimes 2}$. If $-2a \ge -b + q$, then for any $A, B$ we have $h^0(C, B \otimes (A^*)^{\otimes 2}) > 0$. If $2a \le -b + 1 - q$ for any fixed $B \in \operatorname{Pic}^b(C)$, respectively $A \in \operatorname{Pic}^a(C)$, we have $h^0(C, B \otimes (A^*)^{\otimes 2}) = 0$ for a general $A \in \operatorname{Pic}^a(C)$, respectively $B \in \operatorname{Pic}^a(C)$, while we have $h^0(C, B \otimes (A^*)^{\otimes 2}) > 0$ for some $A \in \operatorname{Pic}^a(C)$, respectively $B \in \operatorname{Pic}^a(C)$. Hence we see that when $q > 0$ not only there are restrictions on the possible numerical data $(a, b)$ but for certain pairs $(a, b)$ there are restrictions on the possible pairs of line bundles $(A, B)$ with $(a, b)$ as degrees and such that an associated bundle $E$ for some triple covering fits in (1). As remarked before, by [**9**, Section 6], we have $h^0(C, A \otimes (B^*)^{\otimes 2}) > 0$ and hence $-2b \geq -a$ if (1) splits. Since $b \leq 0$ (and even $b < 0$ by our assumption $f$ not étale) the inequality $-2b \geq -a$ is satisfied if $a \geq b$, i.e., if $E$ is not stable. Now assume $E$ stable, i.e., $a < b$. By a theorem of Segre and Nagata we have $b \leq a + q$ (see, e.g., the introduction of [**7**]). Since $a + b = 3q - 2 - g$, we have $2a \geq 2q - 2 - g$. Hence if $g \geq 6q - 2$ the inequality $-2b \geq -a$ is satisfied for the associated bundle of any triple covering.

**Proposition 5.9.** *Assume* $\mathrm{char}\,(\mathbf{K}) = 0$, $q \geq 1$ *and* $n(f, 1) > 0$. *Then* $q = 1$ *and* $f$ *factors through an étale covering* $b : Y \to C$ *with* $\deg\,(b) \geq n(f, 1) + 1$.

*Proof.* Since $\mathrm{char}\,(\mathbf{K}) = 0$, we have $h^0(C, E) = 0$. For a general $P \in C$ we have $h^0(C, E(p)) = n(f, 1) > 0$. □

Now we will show that if $q \geq 3$ and if $g + q - kq - 1$ is small, there are strong restrictions for a vector bundle $E$ of rank $k - 1$ and degree $kq - q - g + 1$ to be the bundle associated to a covering $f : X \to C$ with invariants $k, g$ and $q$ (see Theorem 5.12). We recall that, for any smooth curve $C$ of genus $q \geq 2$ and all integers $r, d$ with $r > 0$, the moduli scheme $M(C; u, v)$ of all stable vector bundles, $F$, on $C$ with $\mathrm{rank}\,(F) = r$ and $\deg\,(F) = d$ is a smooth irreducible variety of dimension $(r^2 - 1)(q - 1) + 1$. To prove Theorem 5.12 we need the following two results which are variations on the theme played in [**3**]. As far as we know, Proposition 5.11 could be deducted from [**3**], but not Proposition 5.10, except for $r = 2$.

**Proposition 5.10.** *Let* $C$ *be a smooth projective curve of genus* $q \geq 3$. *Fix integers* $r, d$ *with* $r > 0$ *and* $d \leq (2 - q)r$. *Then for a general* $A \in M(C; r, d)$ *we have* $h^0(C, \mathrm{Hom}\,(A \otimes A, A)) = 0$.

*Proof.* Just to fix the notation we assume $d < 0$; indeed, the case $d \geq 0$ is obviously true for every $A \in M(C; r, d)$, except the case $r = 1$,

$d = 0$ and $A = \mathbf{O}_C$. Set $d = ar + b$ with $0 \le b < r$. Take $r-b$ general line bundles $L_1, \dots, L_{r-b}$ on $C$ with $\deg(L_i) = a$ for every $i$ and $b$ general line bundles $L_{r-b+1}, \dots, L_{r-b+1}$ of degree $a+1$. By assumption we have $a \le q-2$ and $a \le q-3$ if $b > 0$. Set $F := \oplus_{1 \le i \le r} L_i$. Hence $\mathrm{rank}\,(F) = r$ and $\deg(F) = d$. The vector bundle $\mathrm{Hom}\,(F \otimes F, F)$ is a direct sum of line bundles, all of degree at most $a + 2$ (case $b > 0$) or degree $a$ (case $b = 0$). Each indecomposable factor, $M$, of $\mathrm{Hom}\,(F \otimes F, F)$ may be considered as a general element of its component of $\mathrm{Pic}\,(C)$ because $M \cong L_k \otimes L_i^* \otimes L_j^*$ for some $i, j, k$. Hence $h^0(C, M) = 0$ because $\deg(M) \le q - 1$. Hence $h^0(C, \mathrm{Hom}\,(F \otimes F, F))$. Now we use the semicontinuity theorem for cohomology and the fact that $F$ is a flat limit of a family of stable vector bundles (see [**10**, Proposition 2.6] or, for an easy proof in arbitrary characteristic, [**5**, Corollary 2.2]).

The same proof gives the following result.

**Proposition 5.11.** *Let $C$ be a smooth projective curve of genus $q \ge 3$. Fix integers $r, d$ with $r > 0$ and $d \le (2-q)r$. Then for a general $A \in M(C; r, d)$ we have $h^0(C, \mathrm{Hom}\,(A \otimes A, O_C)) = 0$.*

**Theorem 5.12.** *Fix integers $k, g$ and $q$ with $q \ge 3$, $k \ge 2$ and $g \le (2q-2)(k-1)+1$. Let $C$ be a smooth projective curve of genus $q$. Then the general $A \in M(C; k-1, kq-q+1-g)$ is not the associated bundle of a degree $k$ covering $f : X \to C$ with $X$ smooth curve of genus $g$.*

*Proof.* Assume the existence of such covering $f : X \to C$ with $A$ as associated bundle. The multiplication map $f_*(\mathbf{O}_X) \otimes f_*(\mathbf{O}_X) \to f_*(\mathbf{O}_X)$ induces two maps $\alpha : A \otimes A \to A$ and $\beta : A \otimes A \to \mathbf{O}_C$ and at least one of them cannot be identically zero, contradicting Propositions 5.10 and 5.11. $\square$

*Remark* 5.12. If $d/r \in \mathbf{Z}$, then the proofs of d6 and d7 work verbatim with the weaker assumption $d \le r(1-q)$. Hence when $g-1$ is divisible by $k-1$ the thesis of Theorem d8, just assuming $g \le (2q-1)(k-1)+1$.

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